Journal of Uncertain Systems
Vol.5, No.3, pp.227-240, 2011
Online at: www.jus.org.uk

# Some New Results about Arithmetic of Type-2 Fuzzy Variables*

Yanju Chen[†], Liwei Zhang
*College of Mathematics and Computer Science, Hebei, University, Baoding 071002, China*

Received 1 October 2010; Revised 23 Feburuary 2011

**Abstract**

Type-2 (T2) fuzzy variable is an extension of an ordinary fuzzy variable. In fuzzy possibility theory, T2 fuzzy variable is defined as a measurable map from the universe to the set of real numbers, and the possibility of a T2 fuzzy variable takes on a real number is a regular fuzzy variable (RFV). T2 fuzziness, which is usually used to handle linguistic uncertainties, can be described as T2 fuzzy variable. In this paper, we discuss some new results about T2 fuzzy arithmetic, which have applications in fuzzy optimization and decision making problems.


**Keywords:** fuzzy possibility theory, regular fuzzy variable, T2 fuzzy variable, fuzzy arithmetic

## 1 Introduction

The concept of a T2 fuzzy set as an extension of an ordinary fuzzy set was introduced by Zadeh [23]. But in the 1970's, there were only a few researchers to study T2 fuzzy sets. For instance, Mizumoto and Tanaka [14] discussed what kinds of algebraic structures the grades of T2 fuzzy sets form under join, meet and negation, and showed that normal convex fuzzy grades form a distributive lattace under the join and meet; Nieminen [16] studied on the algebraic structure of T2 fuzzy sets; Dubois and Prade [2] investigated the operations in a fuzzy-valued logic; Yager [22] applied the T2 fuzzy set to decision making. A T2 fuzzy set is characterized by a fuzzy membership function. A T2 fuzzy set represents the uncertainty in terms of secondary membership function and footprint of uncertainty [12]. Now, T2 fuzzy sets have been applied successfully to T2 fuzzy logic systems [5, 6], pattern recognition [13, 25, 26], and etc. [3, 4, 8, 9].

Liu and Liu [11] presented the fuzzy possibility theory which is a generalization of the usual possibility theory [7, 15, 17, 20, 21, 24]. The paper introduced some fundamental concepts in the proposed theory, such as fuzzy possibility measure defined as a set function from the ample field to a collection of RFV values, fuzzy possibility space (FPS), T2 fuzzy variable defined as a measurable map from the universe to the set of real numbers and the possibility of a T2 fuzzy variable takes on a real number is an RFV, T2 possibility distribution function, secondary possibility distribution function. To characterize the properties of T2 fuzzy variables in some aspects, Chen and Wang [1] presented a scalar representative value operator for T2 fuzzy variable. They also discussed some properties of the representative value operator. For discrete T2 fuzzy variable and T2 triangular fuzzy variable, they obtained the computational formulas of the representative value. To defuzzify type-2 fuzzy variables, Qin et al. [18, 19] gave the mean reduction methods and the critical value reduction methods for the type-2 fuzzy variable. The reference [11] also provided the theoretical foundation for the arithmetic of T2 fuzzy variables. In this paper, for three kinds of common T2 fuzzy variables, we give some new results of T2 fuzzy arithmetic.

The paper is organized as follows. We first recall several required fundamental concepts in Section 2. Section 3 gives the arithmetic results of some T2 triangular fuzzy variables. The arithmetic results of some T2 trapezoid fuzzy variables are described in Section 4. In Section 5, we conclude the arithmetic results of some T2 normal fuzzy variables. Finally, Section 6 gives the conclusions.

*This work is supported by the National Natural Science Foundation of China (No.60974134), the Natural Science Foundation and the Education Department of Hebei Province (No.A2011201007, No.F2010000318, No.2010109), and the Scientific Research andDevelopment Program of Baoding (No.10ZR002).
†Corresponding author. Email: yanjuchen@hbu.edu.cn (Y. Chen).

## 2 Preliminaries

Let $\Gamma$ be the universe of discourse. An ample field $\mathcal{A}$ on $\Gamma$ is a class of subsets of $\Gamma$ that is closed under arbitrary unions, intersections and complement in $\Gamma$. Let $\xi$ be a fuzzy variable which was defined on the possibility space $(\Gamma, \mathcal{A}, \mathrm{Pos})$ [21] with possibility distribution function $\mu : \Re \to [0, 1]$.

An $m$-ary regular fuzzy vector $\xi = (\xi_1, \xi_2, \ldots, \xi_m)$ is defined as a vector from $\Gamma$ to the set $[0, 1]^m$, i.e., for any $\gamma \in \Gamma$, $\xi(\gamma) = (\xi_1(\gamma), \xi_2(\gamma), \ldots, \xi_m(\gamma)) \in [0, 1]^m$. As $m$=1, $\xi$ is called an RFV. For example, $\xi = (r_1, r_2, r_3)$ with $0 \le r_1 < r_2 < r_3 \le 1$ is a triangular RFV. A fuzzy variable which only takes on value 0 with possibility 1 is an RFV, denoted by $\tilde{0}$. A fuzzy variable which only takes on value 1 with possibility 1 is an RFV, denoted by $\tilde{1}$.

In this paper, we denote by $\mathcal{R}([0, 1])$ as the collection of all RFVs on $[0, 1]$.

**Definition 1** [10] *Let $\xi_i, 1 \le i \le m$ be $m_i$-ary regular fuzzy vectors defined on a possibility space* $(\Gamma, \mathcal{A}, \mathrm{Pos})$, *respectively. They are said to be mutually independent if*

$$\mathrm{Pos}\{\gamma \in \Gamma \mid \xi_1(\gamma) = t_1, \ldots, \xi_m(\gamma) = t_m\} = \min_{1 \le i \le m} \mathrm{Pos}\{\gamma \in \Gamma \mid \xi_i(\gamma) = t_i\} \tag{1}$$

*for any $t_i = (t_1^{(i)}, \ldots, t_{m_i}^{(i)}) \in [0, 1]^{m_i}$ and $i = 1, \cdots, m$.*

*Moreover, a family of regular fuzzy vectors $\{\xi_i, i \in I\}$ is said to be mutually independent if for each integer $m$, and $i_1 < i_2 < \cdots < i_m$, the regular fuzzy vectors $\xi_{i_k}, k = 1, 2, \ldots, m$ are mutually independent.*

**Definition 2** [11] *Let $\mathcal{A}$ be an ample field on the universe $\Gamma$, and $\tilde{\mathrm{Pos}} : \mathcal{A} \mapsto \mathcal{R}([0, 1])$ a set function on $\mathcal{A}$ such that $\{\tilde{\mathrm{Pos}}(A) \mid \mathcal{A} \ni A \text{ atom}\}$ is a family of mutually independent RFVs. We call $\tilde{\mathrm{Pos}}$ a fuzzy possibility measure if it satisfies the following conditions:*

**Pos1)** $\tilde{\mathrm{Pos}}(\emptyset) = \tilde{0}$;

**Pos2)** *For any subclass $\{A_i \mid i \in I\}$ of $\mathcal{A}$ (finite, countable or uncountable),*

$$\tilde{\mathrm{Pos}}\left(\bigcup_{i \in I} A_i\right) = \sup_{i \in I} \tilde{\mathrm{Pos}}(A_i).$$

*Moreover, if $\mu_{\tilde{\mathrm{Pos}}(\Gamma)}(1) = 1$, then we call $\tilde{\mathrm{Pos}}$ a regular fuzzy possibility measure.*

The triplet $(\Gamma, \mathcal{A}, \tilde{\mathrm{Pos}})$ is referred to as a fuzzy possibility space (FPS).

If the universe $\Gamma$ is a finite set, then the ample field $\mathcal{A}$ on $\Gamma$ is an algebra containing a finite number of subsets of $\Gamma$. Therefore, the axiom Pos2) in Definition 2 can be replaced by

$$\tilde{\mathrm{Pos}}\left(\bigcup_{i=1}^{n} A_i\right) = \max_{1 \le i \le n} \tilde{\mathrm{Pos}}(A_i)$$

for any finite subclass $\{A_i, i = 1, \ldots, n\}$ of $\mathcal{A}$.

If $\mathcal{A}$ is the power set of the universe $\Gamma$, then the atoms of $\mathcal{A}$ are all single point sets $\{\gamma\}, \gamma \in \Gamma$. Therefore, in order to define a fuzzy possibility measure on $\mathcal{A}$, it suffices to give the value of $\tilde{\mathrm{Pos}}$ at each single point set.

**Definition 3** [11] *Let $(\Gamma, \mathcal{A}, \tilde{\mathrm{Pos}})$ be an FPS. A map $\xi = (\xi_1, \xi_2, \ldots, \xi_m) : \Gamma \mapsto \Re^m$ is called an $m$-ary T2 fuzzy vector if for any $x = (x_1, x_2, \ldots, x_m) \in \Re^m$, the set $\{\gamma \in \Gamma \mid \xi(\gamma) \le x\}$ is an element of $\mathcal{A}$, i.e.,*

$$\{\gamma \in \Gamma \mid \xi(\gamma) \le x\} = \{\gamma \in \Gamma \mid \xi_1(\gamma) \le x_1, \ldots, \xi_m(\gamma) \le x_m\} \in \mathcal{A}. \tag{2}$$

*As $m = 1$, the map $\xi : \Gamma \mapsto \Re$ is called a T2 fuzzy variable.*

**Definition 4** [11] *Let $\xi_i, i = 1, 2, \ldots, m$ be T2 fuzzy variables defined on an FPS $(\Gamma, \mathcal{A}, \tilde{\mathrm{Pos}})$. They are said to be mutually independent if*

$$\tilde{\mathrm{Pos}}\left(\{\gamma \in \Gamma \mid \xi_1(\gamma) \in B_1, \cdots, \xi_m(\gamma) \in B_m\}\right) = \min_{1 \le i \le m} \tilde{\mathrm{Pos}}\left(\{\gamma \in \Gamma \mid \xi_i(\gamma) \in B_i\}\right) \tag{3}$$

*for any $B_i \subset \Re, i = 1, 2, \cdots, m$, where $\tilde{\mathrm{Pos}}(\{\gamma \in \Gamma \mid \xi_i(\gamma) \in B_i\}), i = 1, 2, \cdots, m$ are supposed to be mutually independent RFVs.*

*Moreover, a family of T2 fuzzy variables $\{\xi_i \mid i \in I\}$ is said to be mutually independent if for each integer $m \ge 2$, and $i_1 < i_2 < \cdots < i_m$, the T2 fuzzy variables $\xi_{i_k}, k = 1, 2, \cdots, m$ are mutually independent.*

**Definition 5** [11] *Let $\xi = (\xi_1, \xi_2, \ldots, \xi_m)$ be a T2 fuzzy vector defined on an FPS $(\Gamma, \mathcal{A}, \tilde{\text{Pos}})$. The secondary possibility distribution function of $\xi$, denoted by $\tilde{\mu}_\xi(x)$, is a map $\Re^m \mapsto \mathcal{R}([0,1])$ such that*

$$\tilde{\mu}_\xi(x) = \tilde{\text{Pos}}\{\gamma \in \Gamma \mid \xi(\gamma) = x\}, \quad x \in \Re^m, \tag{4}$$

*while the T2 possibility distribution function of $\xi$, denoted by $\mu_\xi(x,u)$, is a map $\Re^m \times J_x \mapsto [0,1]$ such that*

$$\mu_\xi(x,u) = \text{Pos}\left\{\tilde{\mu}_\xi(x) = u\right\}, \quad (x,u) \in \Re^m \times J_x \tag{5}$$

*where* Pos *is the possibility measure induced by the distribution of $\tilde{\mu}_\xi(x)$, and $J_x \subset [0,1]$ is the support of $\tilde{\mu}_\xi(x)$, i.e., $J_x = \{u \in [0,1] \mid \mu_\xi(x,u) > 0\}$.*

Liu and Liu [11] dealt with the arithmetic of T2 fuzzy variables as following.

**Theorem 1** [11] *If $\xi_i, i = 1, \ldots, n$ are T2 fuzzy variables on an FPS $(\Gamma, \mathcal{A}, \tilde{\text{Pos}})$, and $f$ is a real-valued function from $\Re^n$ to $\Re$, then the function $\eta = f(\xi_1, \xi_2, \ldots, \xi_n)$ defined by*

$$\eta(\gamma) = f(\xi_1(\gamma), \xi_2(\gamma), \ldots, \xi_n(\gamma)), \quad \gamma \in \Gamma \tag{6}$$

*is also a T2 fuzzy variable on the FPS, and its secondary possibility distribution function can be written as*

$$\tilde{\mu}_\eta(y) = \sup_{f(x_1,x_2,\ldots,x_n)=y} \tilde{\mu}_\xi(x_1, x_2, \ldots, x_n), \quad y \in \Re \tag{7}$$

*where $(x_1, x_2, \ldots, x_n) \in \Re^n$, and $\tilde{\mu}_\xi(x_1, x_2, \ldots, x_n)$ is the secondary possibility distribution function of $\xi = (\xi_1, \xi_2, \ldots, \xi_n)$.*

**Definition 6** [11] *The support of a T2 fuzzy vector $\xi$ is defined as*

$$supp\, \xi = \{(x,u) \in \Re^m \times [0,1] \mid \mu_\xi(x,u) > 0\}$$

*where $\mu_\xi(x,u)$ is the T2 possibility distribution function of $\xi$.*

A T2 fuzzy variable $\xi$ is called triangular [18] if its secondary possibility distribution $\tilde{\mu}_\xi(x)$ is

$$\left(\tfrac{x-r_1}{r_2-r_1} - \theta_l \min\{\tfrac{x-r_1}{r_2-r_1}, \tfrac{r_2-x}{r_2-r_1}\}, \tfrac{x-r_1}{r_2-r_1}, \tfrac{x-r_1}{r_2-r_1} + \theta_r \min\{\tfrac{x-r_1}{r_2-r_1}, \tfrac{r_2-x}{r_2-r_1}\}\right)$$

for $x \in [r_1, r_2]$, and

$$\left(\tfrac{r_3-x}{r_3-r_2} - \theta_l \min\{\tfrac{r_3-x}{r_3-r_2}, \tfrac{x-r_2}{r_3-r_2}\}, \tfrac{r_3-x}{r_3-r_2}, \tfrac{r_3-x}{r_3-r_2} + \theta_r \min\{\tfrac{r_3-x}{r_3-r_2}, \tfrac{x-r_2}{r_3-r_2}\}\right)$$

for $x \in [r_2, r_3]$, where $\theta_l, \theta_r \in [0,1]$ are two parameters characterizing the degree of uncertainty that $\xi$ takes the value $x$. For simplicity, we denote the T2 triangular fuzzy variable $\xi$ with the above distribution by $(\tilde{r}_1, \tilde{r}_2, \tilde{r}_3; \theta_l, \theta_r)$.

A T2 fuzzy variable $\xi$ is called trapezoid [18] if its secondary possibility distribution $\tilde{\mu}_\xi(x)$ is $\tilde{1}$ for $x \in [r_2, r_3]$,

$$\left(\tfrac{x-r_1}{r_2-r_1} - \theta_l \min\{\tfrac{x-r_1}{r_2-r_1}, \tfrac{r_2-x}{r_2-r_1}\}, \tfrac{x-r_1}{r_2-r_1}, \tfrac{x-r_1}{r_2-r_1} + \theta_r \min\{\tfrac{x-r_1}{r_2-r_1}, \tfrac{r_2-x}{r_2-r_1}\}\right)$$

for $x \in [r_1, r_2]$, and

$$\left(\tfrac{r_4-x}{r_4-r_3} - \theta_l \min\{\tfrac{r_4-x}{r_4-r_3}, \tfrac{x-r_3}{r_4-r_3}\}, \tfrac{r_4-x}{r_4-r_3}, \tfrac{r_4-x}{r_4-r_3} + \theta_r \min\{\tfrac{r_4-x}{r_4-r_3}, \tfrac{x-r_3}{r_4-r_3}\}\right)$$

for $x \in [r_3, r_4]$, where $\theta_l, \theta_r \in [0,1]$ are two parameters characterizing the degree of uncertainty that $\xi$ takes the value $x$. For simplicity, we denote the T2 trapezoid fuzzy variable $\xi$ with the above distribution by $(\tilde{r}_1, \tilde{r}_2, \tilde{r}_3, \tilde{r}_4; \theta_l, \theta_r)$.

A T2 fuzzy variable $\xi$ is called normal if its secondary possibility distribution $\tilde{\mu}_\xi(x)$ is

$$\Big(\exp(-\tfrac{(x-\mu)^2}{2\sigma^2}) - \theta_l \min\{1 - \exp(-\tfrac{(x-\mu)^2}{2\sigma^2}), \exp(-\tfrac{(x-\mu)^2}{2\sigma^2})\}, \exp(-\tfrac{(x-\mu)^2}{2\sigma^2}),$$
$$\exp(-\tfrac{(x-\mu)^2}{2\sigma^2}) + \theta_r \min\{1 - \exp(-\tfrac{(x-\mu)^2}{2\sigma^2}), \exp(-\tfrac{(x-\mu)^2}{2\sigma^2})\}\Big)$$

for any $x \in \Re$, where $\mu \in \Re$, $\sigma > 0$, and $\theta_l, \theta_r \in [0,1]$ are two parameters characterizing the degree of uncertainty that $\xi$ takes the value $x$. For simplicity, the T2 normal fuzzy variable $\xi$ with the above distribution is denoted by $\tilde{n}(\mu, \sigma^2; \theta_l, \theta_r)$.

# 3 The Linear Combination of T2 Triangular Fuzzy Variables

**Theorem 2** *Let $\xi = (\tilde{r}_1, \tilde{r}_2, \tilde{r}_3; \theta_l, \theta_r)$ be a T2 triangular fuzzy variable. Then for any real number $a \neq 0$, we have*

$$a\xi = \begin{cases} (a\tilde{r}_1, a\tilde{r}_2, a\tilde{r}_3; \theta_l, \theta_r), & \text{if } a > 0 \\ (a\tilde{r}_3, a\tilde{r}_2, a\tilde{r}_1; \theta_l, \theta_r), & \text{if } a < 0. \end{cases}$$

**Proof:** According to the definition of T2 triangular fuzzy variable, we have $\tilde{\mu}_{a\xi}(x) = \tilde{\mu}_\xi(\frac{x}{a})$. If $a > 0$, and $\frac{x}{a} \in [r_1, r_2]$, i.e., $x \in [ar_1, ar_2]$, then

$$\begin{aligned} \tilde{\mu}_{a\xi}(x) = \tilde{\mu}_\xi(\tfrac{x}{a}) &= \left(\tfrac{\frac{x}{a}-r_1}{r_2-r_1} - \theta_l \min\{\tfrac{\frac{x}{a}-r_1}{r_2-r_1}, \tfrac{r_2-\frac{x}{a}}{r_2-r_1}\}, \tfrac{\frac{x}{a}-r_1}{r_2-r_1}, \tfrac{\frac{x}{a}-r_1}{r_2-r_1} + \theta_r \min\{\tfrac{\frac{x}{a}-r_1}{r_2-r_1}, \tfrac{r_2-\frac{x}{a}}{r_2-r_1}\}\right) \\ &= \left(\tfrac{x-ar_1}{ar_2-ar_1} - \theta_l \min\{\tfrac{x-ar_1}{ar_2-ar_1}, \tfrac{ar_2-x}{ar_2-ar_1}\}, \tfrac{x-ar_1}{ar_2-ar_1}, \tfrac{x-ar_1}{ar_2-ar_1} + \theta_r \min\{\tfrac{x-ar_1}{ar_2-ar_1}, \tfrac{ar_2-x}{ar_2-ar_1}\}\right). \end{aligned}$$

If $a > 0$, and $\frac{x}{a} \in [r_2, r_3]$, i.e., $x \in [ar_2, ar_3]$, then

$$\begin{aligned} \tilde{\mu}_{a\xi}(x) = \tilde{\mu}_\xi(\tfrac{x}{a}) &= \left(\tfrac{r_3-\frac{x}{a}}{r_3-r_2} - \theta_l \min\{\tfrac{r_3-\frac{x}{a}}{r_3-r_2}, \tfrac{\frac{x}{a}-r_2}{r_3-r_2}\}, \tfrac{r_3-\frac{x}{a}}{r_3-r_2}, \tfrac{r_3-\frac{x}{a}}{r_3-r_2} + \theta_r \min\{\tfrac{r_3-\frac{x}{a}}{r_3-r_2}, \tfrac{\frac{x}{a}-r_2}{r_3-r_2}\}\right) \\ &= \left(\tfrac{ar_3-x}{ar_3-ar_2} - \theta_l \min\{\tfrac{ar_3-x}{ar_3-ar_2}, \tfrac{x-ar_2}{ar_3-ar_2}\}, \tfrac{ar_3-x}{ar_3-ar_2}, \tfrac{ar_3-x}{ar_3-ar_2} + \theta_r \min\{\tfrac{ar_3-x}{ar_3-ar_2}, \tfrac{x-ar_2}{ar_3-ar_2}\}\right). \end{aligned}$$

Therefore, for any real number $a > 0$, we have

$$a\xi = (a\tilde{r}_1, a\tilde{r}_2, a\tilde{r}_3; \theta_l, \theta_r).$$

If $a < 0$, and $\frac{x}{a} \in [r_1, r_2]$, i.e., $x \in [ar_2, ar_1]$, then

$$\begin{aligned} \tilde{\mu}_{a\xi}(x) = \tilde{\mu}_\xi(\tfrac{x}{a}) &= \left(\tfrac{x-ar_1}{ar_2-ar_1} - \theta_l \min\{\tfrac{x-ar_1}{ar_2-ar_1}, \tfrac{ar_2-x}{ar_2-ar_1}\}, \tfrac{x-ar_1}{ar_2-ar_1}, \tfrac{x-ar_1}{ar_2-ar_1} + \theta_r \min\{\tfrac{x-ar_1}{ar_2-ar_1}, \tfrac{ar_2-x}{ar_2-ar_1}\}\right) \\ &= \left(\tfrac{ar_1-x}{ar_1-ar_2} - \theta_l \min\{\tfrac{ar_1-x}{ar_1-ar_2}, \tfrac{x-ar_2}{ar_1-ar_2}\}, \tfrac{ar_1-x}{ar_1-ar_2}, \tfrac{ar_1-x}{ar_1-ar_2} + \theta_r \min\{\tfrac{ar_1-x}{ar_1-ar_2}, \tfrac{x-ar_2}{ar_1-ar_2}\}\right). \end{aligned}$$

If $a < 0$, and $\frac{x}{a} \in [r_2, r_3]$, i.e., $x \in [ar_3, ar_2]$, then

$$\begin{aligned} \tilde{\mu}_{a\xi}(x) = \tilde{\mu}_\xi(\tfrac{x}{a}) &= \left(\tfrac{ar_3-x}{ar_3-ar_2} - \theta_l \min\{\tfrac{ar_3-x}{ar_3-ar_2}, \tfrac{x-ar_2}{ar_3-ar_2}\}, \tfrac{ar_3-x}{ar_3-ar_2}, \tfrac{ar_3-x}{ar_3-ar_2} + \theta_r \min\{\tfrac{ar_3-x}{ar_3-ar_2}, \tfrac{x-ar_2}{ar_3-ar_2}\}\right) \\ &= \left(\tfrac{x-ar_3}{ar_2-ar_3} - \theta_l \min\{\tfrac{x-ar_3}{ar_2-ar_3}, \tfrac{ar_2-x}{ar_2-ar_3}\}, \tfrac{x-ar_3}{ar_2-ar_3}, \tfrac{x-ar_3}{ar_2-ar_3} + \theta_r \min\{\tfrac{x-ar_3}{ar_2-ar_3}, \tfrac{ar_2-x}{ar_2-ar_3}\}\right). \end{aligned}$$

Therefore, for any real number $a < 0$, we know that

$$a\xi = (a\tilde{r}_3, a\tilde{r}_2, a\tilde{r}_1; \theta_l, \theta_r).$$

The proof is complete.

**Theorem 3** *Let $\xi_1 = (\tilde{r}_1, \tilde{r}_2, \tilde{r}_3; \theta_l, \theta_r)$ and $\xi_2 = (\tilde{l}_1, \tilde{l}_2, \tilde{l}_3; \theta_l, \theta_r)$ be two mutually independent T2 triangular fuzzy variables, and $\xi = \xi_1 + \xi_2$. The secondary possibility distribution function of $\xi_1$ is $\tilde{\mu}_{\xi_1}(x), x \in \Re$, $\{\tilde{\mu}_{\xi_1}(x), x \in [r_1, r_3]\}$ is supposed to be a family of mutually independent RFVs. The secondary possibility distribution function of $\xi_2$ is $\tilde{\mu}_{\xi_2}(x), x \in \Re$, $\{\tilde{\mu}_{\xi_2}(x), x \in [l_1, l_3]\}$ is supposed to be a family of mutually independent RFVs. Then*

$$\xi = (\widetilde{r_1 + l_1}, \widetilde{r_2 + l_2}, \widetilde{r_3 + l_3}; \theta_l, \theta_r).$$

**Proof:** Since $\xi_i, i = 1, 2$ are mutually independent T2 fuzzy variables, according to Theorem 1 and Definition 4, the secondary possibility distribution function of $\xi$ is

$$\tilde{\mu}_\xi(x) = \sup_{x_1+x_2=x} \tilde{\mu}_{\xi_1}(x_1) \wedge \tilde{\mu}_{\xi_2}(x_2), \quad x \in \Re \tag{8}$$

where $\tilde{\mu}_{\xi_i}(x_i)$ is the secondary possibility distribution function of $\xi_i$.

By the definitions of $\tilde{\mu}_{\xi_i}(t),\ i = 1, 2$ and the Extension Principal of Zadeh, we have

$$\tilde{\mu}_{\xi_1}(x_1) \wedge \tilde{\mu}_{\xi_2}(x_2) = \tilde{\mu}_{\xi_1}(x_1)$$

in the following four cases $(i)-(iv)$:

$$(i)\quad r_1 \le x_1 \le r_2, l_1 \le x_2 \le l_2, \text{ and } \frac{x_1-r_1}{r_2-r_1} \le \frac{x_2-l_1}{l_2-l_1},$$

$$(ii)\quad r_1 \le x_1 \le r_2, l_2 \le x_2 \le l_3, \text{ and } \frac{x_1-r_1}{r_2-r_1} \le \frac{l_3-x_2}{l_3-l_2},$$

$$(iii)\quad r_2 \le t_1 \le r_3, l_1 \le t_2 \le l_2, \text{ and } \frac{r_3-x_1}{r_3-r_2} \le \frac{x_2-l_1}{l_2-l_1},$$

$$(iv)\quad r_2 \le x_1 \le r_3, l_2 \le x_2 \le l_3, \text{ and } \frac{r_3-x_1}{r_3-r_2} \le \frac{l_3-x_2}{l_3-l_2}.$$

So, for any $x_1, x_2$ such that $x_1 + x_2 = x$, we have the following results: in case $(i)$, if

$$\frac{x_1-r_1}{r_2-r_1} = \frac{x_2-l_1}{l_2-l_1}, \tag{9}$$

we have

$$\sup_{x_1+x_2=x} \tilde{\mu}_{\xi_1}(x_1) \wedge \tilde{\mu}_{\xi_2}(x_2) = \tilde{\mu}_{\xi_1}(x_1) = \tilde{\mu}_{\xi_2}(x_2);$$

in case $(ii)$, if

$$\frac{x_1-r_1}{r_2-r_1} = \frac{l_3-x_2}{l_3-l_2}, \tag{10}$$

we have

$$\sup_{x_1+x_2=x} \tilde{\mu}_{\xi_1}(x_1) \wedge \tilde{\mu}_{\xi_2}(x_2) = \tilde{\mu}_{\xi_1}(x_1) = \tilde{\mu}_{\xi_2}(x_2);$$

in case $(iii)$, if

$$\frac{r_3-x_1}{r_3-r_2} = \frac{x_2-l_1}{l_2-l_1}, \tag{11}$$

we have

$$\sup_{x_1+x_2=x} \tilde{\mu}_{\xi_1}(x_1) \wedge \tilde{\mu}_{\xi_2}(x_2) = \tilde{\mu}_{\xi_1}(x_1) = \tilde{\mu}_{\xi_2}(x_2);$$

in case $(iv)$, if

$$\frac{r_3-x_1}{r_3-r_2} = \frac{l_3-x_2}{l_3-l_2}, \tag{12}$$

we have

$$\sup_{x_1+x_2=x} \tilde{\mu}_{\xi_1}(x_1) \wedge \tilde{\mu}_{\xi_2}(x_2) = \tilde{\mu}_{\xi_1}(x_1) = \tilde{\mu}_{\xi_2}(x_2).$$

From (9), we have

$$x_1 = \frac{xr_2 - l_1r_2 + r_1l_2 - xr_1}{r_2+l_2-r_1-l_1}, \quad \frac{x_1-r_1}{r_2-r_1} = \frac{x-r_1-l_1}{r_2+l_2-r_1-l_1}.$$

From (11), we have

$$x_1 = \frac{xr_3 + l_1r_2 - r_3l_2 - xr_2}{r_3-r_2-l_2+l_1}, \quad \frac{r_3-x_1}{r_3-r_2} = \frac{x-r_3-l_1}{r_2+l_2-r_3-l_1}.$$

It is easy to know that $\frac{x-r_1-l_1}{r_2+l_2-r_1-l_1} > \frac{x-r_3-l_1}{r_2+l_2-r_3-l_1}$ and $r_1 + l_1 \le x \le r_2 + l_2$. That is to say

$$\begin{aligned}\tilde{\mu}_\xi(x) = \quad & \left(\frac{x-r_1-l_1}{r_2+l_2-r_1-l_1} - \theta_l \min\left\{\frac{x-r_1-l_1}{r_2+l_2-r_1-l_1}, \frac{l_2+r_2-x}{r_2+l_2-r_1-l_1}\right\}, \frac{x-r_1-l_1}{r_2+l_2-r_1-l_1},\right. \\ & \left. \frac{x-r_1-l_1}{r_2+l_2-r_1-l_1} + \theta_r \min\left\{\frac{x-r_1-l_1}{r_2+l_2-r_1-l_1}, \frac{l_2+r_2-x}{r_2+l_2-r_1-l_1}\right\}\right)\end{aligned}$$

for any $x \in [r_1 + l_1, r_2 + l_2]$. In the same way, from (10) and (12), we have

$$\begin{aligned}\tilde{\mu}_\xi(x) = \quad & \left(\frac{r_3+l_3-x}{r_3+l_3-r_2-l_2} - \theta_l \min\left\{\frac{r_3+l_3-x}{r_3+l_3-r_2-l_2}, \frac{x-l_2-r_2}{r_3+l_3-r_2-l_2}\right\}, \frac{r_3+l_3-x}{r_3+l_3-r_2-l_2},\right. \\ & \left. \frac{r_3+l_3-x}{r_3+l_3-r_2-l_2} + \theta_r \min\left\{\frac{r_3+l_3-x}{r_3+l_3-r_2-l_2}, \frac{x-l_2-r_2}{r_3+l_3-r_2-l_2}\right\}\right)\end{aligned}$$

for any $x \in [r_2 + l_2, r_3 + l_3]$.

If $x \le r_1 + l_1$ or $x \ge r_3 + l_3$, then $\tilde{\mu}_\xi(x) = \tilde{0}$. So

$$\xi = (\widetilde{r_1 + l_1}, \widetilde{r_2 + l_2}, \widetilde{r_3 + l_3}; \theta_l, \theta_r).$$

The proof is complete.

Under some wild assumption, according to Theorems 2 and 3, we know that the linear combination of a finite number of T2 triangular fuzzy variables, which have the same parameters $\theta_l$ and $\theta_r$, is also a T2 triangular fuzzy variable. This result can be prescribed as the following theorem.

**Theorem 4** *Let $\xi_1 = (\tilde{r}_1, \tilde{r}_2, \tilde{r}_3; \theta_l, \theta_r)$ and $\xi_2 = (\tilde{l}_1, \tilde{l}_2, \tilde{l}_3; \theta_l, \theta_r)$ be two mutually independent T2 triangular fuzzy variables, and $\xi = a\xi_1 + b\xi_2$ where $a, b \neq 0$ are two any real numbers. The secondary possibility distribution function of $\xi_1$ is $\tilde{\mu}_{\xi_1}(x), x \in \Re$, $\{\tilde{\mu}_{\xi_1}(x), x \in [r_1, r_3]\}$ is supposed to be a family of mutually independent RFVs. The secondary possibility distribution function of $\xi_2$ is $\tilde{\mu}_{\xi_2}(x), x \in \Re$, $\{\tilde{\mu}_{\xi_2}(x), x \in [l_1, l_3]\}$ is supposed to be a family of mutually independent RFVs. Then*

$$\xi = \begin{cases} (\widetilde{ar_1 + bl_1}, \widetilde{ar_2 + bl_2}, \widetilde{ar_3 + bl_3}; \theta_l, \theta_r), & \text{if } a, b > 0 \\ (\widetilde{ar_1 + bl_3}, \widetilde{ar_2 + bl_2}, \widetilde{ar_3 + bl_1}; \theta_l, \theta_r), & \text{if } a > 0 \text{ and } b < 0 \\ (\widetilde{ar_3 + bl_1}, \widetilde{ar_2 + bl_2}, \widetilde{ar_1 + bl_3}; \theta_l, \theta_r), & \text{if } a < 0 \text{ and } b > 0 \\ (\widetilde{ar_3 + bl_3}, \widetilde{ar_2 + bl_2}, \widetilde{ar_1 + bl_1}; \theta_l, \theta_r), & \text{if } a, b < 0. \end{cases}$$

**Example 1:** Let $\xi_1 = (\tilde{1}, \tilde{2}, \tilde{4}; 0.6, 0.8)$ be a T2 triangular fuzzy variable. The support of $\xi_1$ is showed in Figure 1. The secondary possibility distribution function of $\xi_1$ is $\tilde{\mu}_{\xi_1}(x), x \in \Re$, $\{\tilde{\mu}_{\xi_1}(x), x \in [1, 4]\}$ is supposed to be a family of mutually independent RFVs. Let $\xi_2 = (\tilde{2}, \tilde{3}, \tilde{4}; 0.6, 0.8)$ be a T2 triangular fuzzy variable. The support of $\xi_2$ is showed in Figure 2. The secondary possibility distribution function of $\xi_2$ is $\tilde{\mu}_{\xi_2}(x), x \in \Re$, $\{\tilde{\mu}_{\xi_2}(x), x \in [2, 4]\}$ is supposed to be a family of mutually independent RFVs. Also, we suppose that triangular fuzzy variables $\xi_1$ and $\xi_2$ are mutually independent. Then $\xi = \xi_1 + \xi_2 = (\tilde{3}, \tilde{5}, \tilde{8}; 0.6, 0.8)$ is a T2 triangular fuzzy variable. The support of $\xi$ is showed in Figure 3.

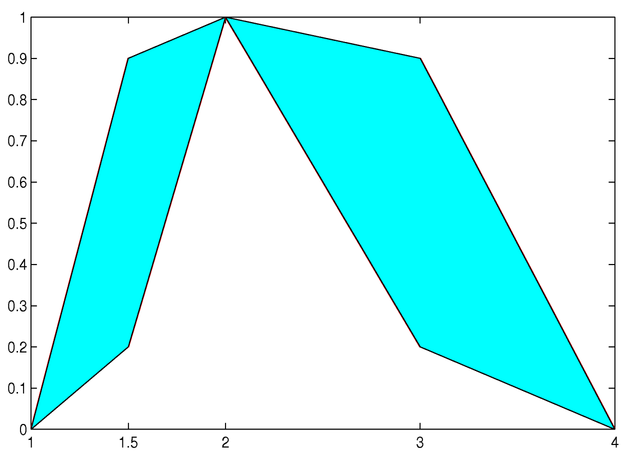


Figure 1: The support of the T2 fuzzy variable $\xi_1$ defined in Example 1

# 4 The Linear Combination of T2 Trapezoid Fuzzy Variables

**Theorem 5** *Let $\zeta = (\tilde{r}_1, \tilde{r}_2, \tilde{r}_3, \tilde{r}_4; \theta_l, \theta_r)$ be a T2 trapezoid fuzzy variable. Then for any real number $a \neq 0$, we have*

$$a\zeta = \begin{cases} (a\tilde{r}_1, a\tilde{r}_2, a\tilde{r}_3, a\tilde{r}_4; \theta_l, \theta_r), & \text{if } a > 0 \\ (a\tilde{r}_4, a\tilde{r}_3, a\tilde{r}_2, a\tilde{r}_1; \theta_l, \theta_r), & \text{if } a < 0. \end{cases}$$

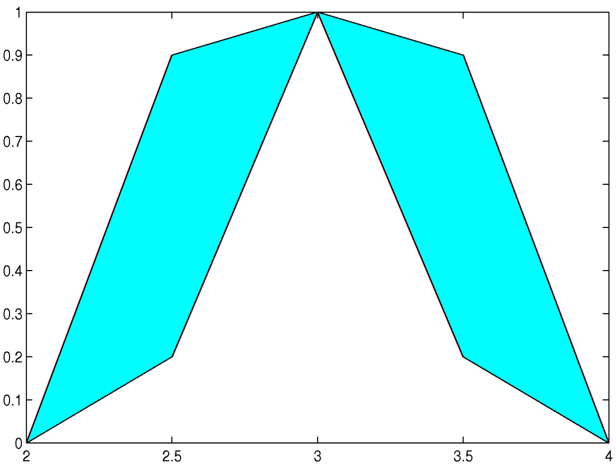


Figure 2: The support of the T2 fuzzy variable $\xi_2$ defined in Example 1

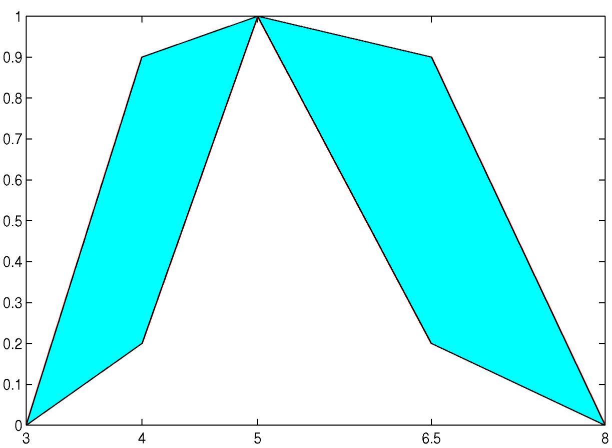


Figure 3: The support of the T2 fuzzy variable $\xi$ defined in Example 1

**Proof:** According to the definition of T2 trapezoid fuzzy variable, we have $\tilde{\mu}_{a\zeta}(x) = \tilde{\mu}_{\zeta}(\frac{x}{a})$. If $a > 0$, and $\frac{x}{a} \in [r_1, r_2]$, i.e., $x \in [ar_1, ar_2]$, then

$$\begin{aligned}&\tilde{\mu}_{a\zeta}(x) = \tilde{\mu}_{\zeta}(\tfrac{x}{a}) = \left(\tfrac{\frac{x}{a}-r_1}{r_2-r_1} - \theta_l \min\{\tfrac{\frac{x}{a}-r_1}{r_2-r_1}, \tfrac{r_2-\frac{x}{a}}{r_2-r_1}\}, \tfrac{\frac{x}{a}-r_1}{r_2-r_1}, \tfrac{\frac{x}{a}-r_1}{r_2-r_1} + \theta_r \min\{\tfrac{\frac{x}{a}-r_1}{r_2-r_1}, \tfrac{r_2-\frac{x}{a}}{r_2-r_1}\}\right)\\ &= \left(\tfrac{x-ar_1}{ar_2-ar_1} - \theta_l \min\{\tfrac{x-ar_1}{ar_2-ar_1}, \tfrac{ar_2-x}{ar_2-ar_1}\}, \tfrac{x-ar_1}{ar_2-ar_1}, \tfrac{x-ar_1}{ar_2-ar_1} + \theta_r \min\{\tfrac{x-ar_1}{ar_2-ar_1}, \tfrac{ar_2-x}{ar_2-ar_1}\}\right).\end{aligned}$$

If $a > 0$, and $\frac{x}{a} \in [r_3, r_4]$, i.e., $x \in [ar_3, ar_4]$, then

$$\begin{aligned}&\tilde{\mu}_{a\zeta}(x) = \tilde{\mu}_{\zeta}(\tfrac{x}{a}) = \left(\tfrac{r_4-\frac{x}{a}}{r_4-r_3} - \theta_l \min\{\tfrac{r_4-\frac{x}{a}}{r_4-r_3}, \tfrac{\frac{x}{a}-r_3}{r_4-r_3}\}, \tfrac{r_4-\frac{x}{a}}{r_4-r_3}, \tfrac{r_4-\frac{x}{a}}{r_4-r_3} + \theta_r \min\{\tfrac{r_4-\frac{x}{a}}{r_4-r_3}, \tfrac{\frac{x}{a}-r_3}{r_4-r_3}\}\right)\\ &= \left(\tfrac{ar_4-x}{ar_4-ar_3} - \theta_l \min\{\tfrac{ar_4-x}{ar_4-ar_3}, \tfrac{x-ar_3}{ar_4-ar_3}\}, \tfrac{ar_4-x}{ar_4-ar_3}, \tfrac{ar_4-x}{ar_4-ar_3} + \theta_r \min\{\tfrac{ar_4-x}{ar_4-ar_3}, \tfrac{x-ar_3}{ar_4-ar_3}\}\right).\end{aligned}$$

If $a > 0$, and $\frac{x}{a} \in [r_2, r_3]$, i.e., $x \in [ar_2, ar_3]$, then

$$\tilde{\mu}_{a\zeta}(x) = \tilde{1}.$$

Therefore, for any real number $a > 0$,

$$a\zeta = (a\tilde{r}_1, a\tilde{r}_2, a\tilde{r}_3, a\tilde{r}_4; \theta_l, \theta_r).$$

In the same way, for any real number $a < 0$, we have

$$a\zeta = (a\tilde{r}_4, a\tilde{r}_3, a\tilde{r}_2, a\tilde{r}_1; \theta_l, \theta_r).$$

The proof is complete.

**Theorem 6** *Let $\zeta_1 = (\tilde{r}_1, \tilde{r}_2, \tilde{r}_3, \tilde{r}_4; \theta_l, \theta_r)$ and $\zeta_2 = (\tilde{l}_1, \tilde{l}_2, \tilde{l}_3, \tilde{l}_4; \theta_l, \theta_r)$ be two mutually independent T2 trapezoid fuzzy variables, and $\zeta = \zeta_1 + \zeta_2$. The secondary possibility distribution function of $\zeta_1$ is $\tilde{\mu}_{\zeta_1}(x), x \in \Re$, $\{\tilde{\mu}_{\zeta_1}(x), x \in [r_1, r_2] \cup [r_3, r_4]\}$ is supposed to be a family of mutually independent RFVs. The secondary possibility distribution function of $\zeta_2$ is $\tilde{\mu}_{\zeta_2}(x), x \in \Re$, $\{\tilde{\mu}_{\zeta_2}(x), x \in [l_1, l_2] \cup [l_3, l_4]\}$ is supposed to be a family of mutually independent RFVs. Then*

$$\zeta = (\widetilde{r_1 + l_1}, \widetilde{r_2 + l_2}, \widetilde{r_3 + l_3}, \widetilde{r_4 + l_4}; \theta_l, \theta_r).$$

**Proof:** Since $\zeta_i, i = 1, 2$ are mutually independent T2 fuzzy variables, according to Theorem 1 and Definition 4, the secondary possibility distribution function of $\zeta$ is

$$\tilde{\mu}_\zeta(x) = \sup_{x_1 + x_2 = x} \tilde{\mu}_{\zeta_1}(x_1) \wedge \tilde{\mu}_{\zeta_2}(x_2), \quad x \in \Re \tag{13}$$

where $\tilde{\mu}_{\zeta_i}(x_i)$ is the secondary possibility distribution function of $\zeta_i$.

By the definitions of $\tilde{\mu}_{\zeta_i}(t),\ i = 1, 2$ and the Extension Principal of Zadeh, we all have

$$\tilde{\mu}_{\zeta_1}(x_1) \wedge \tilde{\mu}_{\zeta_2}(x_2) = \tilde{\mu}_{\zeta_1}(x_1)$$

in the following five cases $(i) - (v)$:

$$(i) \quad r_1 \le x_1 \le r_2, l_1 \le x_2 \le l_2, \text{ and } \frac{x_1 - r_1}{r_2 - r_1} \le \frac{x_2 - l_1}{l_2 - l_1},$$

$$(ii) \quad r_1 \le x_1 \le r_2, l_3 \le x_2 \le l_4, \text{ and } \frac{x_1 - r_1}{r_2 - r_1} \le \frac{l_4 - x_2}{l_4 - l_3},$$

$$(iii) \quad r_3 \le t_1 \le r_4, l_1 \le t_2 \le l_2, \text{ and } \frac{r_4 - x_1}{r_4 - r_3} \le \frac{x_2 - l_1}{l_2 - l_1},$$

$$(iv) \quad r_3 \le x_1 \le r_4, l_3 \le x_2 \le l_4, \text{ and } \frac{r_4 - x_1}{r_4 - r_3} \le \frac{l_4 - x_2}{l_4 - l_3},$$

$$(v) \quad r_1 \le x_1 \le r_4, l_2 \le x_2 \le l_3.$$

So, for any $x_1, x_2$ such that $x_1 + x_2 = x$, we have the following results: in case $(i)$, if

$$\frac{x_1 - r_1}{r_2 - r_1} = \frac{x_2 - l_1}{l_2 - l_1}, \tag{14}$$

we have

$$\sup_{x_1 + x_2 = x} \tilde{\mu}_{\xi_1}(x_1) \wedge \tilde{\mu}_{\xi_2}(x_2) = \tilde{\mu}_{\xi_1}(x_1) = \tilde{\mu}_{\xi_2}(x_2);$$

in case $(ii)$, if

$$\frac{x_1 - r_1}{r_2 - r_1} = \frac{l_4 - x_2}{l_4 - l_3}, \tag{15}$$

we have

$$\sup_{x_1 + x_2 = x} \tilde{\mu}_{\xi_1}(x_1) \wedge \tilde{\mu}_{\xi_2}(x_2) = \tilde{\mu}_{\xi_1}(x_1) = \tilde{\mu}_{\xi_2}(x_2);$$

in case $(iii)$, if

$$\frac{r_4 - x_1}{r_4 - r_3} = \frac{x_2 - l_1}{l_2 - l_1}, \tag{16}$$

we have

$$\sup_{x_1+x_2=x} \tilde{\mu}_{\xi_1}(x_1) \wedge \tilde{\mu}_{\xi_2}(x_2) = \tilde{\mu}_{\xi_1}(x_1) = \tilde{\mu}_{\xi_2}(x_2);$$

in case $(iv)$, if

$$\frac{r_4 - x_1}{r_4 - r_3} = \frac{l_4 - x_2}{l_4 - l_3}, \tag{17}$$

we have

$$\sup_{x_1+x_2=x} \tilde{\mu}_{\xi_1}(x_1) \wedge \tilde{\mu}_{\xi_2}(x_2) = \tilde{\mu}_{\xi_1}(x_1) = \tilde{\mu}_{\xi_2}(x_2);$$

in case $(v)$, if

$$r_2 \le x_1 \le r_3 \quad \text{and} \quad l_2 \le x_2 \le l_3, \tag{18}$$

we have

$$\sup_{x_1+x_2=x} \tilde{\mu}_{\xi_1}(x_1) \wedge \tilde{\mu}_{\xi_2}(x_2) = \tilde{\mu}_{\xi_1}(x_1) = \tilde{\mu}_{\xi_2}(x_2) = \tilde{1}.$$

From (14), we have

$$x_1 = \tfrac{xr_2 - l_1 r_2 + r_1 l_2 - x r_1}{r_2 + l_2 - r_1 - l_1}, \quad \tfrac{x_1 - r_1}{r_2 - r_1} = \tfrac{x - r_1 - l_1}{r_2 + l_2 - r_1 - l_1}.$$

From (16), we have

$$x_1 = \tfrac{xr_4 + l_1 r_3 - r_4 l_3 - x r_3}{r_3 + l_3 - r_1 - l_1}, \quad \tfrac{r_4 - x_1}{r_4 - r_3} = \tfrac{x - r_4 - l_1}{r_3 + l_2 - r_4 - l_1}.$$

It is easy to know that $\frac{x - r_1 - l_1}{r_2 + l_2 - r_1 - l_1} > \frac{x - r_4 - l_1}{r_3 + l_2 - r_4 - l_1}$ and $r_1 + l_1 \le x \le r_2 + l_2$. That is to say

$$\begin{aligned}\tilde{\mu}_\zeta(x) = \ & \Big(\tfrac{x-r_1-l_1}{r_2+l_2-r_1-l_1} - \theta_l \min\left\{\tfrac{x-r_1-l_1}{r_2+l_2-r_1-l_1}, \tfrac{l_2+r_2-x}{r_2+l_2-r_1-l_1}\right\}, \tfrac{x-r_1-l_1}{r_2+l_2-r_1-l_1}, \\ & \tfrac{x-r_1-l_1}{r_2+l_2-r_1-l_1} + \theta_r \min\left\{\tfrac{x-r_1-l_1}{r_2+l_2-r_1-l_1}, \tfrac{l_2+r_2-x}{r_2+l_2-r_1-l_1}\right\}\Big)\end{aligned}$$

for any $x \in [r_1 + l_1, r_2 + l_2]$. In the same way, from (15) and (17), we have

$$\begin{aligned}\tilde{\mu}_\zeta(x) = \ & \Big(\tfrac{r_4+l_4-x}{r_4+l_4-r_3-l_3} - \theta_l \min\left\{\tfrac{r_4+l_4-x}{r_4+l_4-r_3-l_3}, \tfrac{x-l_3-r_3}{r_4+l_4-r_3-l_3}\right\}, \tfrac{r_4+l_4-x}{r_4+l_4-r_3-l_3}, \\ & \tfrac{r_4+l_4-x}{r_4+l_4-r_3-l_3} + \theta_r \min\left\{\tfrac{r_4+l_4-x}{r_4+l_4-r_3-l_3}, \tfrac{x-l_3-r_3}{r_4+l_4-r_3-l_3}\right\}\Big)\end{aligned}$$

for any $x \in [r_3 + l_3, r_4 + l_4]$.

From (18), if $r_2 \le x_1 \le r_3$ and $l_2 \le x_2 \le l_3$, i.e., $r_2 + l_2 \le x \le r_3 + l_3$, we know that $\tilde{\mu}_\zeta(x) = \tilde{1}$. If $x \le r_1 + l_1$ or $x \ge r_4 + l_4$, then $\tilde{\mu}_\zeta(x) = \tilde{0}$. So

$$\zeta = (\widetilde{r_1 + l_1}, \widetilde{r_2 + l_2}, \widetilde{r_3 + l_3}, \widetilde{r_4 + l_4}; \theta_l, \theta_r).$$

The proof is complete.

According to Theorems 5 and 6, we have the following result.

**Theorem 7** *Let $\zeta_1 = (\tilde{r}_1, \tilde{r}_2, \tilde{r}_3, \tilde{r}_4; \theta_l, \theta_r)$ and $\zeta_2 = (\tilde{l}_1, \tilde{l}_2, \tilde{l}_3, \tilde{l}_4; \theta_l, \theta_r)$ be two mutually independent T2 trapezoid fuzzy variables, and $\zeta = a\zeta_1 + b\zeta_2$ where $a, b \neq 0$ are two any real numbers. The secondary possibility distribution function of $\zeta_1$ is $\tilde{\mu}_{\zeta_1}(x), x \in \Re$, $\{\tilde{\mu}_{\zeta_1}(x), x \in [r_1, r_2] \cup [r_3, r_4]\}$ is supposed to be a family of mutually independent RFVs. The secondary possibility distribution function of $\zeta_2$ is $\tilde{\mu}_{\zeta_2}(x), x \in \Re$, $\{\tilde{\mu}_{\zeta_2}(x), x \in [l_1, l_2] \cup [l_3, l_4]\}$ is supposed to be a family of mutually independent RFVs. Then*

$$\zeta = \begin{cases} (\widetilde{ar_1 + bl_1}, \widetilde{ar_2 + bl_2}, \widetilde{ar_3 + bl_3}, \widetilde{ar_4 + bl_4}; \theta_l, \theta_r), & \text{if } a, b > 0 \\ (\widetilde{ar_1 + bl_4}, \widetilde{ar_2 + bl_3}, \widetilde{ar_3 + bl_2}, \widetilde{ar_4 + bl_1}; \theta_l, \theta_r), & \text{if } a > 0 \text{ and } b < 0 \\ (\widetilde{ar_4 + bl_1}, \widetilde{ar_3 + bl_2}, \widetilde{ar_2 + bl_3}, \widetilde{ar_1 + bl_4}; \theta_l, \theta_r), & \text{if } a < 0 \text{ and } b > 0 \\ (\widetilde{ar_4 + bl_4}, \widetilde{ar_3 + bl_3}, \widetilde{ar_2 + bl_2}, \widetilde{ar_1 + bl_1}; \theta_l, \theta_r), & \text{if } a, b < 0. \end{cases}$$

**Example 2:** Let $\zeta_1 = (\tilde{1}, \tilde{2}, \tilde{3}, \tilde{5}; 0.6, 0.8)$ be a T2 trapezoid fuzzy variable. The support of $\zeta_1$ is showed in Figure 4. The secondary possibility distribution function of $\zeta_1$ is $\tilde{\mu}_{\zeta_1}(x), x \in \Re$, $\{\tilde{\mu}_{\zeta_1}(x), x \in [1,2] \cup [3,5]\}$ is supposed to be a family of mutually independent RFVs. Let $\zeta_2 = (\tilde{2}, \tilde{3}, \tilde{4}, \tilde{5}; 0.6, 0.8)$ be a T2 trapezoid fuzzy variable. The support of $\zeta_2$ is showed in Figure 5. The secondary possibility distribution function of $\zeta_2$ is $\tilde{\mu}_{\zeta_2}(x), x \in \Re$, $\{\tilde{\mu}_{\zeta_2}(x), x \in [2,3] \cup [4,5]\}$ is supposed to be a family of mutually independent RFVs. Also, we suppose that trapezoid fuzzy variables $\zeta_1$ and $\zeta_2$ are mutually independent. Then $\zeta = \zeta_1 + \zeta_2 = (\tilde{3}, \tilde{5}, \tilde{7}, \tilde{10}; 0.6, 0.8)$ is a T2 trapezoid fuzzy variable. The support of $\zeta$ is showed in Figure 6.

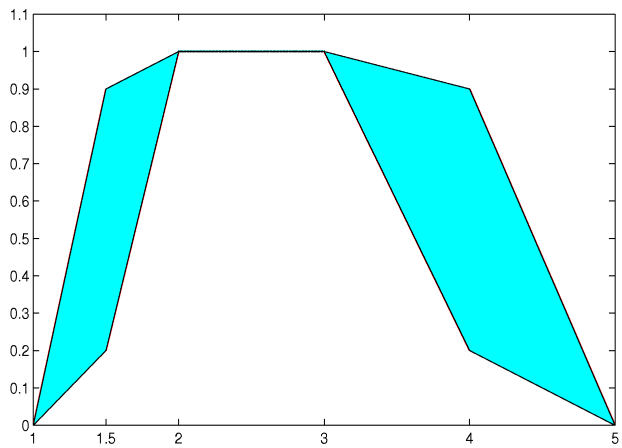


Figure 4: The support of the T2 fuzzy variable $\zeta_1$ defined in Example 2

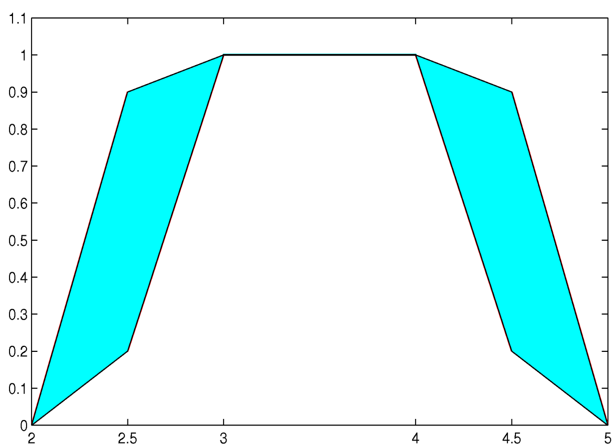


Figure 5: The support of the T2 fuzzy variable $\zeta_2$ defined in Example 2

# 5 The Linear Combination of T2 Normal Fuzzy Variables

**Theorem 8** *Let $\eta_1 = \tilde{n}(\mu_1, \sigma_1^2; \theta_l, \theta_r)$ and $\eta_2 = \tilde{n}(\mu_2, \sigma_2^2; \theta_l, \theta_r)$ be two mutually independent T2 normal fuzzy variables, and $\eta = \eta_1 + \eta_2$. The secondary possibility distribution function of $\eta_1$ is $\tilde{\mu}_{\eta_1}(x), x \in \Re$, $\{\tilde{\mu}_{\eta_1}(x), x \in \Re\}$ is supposed to be a family of mutually independent RFVs. The secondary possibility distribution function of $\eta_2$ is $\tilde{\mu}_{\eta_2}(x), x \in \Re$, $\{\tilde{\mu}_{\xi_2}(x), x \in \Re\}$ is supposed to be a family of mutually independent RFVs. Then*

$$\eta = \tilde{n}(\mu_1 + \mu_2, (\sigma_1 + \sigma_2)^2; \theta_l, \theta_r).$$

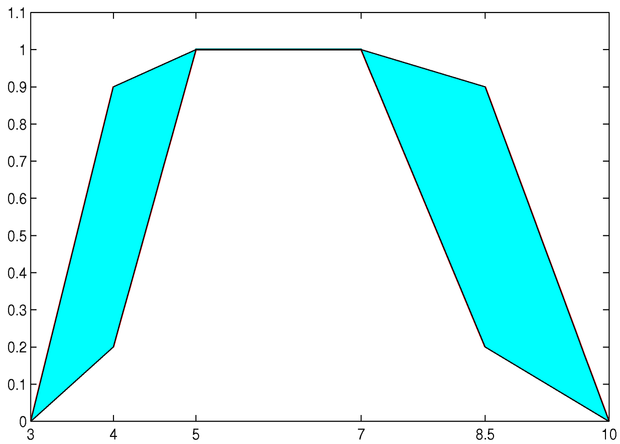


Figure 6: The support of the T2 fuzzy variable $\zeta$ defined in Example 2

**Proof:** Since $\eta_i, i = 1, 2$ are mutually independent T2 fuzzy variables, according to Theorem 1 and Definition 4, the secondary possibility distribution function of $\eta$ is

$$\tilde{\mu}_\eta(x) = \sup_{x_1+x_2=x} \tilde{\mu}_{\eta_1}(x_1) \wedge \tilde{\mu}_{\eta_2}(x_2), \quad x \in \Re \tag{19}$$

where $\tilde{\mu}_{\eta_i}(x_i)$ is the secondary possibility distribution function of $\eta_i$.

By the definitions of $\tilde{\mu}_{\eta_i}(t),\ i = 1, 2$ and the Extension Principal of Zadeh, we have

$$\tilde{\mu}_{\eta_1}(x_1) \wedge \tilde{\mu}_{\eta_2}(x_2) = \tilde{\mu}_{\eta_1}(x_1)$$

whenever $\exp(-\frac{(x_1-\mu_1)^2}{2\sigma_1^2}) \le \exp(-\frac{(x_2-\mu_2)^2}{2\sigma_2^2})$.

So, for any $x_1, x_2$ such that $x_1 + x_2 = x$, if

$$\exp(-\frac{(x_1-\mu_1)^2}{2\sigma_1^2}) = \exp(-\frac{(x_2-\mu_2)^2}{2\sigma_2^2}), \tag{20}$$

i.e.,

$$\frac{x_1-\mu_1}{\sigma_1} = \frac{x_2-\mu_2}{\sigma_2} \quad \text{or} \quad \frac{x_1-\mu_1}{\sigma_1} = -\frac{x_2-\mu_2}{\sigma_2}, \tag{21}$$

we have

$$\tilde{\mu}_\eta(x) = \sup_{x_1+x_2=x} \tilde{\mu}_{\eta_1}(x_1) \wedge \tilde{\mu}_{\eta_2}(x_2) = \tilde{\mu}_{\eta_1}(x_1) = \tilde{\mu}_{\eta_2}(x_2). \tag{22}$$

From (21), we have

$$x_1 = \frac{\mu_1\sigma_2+\sigma_1(x-\mu_2)}{\sigma_1+\sigma_2},\ \frac{x_1-\mu_1}{\sigma_1} = \frac{x-\mu_1-\mu_2}{\sigma_1+\sigma_2},$$

or

$$x_1 = \frac{\mu_1\sigma_2+\sigma_1(\mu_2-x)}{\sigma_2-\sigma_1},\ \frac{x_1-\mu_1}{\sigma_1} = \frac{\mu_1+\mu_2-x}{\sigma_2-\sigma_1}.$$

It is easy to know that $\frac{(x-\mu_1-\mu_2)^2}{(\sigma_1+\sigma_2)^2} < \frac{(\mu_1+\mu_2-x)^2}{(\sigma_2-\sigma_1)^2}$. That is to say $\exp(-\frac{(x-\mu_1-\mu_2)^2}{2(\sigma_1+\sigma_2)^2}) > \exp(-\frac{(\mu_1+\mu_2-x)^2}{2(\sigma_2-\sigma_1)^2})$. Then, for any $x \in \Re$, we have

$$\begin{aligned}\tilde{\mu}_\eta(x) = \Big(&\exp(-\tfrac{(x-\mu_1-\mu_2)^2}{2(\sigma_1+\sigma_2)^2}) - \theta_l \min\{1-\exp(-\tfrac{(x-\mu_1-\mu_2)^2}{2(\sigma_1+\sigma_2)^2}), \exp(-\tfrac{(x-\mu_1-\mu_2)^2}{2(\sigma_1+\sigma_2)^2})\}, \exp(-\tfrac{(x-\mu_1-\mu_2)^2}{2(\sigma_1+\sigma_2)^2}),\\ &\exp(-\tfrac{(x-\mu_1-\mu_2)^2}{2(\sigma_1+\sigma_2)^2}) + \theta_r \min\{1-\exp(-\tfrac{(x-\mu_1-\mu_2)^2}{2(\sigma_1+\sigma_2)^2}), \exp(-\tfrac{(x-\mu_1-\mu_2)^2}{2(\sigma_1+\sigma_2)^2})\}\Big).\end{aligned}$$

So

$$\eta = \tilde{n}(\mu_1+\mu_2, (\sigma_1+\sigma_2)^2; \theta_l, \theta_r).$$

The proof is complete.

**Theorem 9** *Let $\eta = \tilde{n}(\mu, \sigma^2; \theta_l, \theta_r)$ be a T2 normal fuzzy variable. Then for any real number $a \neq 0$, we have*

$$a\eta = \tilde{n}(a\mu, (a\sigma)^2; \theta_l, \theta_r) = \begin{cases} \tilde{n}(a\mu, (a\sigma)^2; \theta_l, \theta_r), & \text{if } a > 0 \\ \tilde{n}(a\mu, (-a\sigma)^2; \theta_l, \theta_r), & \text{if } a < 0. \end{cases}$$

**Proof:** According to the definition of T2 normal fuzzy variable, we have $\tilde{\mu}_{a\eta}(x) = \tilde{\mu}_\eta(\frac{x}{a})$. Then

$$\begin{aligned} &\tilde{\mu}_{a\eta}(x) = \tilde{\mu}_\eta(\tfrac{x}{a}) \\ &= \Big(\exp(-\tfrac{(\frac{x}{a}-\mu)^2}{2\sigma^2}) - \theta_l \min\{1-\exp(-\tfrac{(\frac{x}{a}-\mu)^2}{2\sigma^2}), \exp(-\tfrac{(\frac{x}{a}-\mu)^2}{2\sigma^2})\}, \exp(-\tfrac{(\frac{x}{a}-\mu)^2}{2\sigma^2}), \\ &\qquad \exp(-\tfrac{(\frac{x}{a}-\mu)^2}{2\sigma^2}) + \theta_r \min\{1-\exp(-\tfrac{(\frac{x}{a}-\mu)^2}{2\sigma^2}), \exp(-\tfrac{(\frac{x}{a}-\mu)^2}{2\sigma^2})\}\Big) \\ &= \Big(\exp(-\tfrac{(x-a\mu)^2}{2(a\sigma)^2}) - \theta_l \min\{1-\exp(-\tfrac{(x-a\mu)^2}{2(a\sigma)^2}), \exp(-\tfrac{(x-a\mu)^2}{2(a\sigma)^2})\}, \exp(-\tfrac{(x-a\mu)^2}{2(a\sigma)^2}), \\ &\qquad \exp(-\tfrac{(x-a\mu)^2}{2(a\sigma)^2}) + \theta_r \min\{1-\exp(-\tfrac{(x-a\mu)^2}{2(a\sigma)^2}), \exp(-\tfrac{(x-a\mu)^2}{2(a\sigma)^2})\}\Big). \end{aligned}$$

Therefore, for any real number $a \neq 0$, we know that

$$a\eta = \tilde{n}(a\mu, (a\sigma)^2; \theta_l, \theta_r).$$

It is easy to know that the second equal sign in the result of the theorem holds, too.

The proof is complete.

According to Theorems 8 and 9, we have the following theorem.

**Theorem 10** *Let $\eta_1 = \tilde{n}(\mu_1, \sigma_1^2; \theta_l, \theta_r)$ and $\eta_2 = \tilde{n}(\mu_2, \sigma_2^2; \theta_l, \theta_r)$ be two mutually independent T2 normal fuzzy variables, and $\eta = a\eta_1 + b\eta_2$ where $a, b \neq 0$ are two any real numbers. The secondary possibility distribution function of $\eta_1$ is $\tilde{\mu}_{\eta_1}(x), x \in \Re$, $\{\tilde{\mu}_{\eta_1}(x), x \in \Re\}$ is supposed to be a family of mutually independent RFVs. The secondary possibility distribution function of $\eta_2$ is $\tilde{\mu}_{\eta_2}(x), x \in \Re$, $\{\tilde{\mu}_{\xi_2}(x), x \in \Re\}$ is supposed to be a family of mutually independent RFVs. Then*

$$\eta = \begin{cases} (a\mu_1 + b\mu_2, (a\sigma_1 + b\sigma_2)^2; \theta_l, \theta_r), & \text{if } a, b > 0 \\ (a\mu_1 + b\mu_2, (a\sigma_1 - b\sigma_2)^2; \theta_l, \theta_r), & \text{if } a > 0 \text{ and } b < 0 \\ (a\mu_1 + b\mu_2, (-a\sigma_1 + b\sigma_2)^2; \theta_l, \theta_r), & \text{if } a < 0 \text{ and } b > 0 \\ (a\mu_1 + b\mu_2, (a\sigma_1 + b\sigma_2)^2; \theta_l, \theta_r), & \text{if } a, b < 0. \end{cases}$$

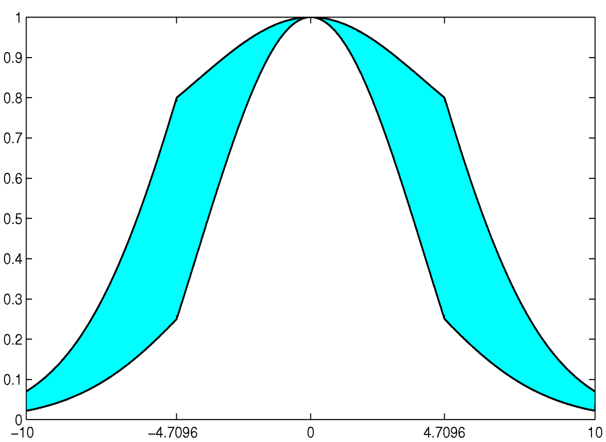


Figure 7: The support of the T2 fuzzy variable $\eta_1$ defined in Example 3

**Example 3:** Let $\eta_1 = \tilde{n}(0, 4^2; 0.5, 0.6)$ be a T2 normal fuzzy variable. The support of $\eta_1$ is showed in Figure 7. The secondary possibility distribution function of $\eta_1$ is $\tilde{\mu}_{\eta_1}(x), x \in \Re$, $\{\tilde{\mu}_{\eta_1}(x), x \in \Re\}$ is supposed to be a family of mutually independent RFVs. Let $\eta_2 = \tilde{n}(1, 2^2; 0.5, 0.6)$ be a T2 normal fuzzy variable. The support of $\eta_2$ is showed in Figure 8. The secondary possibility distribution function of $\eta_2$ is $\tilde{\mu}_{\eta_2}(x), x \in \Re$, $\{\tilde{\mu}_{\eta_2}(x), x \in \Re\}$ is supposed to be a family of mutually independent RFVs. Also, we suppose that normal fuzzy variables $\eta_1$ and $\eta_2$ are mutually independent. Then $\eta = 3\eta_1 - \eta_2 = \tilde{n}(-1, 14^2; 0.5, 0.6)$ is a T2 normal fuzzy variable. The support of $\eta$ is showed in Figure 9.

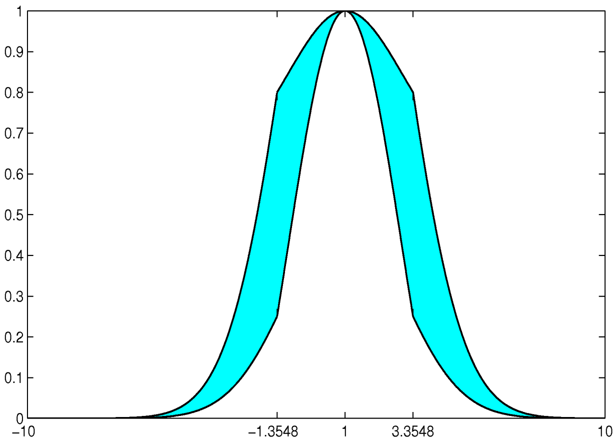


Figure 8: The support of the T2 fuzzy variable $\eta_2$ defined in Example 3

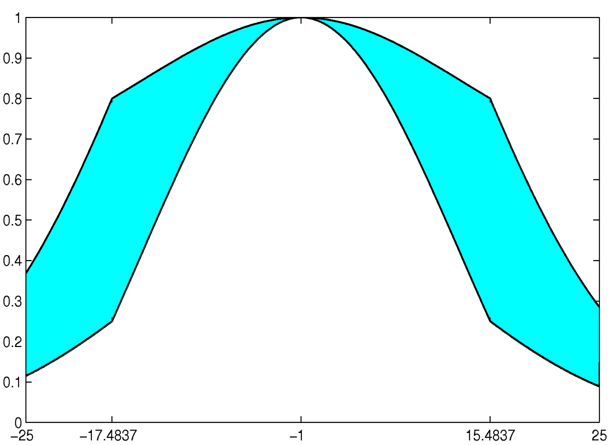


Figure 9: The support of the T2 fuzzy variable $\eta$ defined in Example 3

# 6 Conclusions

Based on the theoretical foundation of T2 fuzzy arithmetic, for three kinds of common T2 fuzzy variables, we have given some new results about T2 fuzzy arithmetic. Under some wild assumption, we have concluded that the linear combination of a finite number of T2 triangular fuzzy variables, which have the same parameters $\theta_l$ and $\theta_r$, is also a T2 triangular fuzzy variable. Also we have proved that the linear combination of a finite number of T2 trapezoid fuzzy variables, which have the same parameters $\theta_l$ and $\theta_r$, is also a T2 trapezoid fuzzy variable. Moreover, we have given that the linear combination of a finite number of T2 normal fuzzy variables, which have the same parameters $\theta_l$ and $\theta_r$, is also a T2 normal fuzzy variable.

# References

[1] Chen, Y., and X. Wang, The possibilistic representative value of type-2 fuzzy variable and its properties, *Journal of Uncertain Systems*, vol.4, no.3, pp.229–240, 2010.

[2] Dubois, D., and H. Prade, Operations in a fuzzy-valued logic, *Information and Control*, vol.43, no.2, pp.224–240, 1979.

[3] John, R.I., Type-2 fuzzy sets: an appraisal of theory and applications, *International Journal of Uncertainty, Fuzziness and Knowledge-based Systems*, vol.6, no.6, pp.563–576, 1998.

[4] John, R.I., P.R. Innocent, and M.R. Barnes, Neuro-fuzzy clustering of radiograpinc tibia image data using type 2 fuzzy sets, *Information Sciences*, vol.125, nos.1-4, pp.65–82, 2000.

[5] Karnik, N.N., and J.M. Mendel, Centroid of a type-2 fuzzy set, *Information Sciences*, vol.132, nos.1-4, pp.195–220, 2001.

[6] Karnik, N.N., J.M. Mendel, and Q. Liang, Type-2 fuzzy logic systems, *IIEEE Transactions on Fuzzy Systems*, vol.7, no.6, pp.643–658, 1999.

[7] Klir, G.J., On fuzzy-set interpretation of possibility theory, *Fuzzy Sets and Systems*, vol.108, no.3, pp.263–273, 1999.

[8] Liang, Q., and J.M. Mendel, Equalization of nonlinear time-varying channels using type-2 fuzzy adaptive filters, *IEEE Transactions on Fuzzy Systems*, vol.8, no.5, pp.551–563, 2000.

[9] Liang, Q., N.N. Karnik, and J.M. Mendel, Connection admission control in ATM networks using survey-based type-2 fuzzy logic systems, *IEEE Transactions on Systems, Man, and Cybernetics, Part C: Applications and Reviews*, vol.30, no.3, pp.329–339, 2000.

[10] Liu, Y.K., and J. Gao, The independence of fuzzy variables with applications to fuzzy random optimization, *International Journal of Uncertainty, Fuzziness and Knowledge-based Systems*, vol.15(suppl.2), pp.1–20, 2007.

[11] Liu, Z.Q., and Y.K. Liu, Type-2 fuzzy variables and their arithmetic, *Soft Computing*, vol.14, no.7, pp.729–747, 2010.

[12] Mendel, J.M., and R.I. John, Type-2 fuzzy sets made simple, *IEEE Transactions on Fuzzy Systems*, vol.10, no.2, pp.117–127, 2002.

[13] Mitchell, H.B., Pattern recognition using type-II fuzzy sets, *Information Sciences*, vol.170, no.2-4, pp.409–418, 2005.

[14] Mizumoto, M., and K. Tanaka, Some properties of fuzzy sets of type-2, *Information and Control*, vol.31, no.4, pp.312–340, 1976.

[15] Nahmias, S., Fuzzy variables, *Fuzzy Sets and Systems*, vol.1, no.1, pp.97–110, 1978.

[16] Nieminen, J., On the algebraic structure of fuzzy sets of type-2, *Kybernetica*, vol.13, no.4, pp.261–273, 1977.

[17] Pedrycz, W., Granular computing–the emerging paradigm, *Journal of Uncertain Systems*, vol.1, no.1, pp.38–61, 2007.

[18] Qin, R., Y. Liu, and Z. Liu, Modeling fuzzy data envelopment analysis by parametric programming method, *Expert Systems with Applications*, vol.38, no.7, 8648–8663, 2011.

[19] Qin, R., Y. Liu, and Z. Liu, Methods of critical value reduction for type-2 fuzzy variables and their applications, *Journal of Computational and Applied Mathematics*, vol.235, no.5, pp.1454–1481, 2011.

[20] Viertl, R., Foundations of fuzzy bayesian inference, *Journal of Uncertain Systems*, vol.2, no.3, pp.187–191, 2008.

[21] Wang, P., Fuzzy contactability and fuzzy variables, *Fuzzy Sets and Systems*, vol.8, no.1, pp.81–92, 1982.

[22] Yager, R.R., Fuzzy subsets of type-II in decisions, *Journal of Cybernetics*, vol.10, nos.1-3, pp.137–159, 1980.

[23] Zadeh, L.A., Concept of a linguistic variable and its application to approximate reasoning–I, *Information Sciences*, vol.8, no.3, pp.199–249, 1975.

[24] Zadeh, L.A., Fuzzy set as a basis for a theory of possibility, *Fuzzy Sets and Systems*, vol.1, no.1, pp.3–28, 1978.

[25] Zeng, J., and Z.Q. Liu, Type-2 fuzzy hidden Markov models and their application to speech recognition, *IEEE Transactions on Fuzzy Systems*, vol.14, no.3, pp.454–467, 2006.

[26] Zeng, J., and Z.Q. Liu, Type-2 fuzzy sets for pattern recognition: the state-of-the-art, *Journal of Uncertain Systems*, vol.1, no.3, pp.163–177, 2007.