Journal of Uncertain Systems
Vol.5, No.1, pp.72-80, 2011
Online at: www.jus.org.uk

Rough Set Approach on Lattice

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Received 21 July 2009; Revised 23 April 2010

Abstract

This paper deals with rough set approach on lattice theory. We represent the lattices for rough sets determined by an equivalence relation. Without any loss of generality, we have defined the rough set as a pair of sets (lower approximation set, upper approximation set) and then we showed that the collection of all rough sets of an approximations by an equivalence relation form a lattice by some order relation. In this paper we are able to deal with information sources in a set-theoretic manner. We also given an integrated approach to form lattices by choice function and lattice structure in rough set theory. The simple notion of this paper is to show the lattice structure in rough set theory by using indiscernible equivalence relation. Some important results are also proved. Finally, some examples are considered to illustrate the paper.

Keywords: rough set, choice function, equivalence class, indiscernibility relation, lattice

1 Preliminaries

The lattice is one of the most widely discussed and studied structure in the classical algebraic theory, both as a specific algebra with a carrier and two binary operations, and as a relational structure-a specific ordered set [2]. The lattice as a poset will be denoted by $(L, \leq)$, and the lattice as an algebra by $(L, \wedge, \vee)$. We write simply L to denote the lattice in both senses. A poset $(L, \leq)$ is a lattice if $sup\{a, b\}$ and $inf\{a, b\}$ exist for all $a, b \in L$.

2 Introduction

The original setting of the theory of rough sets was introduced by Z. Pawlak [15], assumed that sets are chosen from a universe U, but that elements of U can be specified only upto an indiscernibility equivalence relation E on U. If a subset $X \subseteq U$ contains an element indiscernible from some elements not in X, then X is rough. Also a rough set X is described by two approximations. Basically, in rough set theory, it is assume that our knowledge is restricted by an indiscernibility relation. An indiscernibility relation is an equivalence relation E such that two elements of an universe of discourse U are E-equivalent if we can not distinguish these two elements by their properties known by us. By means of an indiscernibility relation E, we can partition the elements of U into three disjoint classes respect to any set $X \subseteq U$, defined as follows:

- The elements which are certainly in X. These are elements $x \in U$ whose E-class x/E is included in X.
- The elements which certainly are not in X. These are elements $x \in U$ such that their E-class x/E is included in X^{co}, which is the complement of X.
- The elements which are possibly belongs to X. These are elements whose E-class intersects with both X and X^{co}. In other words, x/E is not included in X nor in X^{co}.

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From this observation, Pawlak [15] defined lower approximation set $X\downarrow$ of X to be the set of those elements $x \in U$ whose E-class is included in X, i.e, $X\downarrow = \{x \in U : x/E \subseteq X\}$ and for the upper approximation set $X\uparrow$ of X consists of elements $x \in U$ whose E-class intersect with X, i.e, $X\uparrow = \{x \in U : x/E \cap X \neq \emptyset\}$. The difference between $X\downarrow$ and $X\uparrow$ treated as the actual area of uncertainty.

The study of lattices in rough set theory was initiated by Iwiniski [7]. He noticed that rough sets can be represented by their approximations. The set of rough sets can be defined as $\Re\text{s} = \{(X\downarrow, X\uparrow) \ : X \subseteq U\}$. It is also noticed that $\Re$s may be canonically ordered by the co-ordinatewise as $(X\downarrow, X\uparrow) \leq (Y\downarrow, Y\uparrow) \Leftrightarrow X\downarrow \subseteq Y\downarrow$ and $X\uparrow \subseteq Y\uparrow$. Pomykala et al. [18] showed that $R\text{s} = (\Re\text{s}, \leq)$ is a stone lattice and also they meant that $\Re$s is a distributive lattice such that each element in the ordered pair $(X\downarrow, X\uparrow)$ has a pseudo complement $(X\downarrow^{co}, X\uparrow^{co})$.

In early eighties, Pawlak [15] introduced the theory of rough sets as an extension of set theory for the study of intelligent systems characterized by insufficient and incomplete information. One may regard the theory of rough sets to be complementary to other generalizations of set theory, such as fuzzy set and multisets. Gehrke and Walker [5] suggested a precise structure theorem for the stone algebra of rough sets and a characterization of them in the category of all stone algebras. Davey and Priestly [3] introduced the concept of lattice theory and order. Yao [29] introduced the notion of formal concept analysis to rough set theory, which leads to new, different interpretations and representations of formal concepts. Based on the approximation operators, three concept lattices are introduced and examined. Recently Jarvinen [10] proposed the lattice-theoretical background of rough sets which contains the necessary part of lattice theory and shows how to formulate in an elegant way various concepts and facts about rough sets and Pawlak's information systems.

Several methodologies have been proposed to define lattices in terms of rough set approach. Most of these papers are based on tolerence relation. However, no vast studies have been made on lattices determined by an equivalence relation. The main objective of this paper is to present a comparative study of lattice in rough set represented by the region of choice function and lattices determined by an equivalence relation.

3 Axiom of Choice

Let P be any set, then there exists a function which selects from each non-empty subset $S \subseteq P$ to a member $\partial(S)$ of S. Let E be an equivalence relation on U. We denote by U/E the set of all equivalence classes of E. By the axiom of choice, there exists a function $\partial : U/E \longrightarrow U$ such that $\partial(D) \in D$ for every E-class in D. Any such function ∂ is called a choice function for E. The range set $\{\partial(D) \ : \ D \in U/E\}$ of ∂ is denoted by Reg(∂). Let us denote for any $X \subseteq U$, $\overline{X} = X\downarrow \cup (X\uparrow \cap Reg(\partial))$. This means that $\overline{X}$ contains all equivalence classes included in X, and from the classes intersecting X only one element is chosen. Let us also denote $P^{\partial}(U) = \{\overline{X} \ : \ X \subseteq U\}$. It is clear that for any definable set X, $\overline{X} = X$ because $X\downarrow = X\uparrow$. If $X, Y \in Rs$, then $B \leq D \Leftrightarrow X\downarrow \subseteq Y\downarrow$ and $X\uparrow \subseteq Y\uparrow$, where $X \in B$, and $Y \in D$.

Lemma 1 *If $E \in Eq(U)$ and $\partial : U/E \to U$ is a choice function, then $(\mathbf{A}(E), \leq) \cong (D^{\partial} : D \in \mathbf{A}(E), \ \subseteq)$.*

Proof: It is obvious that the map $D \to D^{\partial}$ is onto $\{D^{\partial} : D \in \mathbf{A}(E)\}$. Suppose $B \leq D$ holds in $(\mathbf{A}(E), \leq)$ where $X \in B, Y \in D$. Then $X\downarrow \subseteq Y\downarrow$ and $X\uparrow \subseteq Y\uparrow$. This implies that $B^{\partial} = X^{\partial} = X\downarrow \cup (X\uparrow \cap Reg(\partial)) \subseteq Y\downarrow \cup (Y\uparrow \cap Reg(\partial)) = Y^{\partial} = D^{\partial}$. On the other hand, if $B^{\partial} \subseteq D^{\partial}$, then for all $X \in B$ and $Y \in D, X\downarrow = X^{\partial}\downarrow = B^{\partial}\downarrow \subseteq D^{\partial}\downarrow = Y^{\partial}\downarrow = Y\downarrow$ and $X\uparrow = X^{\partial}\uparrow = B^{\partial}\uparrow \subseteq D^{\partial}\uparrow = Y^{\partial}\uparrow = Y\uparrow$. Hence, $B \leq D$. Thus, $(\mathbf{A}(E), \leq) \cong (D^{\partial} : D \in \mathbf{A}(E), \ \subseteq)$.

Lemma 2 *If $E \in Eq(U)$ and $\partial : U/E \to U$ is a choice function, then $\{D^{\partial} : D \in \mathbf{A}(E), \ \subseteq\}$ is a complete sublattice of $(P(U), \ \subseteq)$,where $P(U)$ is the power set of U.*

Proof: It is obvious that $\emptyset^{\partial} = \emptyset$ and $U^{\partial} = U$, and hence $\bigcup\emptyset$ and $\bigcap\emptyset$ are in $\{D^{\partial} : D \in \mathbf{A}(E)\}$. Let $\{X^{\partial} : X \in H\}$ be a nonempty subset of $\{D^{\partial} : D \in \mathbf{A}(E)\}$, where $H \subseteq P(U)$. Then by Lemma 1 we can write

$$\begin{aligned}
\bigcup_{X\in H} X^{\partial} &= \bigcup_{X\in H}(X\downarrow\cup(X\uparrow\cap Reg(\partial))) \\
&= \bigcup_{X\in H}(X\downarrow)\cup\bigcup_{X\in H}(X\uparrow\cap Reg(\partial)) \\
&= (\bigcup_{X\in H}X^{\partial})\downarrow\cup(\bigcup_{X\in H}(X\uparrow\cap Reg(\partial)) \\
&= (\bigcup_{X\in H}X^{\partial})\downarrow\cup(\bigcup_{X\in H}(X^{\partial}\uparrow\cap Reg(\partial)) \\
&= (\bigcup_{X\in H}X^{\partial})\downarrow\cup((\bigcup_{X\in H}X^{\partial})\uparrow\cap Reg(\partial)) \\
&= (\bigcup_{X\in H}X^{\partial})^{\partial}.
\end{aligned}$$

Hence, $\bigcup_{X\in H} X^{\partial}\in\{D^{\partial}: D\in \mathbf{A}(E)\}$.

Again,

$$\begin{aligned}
\bigcap_{X\in H} X^{\partial} &= \bigcap_{X\in H}(X\downarrow\cup(X\downarrow\cap Reg(\partial)) \\
&= \bigcap_{X\in H}((X\downarrow\cup X\uparrow)\cap(X\downarrow\cup Reg(\partial))) \\
&= \bigcap_{X\in H}X\uparrow\cap\bigcap_{X\in H}(X\downarrow\cup Reg(\partial)) \\
&= \bigcap_{X\in H}X\uparrow\cap(\bigcap_{X\in H}(X\downarrow\cup Reg(\partial)) \\
&= (\bigcap_{X\in H}X^{\partial})\uparrow\cap(\bigcap(X^{\partial})\downarrow\cup Reg(\partial)) \\
&= (\bigcap_{X\in H}(X^{\partial})\uparrow\cap((\bigcap_{X\in H}X^{\partial})\downarrow\cup Reg(\partial)) \\
&= ((\bigcap_{X\in H}X^{\partial})\uparrow\cap(\bigcap_{X\in H}X^{\partial}\downarrow)\cup((\bigcap_{X\in H}X^{\partial}\uparrow)\cap Reg(\partial)) \\
&= (\bigcap_{X\in H}(X^{\partial})\downarrow\cup((\bigcap_{X\in H}(X^{\partial})\uparrow\cap Reg(\partial)) \\
&= (\bigcap_{X\in H}X^{\partial})^{\partial}.
\end{aligned}$$

Example 1: Let U $=\{1,2,3\}$ and let E be an equivalence relation on U such that $1/E=2/E=\{1,2\}$, and $3/E=\{3\}$. Let ∂ be a choice function $\partial: U/E\longrightarrow U$ which picks from each E-class its first element. Then $Reg(\partial)=\{1,3\}$. The sets $X\downarrow, X\uparrow, X^{\partial}$ are represented in the following Table 1, $\forall\ X\subseteq U$. In Table 1, $X^{\partial}=\{\overline{X}:\overline{X}\subseteq U\}$ and the range $\{\partial(D): D\in U/E\}$. The Hasse diagram of $\{\partial(D): D\in U/E,\subseteq\}$ is represented in the following Figure 1.

Table 1:

X	$X\downarrow$	$X\uparrow$	X^{∂}
$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
$\{1\}$	$\emptyset$	$\{1,2\}$	$\{1\}$
$\{2\}$	$\emptyset$	$\{1,2\}$	$\{1\}$
$\{3\}$	$\{3\}$	$\{3\}$	$\{3\}$
$\{1,2\}$	$\{1,2\}$	$\{1,2\}$	$\{1,2\}$
$\{1,3\}$	$\{3\}$	U	$\{1,3\}$
$\{2,3\}$	$\{3\}$	U	$\{1,3\}$
$\{1,2,3\}$	U	U	U

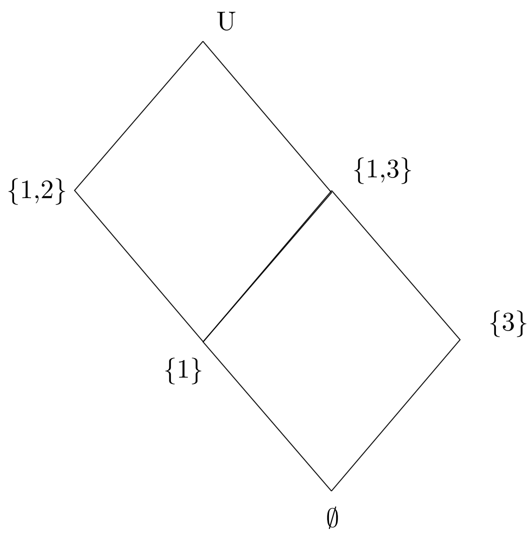

Figure 1:

Lemma 3 *Let ∂ be a choice function for an equivalence relation on U. Then for any families $\{X_i, i \in I\} \subseteq P(U)$, $(\bigcup_{i\in I} \overline{X}_i)\downarrow = \bigcup_{i\in I} X_i\downarrow$ and $(\bigcap_{i\in I} \overline{X}_i)\uparrow = \bigcap_{i\in I} X_i\uparrow$.*

Proof: Let us omit the subscripts $i \in I$(index set). Then

$$\begin{aligned}
(\bigcup_{i\in I} \overline{X}_i)\downarrow &= (\bigcup (X_i\downarrow \cup (X_i\uparrow \cap Reg(\partial))))\downarrow \\
&= (\bigcup X_i\downarrow \cup \bigcup (X_i\uparrow \cap Reg(\partial)))\downarrow \\
&= (\bigcup X_i\downarrow)\downarrow \cup (\bigcup (X_i\uparrow \cap Reg(\partial)))\downarrow \\
&= \bigcup X_i\downarrow \cup (\bigcup (X_i\uparrow \cap Reg(\partial)))\downarrow \\
&= \bigcup_{i\in I} X_i\downarrow .
\end{aligned}$$

Recall that $\bigcup_{i\in I} X_i\downarrow$ is definable. If $x \in \bigcup (X_i\uparrow \cap Reg(\partial))\downarrow$, then necessarily $E(x) \subseteq Reg(\partial)$. It is clear that $x \in X_i\downarrow$ for some $i \in I$, which implies $x \in X_i\downarrow$ with any such i. So $x \in \bigcup X_i\downarrow$. Hence, $(\bigcup X_i\uparrow \cap Reg(\partial))\downarrow \subseteq$

$\bigcup X_i \downarrow$ which implies the desired equality. For the other part we proved in the following way:

$$\begin{aligned}
(\bigcap_{i\in I} \overline{X}_i)\downarrow &= (\bigcap (X_i \downarrow \cup (X_i \uparrow \cap Reg(\partial))))\uparrow \\
&= (\bigcap ((X_i \downarrow \cup X_i \uparrow) \cap (X_i \downarrow \cup Reg(\partial))))\uparrow \\
&= (\bigcap (X_i \uparrow \cap (X_i \downarrow \cup Reg(\partial))))\uparrow \\
&= (\bigcap X_i \uparrow)\uparrow \cap (\bigcap (X_i \downarrow \cup Reg(\partial)))\uparrow \\
&= \bigcap X_i \uparrow \cap (\bigcap X_i \downarrow \cup Reg(\partial))\uparrow \\
&= \bigcap X_i \uparrow \cap U \\
&= \bigcap_{i\in I} X_i \uparrow .
\end{aligned}$$

Theorem 3.1 *For any equivalence relation on U, $\Re s$ is a complete sublattice of $P(U) \times P(U)$.*

Proof: Let $\{(X_i \downarrow, X_i \uparrow) : i \in I\}$ be a subset of $\Re$s. Then by Lemma 1 we have

$$\begin{aligned}
(\bigcup_{i\in I} X_i \downarrow, \bigcup_{i\in I} X_i \uparrow) &= (\bigcup_{i\in I} X_i \downarrow, \bigcup_{i\in I} \overline{X}_i \uparrow) \\
&= ((\bigcup_{i\in I} \overline{X}_i)\downarrow, (\bigcup_{i\in I} \overline{X}_i)\uparrow)
\end{aligned}$$

and

$$\begin{aligned}
(\bigcap_{i\in I} X_i \downarrow, \bigcap_{i\in I} X_i \uparrow) &= (\bigcap_{i\in I} \overline{X}_i \downarrow, \bigcap_{i\in I} X_i \uparrow) \\
&= ((\bigcup_{i\in I} \overline{X}_i \downarrow), (\bigcup_{i\in I} \overline{X}_i \uparrow)).
\end{aligned}$$

Thus

$$\bigvee_{i\in I} (X_i \downarrow, X_i \uparrow) = (\bigcup_{i\in I} X_i \downarrow, \bigcup_{i\in I} X_i \uparrow)$$

and

$$\bigwedge_{i\in I} (X_i \downarrow, X_i \uparrow) = (\bigcap_{i\in I} X_i \downarrow, \bigcap_{i\in I} X_i \uparrow).$$

Proposition 3.2 *If X is a rough set in the form of $(X \downarrow, X \uparrow)$, then i) $X \wedge X = X$ and ii) $X \vee X = X$.*

Proof: i) We have

$$\begin{aligned}
X \wedge X &= (X \downarrow, X \uparrow) \wedge (X \downarrow, X \uparrow) \\
&= ((X \cap X)\downarrow, (X \cap X)\uparrow) \\
&= (X \downarrow, X \uparrow) \\
&= X.
\end{aligned}$$

ii) We have

$$\begin{aligned}
X \vee X &= (X \downarrow, X \uparrow) \vee (X \downarrow, X \uparrow) \\
&= ((X \cup X)\downarrow, (X \cup X)\uparrow) \\
&= (X \downarrow, X \uparrow) \\
&= X.
\end{aligned}$$

Proposition 3.3 *If X, Y are two rough sets, then i) $X \wedge Y = Y \wedge X$ and ii) $X \vee Y = Y \vee X$ where $X = (X\downarrow, X\uparrow)$ and $Y = (Y\downarrow, Y\uparrow)$.*

Proof: i) We have

$$\begin{aligned} X \wedge Y &= (X\downarrow, X\uparrow) \wedge (Y\downarrow, Y\uparrow) \\ &= ((X \cap Y)\downarrow, (X \cap Y)\uparrow) \\ &= ((Y \cap X)\downarrow, (Y \cap X)\uparrow) \\ &= Y \wedge X. \end{aligned}$$

ii) We have

$$\begin{aligned} X \vee Y &= (X\downarrow, X\uparrow) \vee (Y\downarrow, Y\uparrow) \\ &= ((X \cup Y)\downarrow, (X \cup Y)\uparrow) \\ &= ((Y \cup X)\downarrow, (Y \cup X)\uparrow) \\ &= Y \vee X. \end{aligned}$$

4 Scheme of Approximation

Let E be an equivalence relation on U. We denote for any $x \in U$ *and* $E(x) = \{y \in E : xEy\}$. For any subset $X \subseteq U$, the lower approximation of X is $X\downarrow = \{X \in U : E(x) \subseteq X\}$. And the upper approximation is $X\uparrow = \{X \in U : E(x) \cap X \neq \emptyset\}$. Let $X^{co} = U - X$, which is the complement of X. Then $X\uparrow^{co} = X\downarrow^{co}$ and $X\downarrow^{co} = X\uparrow^{co}$, i.e., $\downarrow$ and $\uparrow$ are dual. In addition $(\bigvee D)\uparrow = \bigcup\{X\uparrow : X \in D\}$ and $(\bigwedge D)\downarrow = \bigcap\{X\downarrow : X \in D\}$ for all $D \subseteq P(U)$. Let us denote for any $X \subseteq U$, $\mathbf{A}(X) = (X\downarrow, X\uparrow)$ and call it the rough set of X. Furthermore we denote by $\Re s = \{\mathbf{A} : X \subseteq U\}$ the set of all rough set. The set $\Re$s can be ordered co-ordinate wise by $(X\downarrow, X\uparrow) \leq (Y\downarrow, Y\uparrow) \Leftrightarrow X\downarrow \subseteq Y\downarrow$ and $X\uparrow \subseteq Y\uparrow$. Obtaining in this way a partially ordered set Rs = ($\Re$s, $\leq$) with $A(\emptyset) = (\emptyset\downarrow, \emptyset)$ as the least element and $A(U) = (U, U\uparrow)$ as the greatest element. A rough set $\mathbf{A}$ is called an exact element of Rs if $X\downarrow = X = X\uparrow$. Let us define the mapping ϕ :R$s \longrightarrow$Rs, $A(X) \longmapsto A(X^{co})$, since $\phi((X\downarrow, X\uparrow)) = (X\uparrow^{co}, X\downarrow^{co})$ for any $X \subseteq U$, the mapping ϕ is well defined and it is easy to see that the pair (ϕ, ϕ) is an order reversing Galois connection on Rs [9].

Lemma 4 *The partial order set Rs is self dual, that is, Rs is order-isomorphic to its dual Rs^{du}.*

Proof: For any binary relation E on U, a set C is called connected component of E, if C is an equivalence class of smallest equivalence relation containing E. Clearly, for any connected component C of E and any elements $c \in C$ and $d \in U$, $E(c) \subseteq C$ and $E(d) \cap C \neq \emptyset$ implies $d \in C$. Hence $C\uparrow \subseteq C \subseteq C\downarrow$. Since any completely distributive complete lattice is both pseudocomplemented and dually pseudocomplemented.

5 Lattices of Rough Sets Determined by Equivalence Relation

In this section, we present a condition under which the partial ordered set of equivalence relation based on rough sets forms a complete lattice. Let $E \subseteq U \times U$ be an equivalence relation and denote by U/E the set of all equivalence classes E. It is well known that E defines a partial order relation defined by the symbol $\leq$, and the quotient set U/E by setting $A, B \in U/E$ where $A \leq B$ if there exist $a \in A$ and $b \in B$ with aEb. Noticed that in fact $A \leq B$ if and only if aEb, for all $a \in A$ and $b \in B$.

Example 2: Let $U = \{a, b, c, d\}$ and let E be a equivalence relation on U such that $a/E = b/E = \{a, b\}$, $c/E = d/E = \{c, d\}$. The lower and upper approximations are defined by E and presented in Table 2.

Table 2:

X	$X\downarrow$	$X\uparrow$
$\emptyset$	$\emptyset$	$\emptyset$
$\{a\}$	$\emptyset$	$\{a,b\}$
$\{b\}$	$\emptyset$	$\{a,b\}$
$\{c\}$	$\emptyset$	$\{c,d\}$
$\{d\}$	$\emptyset$	$\{c,d\}$
$\{a,b\}$	$\{a,b\}$	$\{a,b\}$
$\{a,c\}$	$\emptyset$	U
$\{a,d\}$	$\emptyset$	U
$\{b,c\}$	$\emptyset$	U
$\{b,d\}$	$\emptyset$	U
$\{c,d\}$	$\{c,d\}$	$\{c,d\}$
$\{a,b,c\}$	$\{a,b\}$	U
$\{a,b,d\}$	$\{a,b\}$	U
$\{a,c,d\}$	$\{c,d\}$	U
$\{b,c,d\}$	$\{c,d\}$	U
$\{a,b,c,d\}$	U	U

The Hasse diagram of $(\mathbf{A}(E),\leq)$ is given in Figure 2. Noticed that $(\mathbf{A}(E),\leq)$ is a rough-lattice with $(\emptyset,\emptyset)$ as the infimum and (U,U) as the supremum. For simplicity we denote the subsets of U, which is differ from $\emptyset$ and U by sequence of letters. For example $\{a,b,c\}$ is written as abc.

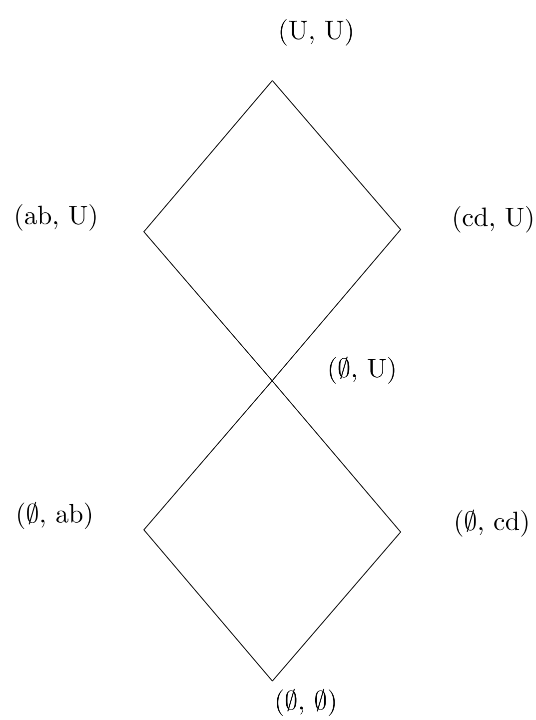

Figure 2:

6 Conclusions

The lattice theory provides an important interpretation on the rough set theory. It allows us to study rough set concept to lattices. This leads naturally to the generalization of rough set approximations. In this paper, though we have showed that lattice theoretic representation by the means of equivalence indiscernibility relation and providing a comparative study of lattice in rough representation by the region of choice function and lattices determined by an equivalence relation. And also represent their corresponding Hasse diagram which is an important concept in discrete mathematics, as well as in computer science. The works are reviewed using a very simple, and unified view. That is rough lattice models are constructed and interpreted based on the equivalence relations. This view may be useful in the application of knowledge representation in the theory of rough sets. The lattice for rough set is a generalization of the set theory in which a pair of new set theoretic operators have been introduced. In this paper, unlike classical set algebra, a pair of sets (lower approximation set, upper approximation set) represented by each concept. Thus based on the lattice structure, we provide an extension of the classical set algebra. As was illustrated by an example, we are able to deal with several information sources in set theoretic manner. We also give an integrated approach to choice function and lattice structure in rough set by an equivalence relation which is another view on the theory of rough sets.

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