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Uncertainty in Partially Ordered Sets as a Natural Generalization of Intervals: Negative Information is Sufficient, Positive is Not

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Abstract

In many real-life applications, we have an ordered set: a set of all space-time events, a set of all alternatives, a set of all degrees of confidence. In practice, we usually only have a partial information about an element x of this set. This partial information includes positive knowledge: that $a \le x$ or $x \le a$ for some known a, and negative knowledge: that $a \not\le a$ or $x \not\le a$ for the known a. In the case of a total order, the set of all elements satisfying this partial information is an interval. We show that in the general case of a partial order, the corresponding analogue of an interval is a convex set. We also show that in general, to describe partial knowledge, it is sufficient to have only negative information about x but it is not sufficient to have only positive information.

Keywords: positive information, negative information, partially ordered sets, intervals

1 Formulation of the Problem

Ordered Sets are Practically Important

In many real-life situations, we have a natural order relation. For example:

- causality relation $a \le b$, meaning that an event a can influence an event b, is a natural order relation on the set of all space-time events;
- a preference relation $a \le b$, meaning that a user prefers an alternative b to the alternative a;
- a relation $a \le b$ between different terms describing the user's degree of confidence, meaning that the degree b describes more confidence than the degree a.

In all these cases, we have a set X – of events, of alternatives, of degrees of confidence – with an order relation $\le$.

For each ordered set $(X, \le)$, we will use the usual notations $a \ge b \stackrel{\text{def}}{=} b \le a$, $a < b \stackrel{\text{def}}{=} (a \le b \,\&\, a \ne b)$, and $a > b \stackrel{\text{def}}{=} b < a$.

How to Describe Uncertainty in an Ordered Set

In the ideal case of a *complete* knowledge, we know the exact element $x \in X$. For example, we may know the event, we may know the alternative, or we may know the degree of confidence.

However, frequently, we only have a *partial* information about an element $x \in X$. This partial information comes in the form of a known relation between this (unknown) element and some known elements $x_1, \ldots, x_n$.

For example, about an unknown event e, we know

- that this event e was influenced by events $e_1, \ldots, e_m$;
- that this event e, in turn, influenced some other events $e'_1, \ldots, e'_{m'}$;

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- that this event e was *not* influenced by events $e''_1, \ldots, e''_{m''}$;
- that events $e'''_1, \ldots, e'''_{n'''}$ did not influence this event.

The first two conditions describe *positive* information about the event e, the last two conditions describe *negative* information about the event e. We want to characterize the set of all the events which satisfy these conditions – both positive and negative.

Let us describe this situation in the general terms.

Definition 1 *Let $(X, \leq)$ be an ordered set.*

- *By a* partial knowledge*, we mean a tuple $K = (P^-, P^+, N^+, N^-)$ of subsets of the set X.*
- *We say that an element $x \in X$ is* consistent *with this partial knowledge if the following four conditions are satisfied:*
  - *for every $p \in P^-$, we have $p \leq x$;*
  - *for every $p \in P^+$, we have $x \leq p$;*
  - *for every $n \in N^-$, we have $n \not\leq x$;*
  - *for every $n \in N^+$, we have $x \not\leq n$.*
- *The set of all elements which are consistent with the partial knowledge K will be called an* uncertainty set *corresponding to K – and denoted by $U(K)$.*

Our objectives are:

- to describe sets which can be represented in the form $U(K)$ for some partial knowledge K; and
- if possible, to find classes of partial knowledges K which are sufficient to describe arbitrary sets $U(K)$.

2 Definitions and the Main Results

Simplest Case: Total Order, Finite Sets $P^\pm$ and $N^\pm$

Let us start with the simplest case when the order $\leq$ is a total order (i.e., for every a and b either $a \leq b$ or $b \leq a$), and all four sets P^-, P^+, N^+, and N^- forming a partial knowledge are finite.

The finiteness requirement is natural since in practice, at any given moment of time, we only have finite amount of information.

Definition 2 *We say that a partial knowledge $K = (P^-, P^+, N^+, N^-)$ is* finite *if all four sets P^-, P^+, N^+, and N^- are finite.*

Uncertainty sets corresponding to finite partial knowledge K can be easily described as *intervals*:

Definition 3 *By an* interval*, we mean one of the following sets:*

$$[a, b] = \{x : a \leq x \leq b\}; \quad (a, b] = \{x : a < x \leq b\};$$

$$[a, b) = \{x : a \leq x < b\}; \quad (a, b) = \{x : a < x < b\};$$

$$[a, +\infty) = \{x : a \leq x\}; \quad (a, +\infty) = \{x : a < x\};$$

$$(-\infty, b] = \{x : x \leq b\}; \quad (-\infty, b) = \{x : x < b\};$$

$$(-\infty, +\infty) = X.$$

Proposition 1 *Let $(X, \leq)$ be a totally ordered set. Then:*

- *For every finite partial knowledge K, the uncertainty set $U(K)$ is an interval.*
- *Every interval I can be represented as $U(K)$ for an appropriate partial knowledge K, namely for a knowledge K formed by at most two one-points sets.*

Comment. For readers' convenience, all the proofs are placed at the end of this paper.

Case of Arbitrary (Not Necessarily Finite) Sets $P^\pm$ and $N^\pm$

For totally ordered sets, it is also possible to characterize sets which can be represented as uncertainty sets $U(K)$ for some (not necessarily finite) partial knowledge K.

It turns out that the same characterization holds for an arbitrary (not necessarily totally ordered) ordered set $(X, \le)$.

Definition 4 *A subset $S \subseteq X$ of an ordered set $(X, \le)$ is called* convex *if it satisfies the following property:*

$$\text{if } s \in S,\ s' \in S, \text{ and } s \le x \le s', \text{ then } x \in S.$$

Proposition 2 *Let $(X, \le)$ be an ordered set. Then:*

- *For every partial knowledge K, the uncertainty set $U(K)$ is convex.*
- *Every convex set $S \subseteq X$ can be represented as an uncertainty set $U(K)$ for an appropriate partial knowledge K.*

Comment. Thus, convex sets are natural generalizations of intervals – to the case when the partial knowledge is not necessarily finite and the order is not necessarily total.

Positive and Negative Knowledge

A general partial knowledge includes both *positive* knowledge P^- and P^+ and negative knowledge N^- and N^+. When is each sufficient?

Definition 5 *We say that partial knowledge $K = (P^-, P^+, N^-, N^+)$ is:*

- positive *if $N^- = N^+ = \emptyset$;*
- negative *if $P^- = P^+ = \emptyset$.*

In general, positive knowledge is not sufficient, but negative knowledge is sufficient.

Proposition 3 *There exists an ordered set $(X, \le)$ and a convex set $S \subseteq X$ that cannot be represented as $U(K)$ for any positive partial knowledge K.*

Proposition 4 *Let $(X, \le)$ be an ordered set, and let S be a convex subset of X. Then there exists a negative partial knowledge K for which $S = U(K)$.*

3 Proofs

Proof of Proposition 1

Let us first prove that for every finite partial knowledge K, the uncertainty set $U(K)$ is an interval. For a totally ordered set, conditions $p_1 \le x, \ldots, p_i \le x$ corresponding to a set P^- are equivalent to a single condition $p^- \le x$, where $p^- \stackrel{\text{def}}{=} \max(p_1, \ldots, p_i)$.

Similarly, conditions $x \le q_1, \ldots, x \le q_j$ corresponding to the set P^+ are equivalent to a single condition $x \le p^+$, where $p^+ \stackrel{\text{def}}{=} \min(q_1, \ldots, q_j)$.

Since $\le$ is a total order, each condition $n_k \not\le x$ corresponding to the set N^- is equivalent to $x < n_k$. Thus, the conditions $n_1 \not\le x, \ldots, n_l \not\le s$ corresponding to the set N^- are equivalent to $x < n_1, \ldots, x < n_l$, i.e., to $x < n^-$, where $n^- \stackrel{\text{def}}{=} \min(n_1, \ldots, n_l)$.

Similarly, each condition $x \not\le m_k$ corresponding to the set N^+ is equivalent to $m_k < x$. Thus, the conditions $x \not\le m_1, \ldots, x \not\le m_a$ corresponding to the set N^+ are equivalent to $m_1 < x, \ldots, m_a < x$, i.e., to $n^+ < x$, where $n^+ \stackrel{\text{def}}{=} \max(m_1, \ldots, m_a)$.

Thus, all the conditions are equivalent to (at most) four inequalities: $p^- \le x$, $x \le p^+$, $x < n^-$, and $n^+ < x$ (at most four, since some of the four sets forming the partial knowledge may be empty, in which case there is no inequality to constrain the value x).

The conditions $p^- \le x$ and $n^+ < x$ are equivalent to a single condition:

- if $p^- \le n^+$, then these two conditions are equivalent to a single condition $n^+ < x$;
- if $p^- > n^+$, then these two conditions are equivalent to a single condition $p^- \le x$.

Similarly, the conditions $x \le p^+$ and $x < n^-$ are equivalent to a single condition:

- if $p^+ \ge n^-$, then these two conditions are equivalent to a single condition $x < n^-$;
- if $p^+ < n^-$, then these two conditions are equivalent to a single condition $x \le p^+$.

So, the set of (at most) four inequalities are equivalent to (at most) two, and it is easy to check that the set of all the elements x satisfying these two inequalities is an interval.

Vice versa, every interval I is described by two conditions of the type $a \le x$, $x \le b$, $a < x$, and $x < b$. Each condition $a < x$ is equivalent to $x \not\le a$, and each condition $x < b$ is equivalent to $b \not\le x$. Thus, each interval I can indeed be described as an uncertainty set $U(K)$ for some partial knowledge that consists of at most two one-point sets.

Proof of Proposition 2

Let us first prove that for every partial knowledge K, the uncertainty set $U(K)$ is convex. In other words, we want to prove that if $s \in U(K)$, $s' \in U(K)$, and $s \le x \le s'$, then $x \in U(K)$. The fact that $s \in U(K)$ means that the element s satisfies all the necessary conditions of the type $a \le s$, $s \le b$, $a \not\le s$, and $s \not\le b$. Thus, to prove that $x \in U(K)$, it is sufficient to prove that the element x also satisfies each of these conditions.

We will prove that for each of the above four conditions, if s and s' satisfies the corresponding condition, then the intermediate element x also satisfies this same condition.

- First, we consider the case when $a \le s$ and $a \le s'$. Since $a \le s$, then from $s \le x$ and transitivity we conclude that $a \le x$.
- Next, we consider the case when $s \le b$ and $s' \le b$. Since $s' \le b$, then from $x \le s'$ and transitivity, we conclude that $x \le b$.
- Third, we consider the case when $a \not\le s$ and $a \not\le s'$. We want to prove that in this case, $a \not\le x$. We can prove this by contradiction. Indeed, if $a \le x$, then from $x \le s'$ and transitivity, we would conclude that $a \le s'$ – and we know that $a \not\le s'$. Thus, $a \not\le x$.
- Finally, we consider the case when $s \not\le b$ and $s' \not\le b$. We want to prove that in this case, $x \not\le b$. We can prove this by contradiction. Indeed, if $x \le b$, then from $s \le x$ and transitivity, we would conclude that $s \le b$ – and we know that $s \not\le b$. Thus, $x \not\le b$.

Let us now prove that every convex set S can be represented as an uncertainty set $U(K)$ for some partial knowledge K. Indeed, let us show that $S = U(K)$, where $P^- = P^+ = \emptyset$, and

$$N^- = \{x : x \not\le s \text{ for all } s \in S\}, \quad N^+ = \{x : s \not\le x \text{ for all } s \in S\}.$$

By definition of the sets N^- and N^+, every element $s \in S$ satisfies

- the condition $x \not\le s$ for all $x \in N^-$ and
- the condition $s \not\le x$ for all $x \in N^+$.

So, every element $s \in S$ belongs to the corresponding uncertainty set $U(K)$: $S \subseteq U(K)$.

To complete our proof, we must also show that, vice versa, every element u of the uncertainty set $U(K)$ belongs to S. By definition, $u \in U(K)$ means that:

- $n \not\le u$ for all $n \in N^-$, and

- $u \not\leq n$ for all $n \in N^+$.

If we have $s \leq u \leq s'$ for some $s \in S$ and $s' \in S$, then, by convexity, $u \in S$. Thus, to prove our result, it is sufficient to prove the following two statements:

- that there exists an element $s \in S$ for which $s \leq u$, and
- that there exists an element $s' \in S$ for which $u \leq s'$.

We will prove both statements by contradiction.

Let us start with the first statement. Assume that this statement is not true, i.e., that $s \not\leq u$ for all $s \in S$. By definition of the set N^+, this means that $u \in N^+$. Since $u \leq u$, we conclude that $u \leq n$ for some $n \in N^+$ – namely, for $n = u$. This contradicts to the fact that $u \not\leq n$ for all $n \in N^+$.

Similarly, we can prove the second statement. Assume that this statement is not true, i.e., that $u \not\leq s$ for all $s \in S$. By definition of the set N^-, this means that $u \in N^-$. Since $u \leq u$, we conclude that $n \leq u$ for some $n \in N^-$ – namely, for $n = u$. This contradicts to the fact that $n \not\leq u$ for all $n \in N^-$.

Both statements are proven, thus $u \in S$. The proposition is proven.

Proof of Proposition 3

As the first example let us take, as the ordered set, a real line R with a natural order. Let us show that the convex set $S = (0, +\infty)$ of all positive numbers cannot be represented as an uncertainty set $U(K)$ for any positive partial knowledge K.

Indeed, let us assume that $S = U(K)$ for some positive partial knowledge $K = (P^-, P^+, \emptyset, \emptyset)$. By definition of the set P^+, every element $p \in P^+$ of this set must be larger than or equal to every element of S. Since no number is larger than every positive real number, we thus conclude that there are no such elements p, i.e., that $P^+ = \emptyset$.

By definition of the set P^-, every element $p \in P^-$ of this set must be larger than or equal to every positive real number. Thus, p cannot be positive, it has to be non-positive.

Now, the value 0 is larger than or equal to every non-positive number, thus it is larger than or equal to every element from the set P^- and so, is sin the uncertainty set $U(K)$. However, $0 \notin S$. So, $S \neq U(K)$.

In this example, the two sets are "almost" equal, in the sense that for $P^- = \{x : x \leq 0\}$ and $P^+ = \emptyset$, the corresponding uncertainty set is $[0, +\infty)$ is the closure of the original convex set $S = (0, +\infty)$. Let us give another example where the difference is more drastic.

Let us consider a 2-D analogue of Minkowski space-time R^2, with the causality-related order

$$(t, x) \leq (t', x') \Leftrightarrow c \cdot (t' - t) \geq |x' - x|,$$

where c is a speed of light. The meaning of this relation is that for an event (t, x) to influence the event (t', x'), it must be possible to get to x' from x in time $t' - t$ with a speed $\dfrac{|x - x'|}{t' - t}$ not exceeding the speed of light.

For this order, $(t, x) \leq (t', x')$ implies $t \leq t'$.

Let us show that a convex set $S = \{(t, x) : c \cdot t = x\}$ cannot be represented as an uncertainty set $U(K)$ for any positive partial knowledge. Indeed, by definition of the set P^+, every element $p = (t_p, x_p) \in P^+$ of this set must be causally following every element of S. Since the set S includes events with arbitrarily values of time t, we would thus conclude that $t_p \geq t$ for all real numbers t. Such a value does not exist, so $P^+ = \emptyset$.

Similarly, by definition of the set P^-, every element $p = (t_p, x_p) \in P^-$ of this set must be causally preceding every element of S. Since the set S includes events with arbitrarily values of time t, we would thus conclude that $t_p \leq t$ for all real numbers t. Such a value does not exist, so $P^- = \emptyset$.

Since both P^+ and P^- are empty sets, the corresponding uncertainty set $U(K)$ coincides with the entire space X and is, thus, different from the set $S \neq X$.

Proof of Proposition 4

This statement was, in effect, proven, when we proven Proposition 2.

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