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On Another Decomposition of Fuzzy Automata*

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Abstract

After introducing the notion of an f-primary of a fuzzy automaton, this paper presents a decomposition of a fuzzy automaton (possibly with infinite state-set) as a union of its f-primaries. En route, it is also shown that the concepts of primaries and f-primaries of a fuzzy automaton coincide in the case of compact fuzzy automaton.
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1 Introduction

The decompositions of finite and infinite automata (as a union of its 'simpler' subautomata) have been studied in [1, 3, 4, 7]. A well-known result in this context about (crisp) *finite* automata, is the so called *"Primary Decomposition Theorem"*. It says that such automata can be so decomposed uniquely as a union of their *primaries*. Its counterpart for *fuzzy* automata, having *finite* state-sets, has been provided in [5]. It turns out that this theorem may not hold good for fuzzy automata having *infinite* state-sets. However, in [8], it was shown that a counterpart of this result holds for fuzzy automata, having even *infinite* state-sets, *provided* their state-set topologies are *compact* (cf. [6] for a similar observation). The present paper continues similar study for fuzzy automata and is mainly motivated by the work of Bavel and Thomas in [2], in which the authors have shown that (crisp) automata, whose state-sets and input-sets may be of arbitrary cardinality (such automata have been called 'monadic algebras' in [2]), admit a certain type of decomposition by using a suitable generalization of the concept of primaries of (crisp) automata. In particular, we produce a decomposition of an arbitrary fuzzy automaton in terms of its f-*primaries*. En route, we make some related observations also.

2 Preliminaries

In this section we collect some results, which are useful in the next section. We start from the following concept of fuzzy automata which resembles the concept of fuzzy machines, as given in [5].

Definition 2.1 *A* **fuzzy automaton** *is a triple* $M = (Q, X, \delta)$*, where* Q *is a nonempty set (of* **states** *of* M*),* X *is a monoid (the* **input monoid** *of* M*), whose identity shall be denoted as* e*, and* δ *is a fuzzy subset of* $Q \times X \times Q$*, i.e., a map* $\delta : Q \times X \times Q \to [0, 1]$*, such that* $\forall q, p \in Q, \forall x, y \in X$,

$$\delta(q, e, p) = \begin{cases} 1 & \text{if } q = p \\ 0 & \text{if } q \neq p \end{cases}$$

$$\text{and} \quad \delta(q, xy, p) = \vee\{\delta(q, x, r) \wedge \delta(r, y, p) : r \in Q\}.$$

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Remark 2.1 *As in* [6]*, we take the view that a (crisp) automaton is a triple* (Q, X, δ)*, where* Q *is a (possibly infinite) set,* X *is a monoid with identity* e*, and* $\delta : Q \times X \to Q$ *is a function such that* $\delta(q, e) = q$ *and* $\delta(q, xy) = \delta(\delta(q, x), y), \forall q \in Q, \forall x, y \in X$*. By identifying* δ *with the function* $\hat{\delta} : Q \times X \times Q \to [0, 1]$*, given by* $\hat{\delta}(q, x, p) = 1$ *or* 0 *according as* $\delta(q, x) = p$ *or* $\delta(q, x) \neq p$*, we see that the concept of a fuzzy automaton generalizes that of a (crisp) automaton.*

Let X be a set and 2^X be its power set. Recall (cf. [10]) that a function $k : 2^X \to 2^X$ is called a *Kuratowski closure operator* on X if $\forall A, B \in 2^X$, (i) $A \subseteq k(A)$, (ii) $k(k(A)) = k(A)$, (iii) $k(A \cup B) = k(A) \cup k(B)$, and (iv) $k(\phi) = \phi$. Each such k gives rise to a topology, say T, such that A is T-*closed* if $A = k(A)$ (and consequently $G \subseteq X$ is T-*open* if $X - G = k(X - G)$). A useful consequence of (iii) (used throughout this paper) is: $A \subseteq B \Rightarrow k(A) \subseteq k(B)$.

Definition 2.2 ([8]) *Given a fuzzy automaton* (Q, X, δ) *and* $A \subseteq Q$*, the* **source** *and the* **successor** *of* A *are respectively the sets*

$$\sigma_Q(A) = \{q \in Q : \delta(q, x, p) > 0, \textit{ for some } (x, p) \in X \times A\}, \text{ and}$$
$$s_Q(A) = \{p \in Q : \delta(q, x, p) > 0, \textit{ for some } (x, q) \in X \times A\}.$$

We shall frequently write $\sigma_Q(A)$ and $s_Q(A)$ as just $\sigma(A)$ and $s(A)$, and $\sigma(\{q\})$ and $s(\{q\})$ as just $\sigma(q)$ and $s(q)$.

Proposition 2.1 ([8]) *Let* (Q, X, δ) *be a fuzzy automaton.*

(a) The source and the successor, viewed as functions $\sigma : 2^Q \to 2^Q$ *and* $s : 2^Q \to 2^Q$*, turn out to be Kuratowski closure operators on* Q*, inducing two topologies on* Q *(which we shall respectively denote as* $T(Q)$ *and* $T^*(Q)$*).*

(b) The topologies $T(Q)$ *and* $T^*(Q)$ *are dual in the sense that each* $A \subseteq Q$ *is* $T(Q)$*-open if and only if* A *is* $T^*(Q)$*-closed.*

Definition 2.3 ([5]) *A fuzzy automaton* (R, X, λ) *is called a* **subautomaton** *of a fuzzy automaton* (Q, X, δ) *if* $R \subseteq Q, s_Q(R) = R$*, and* $\delta_{/R \times X \times R} = \lambda$*.*

Proposition 2.2 ([8]) *Let* $M = (Q, X, \delta)$ *and* $N = (R, Y, \lambda)$ *be fuzzy automata. Then* N *is subautomaton of* M *if and only if* R *is* $T(Q)$*-open.*

Now, we recall the concept of primaries, which plays a key role in the study of decomposition of fuzzy automata.

Definition 2.4 ([5]) *Let* (Q, X, δ) *be a fuzzy automaton. A subset* $R \subseteq Q$ *is called*

(i) **genetically closed** *if* $\exists P \subseteq R$ *such that* $\sigma(P) \subseteq s(P)$ *and* $s(P) = R$*;*

(ii) a **primary** *subset of* Q *if* R *is a nonempty minimal genetically closed subset of* Q*.*

Definition 2.5 ([8]) *A* **primary** *of a fuzzy automaton* $M = (Q, X, \delta)$ *is a subautomaton of* M *whose state-set is a primary subset of* Q*.*

In [8], the primaries of a fuzzy automaton have been characterized in terms of the topology on its state-set by using the concept of *core*, which is reproduced below.

Definition 2.6 *The* **core** *of any subset* R *of the state-set* Q *of a fuzzy automaton is the set*

$$\mu(R) = \{q \in R : \sigma(q) \subseteq R\}.$$

We shall frequently write $\mu(\{q\})$ as just $\mu(q)$. Before stating the next proposition, we first recall the following definition from Willard [10].

Definition 2.7 *A closed subset of topological space is called* **regular closed** *if it is equal to the closure of its interior.*

For example, the closures of open subsets are always regular closed.

Proposition 2.3 ([8]) *A closed subset R of Q is genetically closed if and only if R is a $T^*(Q)$-regular closed subset of Q.*

Proposition 2.4 ([8]) *The state-set of a primary of a fuzzy automaton (Q, X, δ) is a minimal $T^*(Q)$-regular closed subset of Q.*

Proposition 2.5 ([8]) *Let (Q, X, δ) be a fuzzy automaton and $p \in Q$. Then $s(p)$ is a primary of Q if and only if $p \in \mu(s(p))$.*

We close this section by recalling the definition of a compact fuzzy automaton and some of the associated results from [8].

Definition 2.8 *A fuzzy automaton (Q, X, δ) is called* **compact** *if the topology $T(Q)$ is compact.*

Proposition 2.6 *A fuzzy automaton (Q, X, δ) is compact if and only if there exists a finite subset Q' of Q such that $s(Q') = Q$, or equivalently, if and only if $\sigma(Q)$ is finite.*

Proposition 2.7 *A primary of a compact fuzzy automaton is a maximal singly generated subautomaton.*

3 f-Primaries and Decomposition of a Fuzzy Automaton

In this section, chiefly inspired from [2], we have given the concept of f-primaries of a fuzzy automaton and shown that each fuzzy automaton can be expressed, *though non-uniquely*, as a union of its 'f-primaries'. In this section, unless mentioned otherwise, Q is assumed to be equipped with the topology $T^*(Q)$.

Definition 3.1 *Let $M = (Q, X, \delta)$ be a fuzzy automaton. Then a subset R of Q is called* **finitely closed** *if for each finite subset T of R there exists $q \in Q$ such that $T \subseteq s(q)$.*

Remark 3.1 *For each $q \in Q, s(q)$ is obviously a finitely closed subset of Q.*

Lemma 3.1 *Let (R, X, λ) be a primary of a fuzzy automaton (Q, X, δ). Then $s(\sigma(p)) = R, \forall p \in \mu(R)$.*

Proof: Let $p \in \mu(R)$. Then $\sigma(p) \subseteq R$ and so $s(\sigma(p) \subseteq R$ (as $s(R) = R$). But as $s(\sigma(p))$ is a regular closed subset (since the closures of open subsets are always regular closed) and R is a minimal regular closed subset of $Q, s(\sigma(p)) = R$.

Lemma 3.2 *Let (R, X, λ) be a primary of a fuzzy automaton (Q, X, δ). Then for every finite subset T of R, $T \subseteq s(r)$ for some $r \in R$.*

Proof: We prove this lemma by induction on the number of elements in T. The result is obvious if $|T| = 1$. Now assume the result to be true for all T having $k-1$ elements. Consider some $T \subseteq R$ having k elements. Pick any $p \in T$. By induction hypothesis, $\exists q \in R$ such that $T - \{p\} \subseteq s(q)$. Thus $T \subseteq s(q) \cup \{p\} \subseteq s(\{q,p\})$. Put $S = \{q,p\}$ and let $m \in \mu(R)$. Then by Lemma 3.1, $R = s(\sigma(m))$. Note that $q \in R \Rightarrow q \in s(\sigma(m)) \Rightarrow \delta(m', x, q) > 0$ for some $(m', x) \in \sigma(m) \times X$. Now $m' \in \sigma(m) \Rightarrow \sigma(m') \subseteq \sigma(m) \subseteq R$ (as $m \in \mu(R)$) $\Rightarrow m' \in \mu(R) \Rightarrow s(\sigma(m')) = R$ (by Lemma 3.1). Consequently, $p \in R \Rightarrow p \in s(\sigma(m')) \Rightarrow \delta(r, y, p) > 0$ for some $(r, y) \in \sigma(m') \times X$. From $r \in \sigma(m')$, we get $m' \in s(r)$, whereby $s(m') \subseteq s(r)$. This, together with the facts that $q \in s(m')$ and $p \in s(r)$, gives $q, p \in s(r)$. Hence $S = \{q,p\} \subseteq s(r)$. Thus $s(\{q,p\}) \subseteq s(s(r))$, i.e., $s(\{q,p\}) \subseteq s(r)$. Hence $T \subseteq s(r)$.

Proposition 3.1 *Let (R, X, λ) be a primary of a fuzzy automaton (Q, X, δ). Then R is a maximal finitely closed subset of Q.*

Proof: Lemma 3.2 already shows that R must be finitely closed. So only the maximality of R needs to be shown. Let S be another finitely closed subset of Q such that $R \subseteq S$. Let $q \in S$ and $p \in \mu(R)$. Since $\mu(R) \subseteq R \subseteq S, p \in S$. Thus $p, q \in S$. As S is a finitely closed subset of Q, there exists $r \in Q$ such that $\{p,q\} \subseteq s(r)$, whereby $p, q \in s(r)$. Also, as $p \in \mu(R), \sigma(p) \subseteq R$. Now $p \in s(r) \Rightarrow r \in \sigma(p)$. Hence $r \in R$, whereby $s(r) \subseteq s(R) = R$. But $q \in R$ (as $q \in s(r)$), showing that $S \subseteq R$. Hence $S = R$, proving that R is a maximal finitely closed subset of Q.

Remark 3.2 *The converse of the above theorem need not be true, even for a (crisp) automaton, as the following counter-example shows.*

Counter-example 3.1 *Let $M = (Q, X, \delta)$ be a (crisp) automaton, for which Q is the set of integers, $X = \{0, 1, 2, ...\}$, and $\delta : Q \times X \to Q$ be given by $\delta(n, r) = n - r, \forall n \in Q, \forall r \in X$.*

We note that $\forall m, n \in Q, n \in \sigma(m)$ if and only if $n \geq m$. First, we show that M is the only primary of itself.

Let A be a nonempty subset of Q. Then two cases arise.

Case 1: *When A is finite*

In this case $\mu(A) = \phi$, as there does not exist any $n \in A$ such that $\sigma(n) \subseteq A$ (since $\forall n \in Q$, $\sigma(n)$ is infinite). Thus $s(\mu(A)) = \phi$, whereby A is not a regular closed subset of Q.

Case 2: *When A is infinite*

In this case, if $\mu(A) = \phi$, then A cannot be regular closed as indicated above. So let $\mu(A) \neq \phi$ and $n \in \mu(A)$. Then $n \in A$ and $\sigma(n) \subseteq A$. Also, $\forall m \in \sigma(n), \sigma(m) \subseteq \sigma(\sigma(n)) = \sigma(n) \subseteq A$. Thus $m \in \mu(A) \subseteq s(\mu(A)), \forall m \in \sigma(n)$, i.e., $m \in s(\mu(A)), \forall m \geq n$. Next, as $n \in \mu(A)$, we have $s(n) \subseteq s(\mu(A))$, whereby $m \in s(\mu(A)), \forall m \in s(n)$, i.e., $m \in s(\mu(A)), \forall m \leq n$. Thus we have shown that $m \in s(\mu(A)), \forall m \geq n$ and $\forall m \leq n$, i.e., $m \in s(\mu(A)), \forall m \in Q$, showing that $s(\mu(A)) = Q$.

Hence $\forall A \subseteq Q, s(\mu(A)) \neq A$ if $A \neq Q$. Thus Q is the only regular closed subset of itself. Let $R = \{1, 2, 3, ...\}$ be a subset of Q. Then obviously it is finitely closed. But R cannot be regular closed, as Q is the only regular closed subset of itself. Thus a finitely closed subset is not neccesarily regular closed, and hence, not a primary.

Proposition 3.2 *Let $M = (Q, X, \delta)$ be a fuzzy automaton. Then each maximal finitely closed subset of Q is $T(Q)$-open.*

Proof: Let R be a maximal finitely closed subset of Q. We have to show that $s(R) = R$. Let T be a finite subset of $s(R)$. For each $q \in T$, we can find some $p \in R$ with $q \in s(p)$. The set S of all such $p's$ must be finite as T is finite. Also, $S \subseteq R$. Obviously, $T \subseteq s(S)$. Now as R is finitely closed, $\exists t \in Q$ such that $S \subseteq s(t)$. Hence $s(S) \subseteq s(t)$ (since $s(s(t)) = s(t)$), whereby $T \subseteq s(t)$. Thus for each finite subset T of $s(R)$, $\exists t \in Q$ such that $T \subseteq s(t)$. Hence $s(R)$ is a finitely closed subset of Q. But as R is a maximal finitely closed set and $R \subseteq s(R)$, we find that $s(R) = R$.

Definition 3.2 *An* **f-primary** *of a fuzzy automaton $M = (Q, X, \delta)$ is a subautomaton of M whose state-set is a maximal finitely closed subset of Q.*

Proposition 3.3 *Let $M = (Q, X, \delta)$ be a fuzzy automaton. Then every primary of M is an f-primary of M.*

Proof: Follows trivially from Proposition 3.1 and Definition 3.2.

Remark 3.3 *From Remark 3.2, it is clear that the converse of above proposition need not be true.*

Proposition 3.4 *Let $M = (Q, X, \delta)$ be a fuzzy automaton and $p \in Q$. Then $s(p)$ is a primary of M if and only if $s(p)$ is an f-primary of M.*

Proof: In view of Proposition 3.3, we only need to prove that if $s(p)$ is an f-primary then it is a primary. So let $s(p)$ be an f-primary of M. In view of Proposition 2.5, it suffices to show that $p \in \mu(s(p))$, or that $\sigma(p) \subseteq s(p)$. Let $q \in \sigma(p)$. Then $p \in s(q)$, whereby $s(p) \subseteq s(q)$. As $s(q)$ is finitely closed and $s(p)$, being an f-primary, is maximal finitely closed, $s(q) = s(p)$. Thus $q \in s(p)$, showing that $\sigma(p) \subseteq s(p)$.

Proposition 3.5 *Let $M = (Q, X, \delta)$ be a compact fuzzy automaton and $N = (R, X, \lambda)$ be its subautomaton. Then N is an f-primary of M if and only if N is a primary of M.*

Proof: Once again, in view of Proposition 3.3, we only need to prove that if N is an f-primary of M then it is a primary. Now, as M is compact, there exists a finite subset R' of R such that $s(R') = R$ (cf. Proposition 2.6). Also, as R is finitely closed, $\exists q \in Q$ such that $R' \subseteq s(q)$, whereby $s(R') \subseteq s(q)$. Thus $R \subseteq s(q)$. But as $s(q)$ is finitely closed and R is maximal finitely closed (as N is an f-primary), $s(q) = R$. Hence N is a primary of M (cf. Proposition 2.7).

Theorem 3.1 *Every fuzzy automaton can be expressed as the union of its f-primaries.*

Proof: From Remark 3.1, for each $q \in Q, s(q)$ is a finitely closed subset of Q, whereby each $q \in Q$ will be contained in some finitely closed subset of Q (as $q \in s(q), \forall q \in Q$). Let $q \in Q$ and let $\mathcal{F}_q = \{R \subseteq Q : R$ is finitely closed and $q \in R\}$. $\mathcal{F}_q$ is nonempty as, at least, it contains $s(q)$ (cf. Remark 3.1). Let $R_1 \subseteq R_2 \subseteq R_3 \subseteq ...$ be any chain in $\mathcal{F}_q$. Then $\cup_{j \in J} R_j$ is also a finitely closed subset of Q, because for any finite $T \subseteq \cup_{j \in J} R_j, T \subseteq R_j$, for some $j \in J$, whereby $\exists p \in Q$ such that $T \subseteq s(p)$. Thus $\cup_{j \in J} R_j$ is an upper bound in $\mathcal{F}_q$ of the chain $R_1 \subseteq R_2 \subseteq R_3 \subseteq ....$ Thus it has been shown that every chain in the partially ordered set $\mathcal{F}_q$ has an upper bound. Therefore by Zorn's lemma, $\mathcal{F}_q$ has a maximal element. Thus for each $q \in Q$, there exists a maximal finitely closed subset containing it, i.e., an f-primary of Q containing q. Also, $\cup_{p \in Q} s(p) = Q$. Hence every fuzzy automaton can be expressed as the union of its f-primaries.

Remark 3.4 *The decomposition obtained in the previous theorem need not be unique even for a (crisp) automaton, as the following counter-example will show.*

Counter-example 3.2 *Let $M = (Q, X, \delta)$ be the (crisp) automaton, where Q is the set of integers, $X = \{0, 1, 2, ...\}$, and $\delta : Q \times X \to Q$ is given by*

$$\delta(n, x) = \begin{cases} n + x & \text{if } n > 0 \\ -|n + x| & \text{if } n = 0 \\ n & \text{if } n < 0, x = 0 \\ |n + x| & \text{if } n < 0, x \neq 0 \end{cases}$$

$\forall n \in Q, \forall x \in X$.

We note that each $s(-i), i \in N$ (the set of natural numbers), and $s(0)$ are f-primaries of M. Here Q can be expressed in two ways, given by $Q = \cup_{i \in N} s(-i)$ and $Q = (\cup_{i \in N} s(-i)) \cup s(0)$.

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