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Uncertain Logic for Modeling Human Language

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Abstract

This paper discusses uncertain set theory and proposes the concepts of membership function, variance, entropy and distance of uncertain sets. In order to determine membership functions, uncertain statistics is also suggested. Based on uncertain set theory, uncertain logic is presented for dealing with human language by using uncertain quantifier, uncertain subject and uncertain predicate. As an application, uncertain logic provides a means for extracting linguistic summary from a collection of raw data. Finally, fuzzy logic will be discussed at the end of this paper.

Keywords: uncertain set, uncertain statistics, uncertain logic, data mining, linguistic summary

1 Introduction

In order to model human language like "about 100km" or "approximately 39 °C", uncertainty theory was founded by Liu [8] in 2007 and refined by Liu [14] in 2010. Nowadays uncertainty theory has become a branch of mathematics based on normality, self-duality, countable subadditivity, and product measure axioms. Peng and Iwamura [17] gave a sufficient and necessary condition of uncertainty distribution. Gao [5] provided some mathematical properties of uncertain measure. You [21] proved some convergence theorems of uncertain sequences. In addition, Liu and Ha [15] proved some expected value formulas for functions of uncertain variables.

As an application of uncertainty theory, Liu [11] proposed a spectrum of uncertain programming that is a type of mathematical programming involving uncertain variables. In addition, Liu [13] applied uncertainty theory to the area of risk analysis and reliability analysis. In order to deal with differentiation and integration of functions of uncertain processes, Liu [10] developed a theory of uncertain calculus. Based on uncertain calculus, Liu [9] proposed a tool of uncertain differential equations. After that, an existence and uniqueness theorem of solution of uncertain differential equation was proved by Chen and Liu [2]. An uncertain optimal control theory was proposed by Zhu [24] by using uncertain differential equations.

Uncertain set theory was proposed by Liu [12] in 2010 as a generalization of uncertainty theory to the domain of uncertain sets. Roughly speaking, uncertain set is a measurable function from an uncertainty space to a collection of sets of real numbers. Thus uncertain set is a concept different from random set [18], fuzzy set [22] and rough set [16]. This paper will propose the concepts of membership function, variance, entropy and distance of uncertain sets. In order to determine membership functions, uncertain statistics is also suggested.

Based on uncertain set theory, uncertain logic is designed for dealing with human language by using uncertain quantifier, uncertain subject and uncertain predicate. Uncertain logic provides a flexible means for extracting linguistic summary from a collection of raw data. Finally, fuzzy logic will be discussed.

2 Uncertain Set

Definition 1 *([12]) An uncertain set is a measurable function ξ from an uncertainty space $(\Gamma, \mathcal{L}, \mathcal{M})$ to a collection of sets of real numbers, i.e., for any Borel set B of real numbers, the set*

$$\{\xi \subset B\} = \{\gamma \in \Gamma \mid \xi(\gamma) \subset B\} \tag{1}$$

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is an event.

Definition 2 *Let ξ be an uncertain set. Then its membership function is defined as*

$$\mu(x) = \mathcal{M}\{x \in \xi\} \tag{2}$$

for any $x \in \Re$.

Theorem 1 *(Sufficient and Necessary Condition) A real-valued function μ is a membership function of uncertain set if and only if*

$$0 \le \mu(x) \le 1. \tag{3}$$

Proof: If μ is a membership function of uncertain set ξ, then $\mu(x) = \mathcal{M}\{x \in \xi\}$ and $0 \le \mu(x) \le 1$. Conversely, suppose μ is a function such that $0 \le \mu(x) \le 1$. We take an uncertainty space $(\Gamma, \mathcal{L}, \mathcal{M})$ to be $[0,1]$ with $\mathcal{M}\{[0,\gamma]\} = \gamma$ for each $\gamma \in [0,1]$. Then the set-valued function

$$\xi(\gamma) = \left\{x \in \Re \,\middle|\, \mu(x) \ge \gamma\right\} \tag{4}$$

on the uncertainty space $(\Gamma, \mathcal{L}, \mathcal{M})$ is just an uncertain set. It is easy to verify that the uncertain set ξ has a membership function μ.

Theorem 2 *Let ξ be an uncertain set with membership function μ. Then for any real number x, we have*

$$\begin{aligned} &\mathcal{M}\{x \in \xi\} = \mu(x), \quad \mathcal{M}\{x \notin \xi\} = 1 - \mu(x), \\ &\mathcal{M}\{x \notin \xi^c\} = \mu(x), \quad \mathcal{M}\{x \in \xi^c\} = 1 - \mu(x). \end{aligned} \tag{5}$$

Proof: Since μ is the membership function of ξ, we have $\mathcal{M}\{x \in \xi\} = \mu(x)$ immediately. In addition, it follows from the self-duality of uncertain measure that

$$\mathcal{M}\{x \notin \xi\} = 1 - \mathcal{M}\{x \in \xi\} = 1 - \mu(x).$$

Finally, it is easy to verify that $\{x \in \xi^c\} = \{x \notin \xi\}$. Hence $\mathcal{M}\{x \in \xi^c\} = 1 - \mu(x)$ and $\mathcal{M}\{x \notin \xi^c\} = \mu(x)$.

A membership function $\mu(x)$ is called unimodal if there is a point x_0 such that $\mu(x)$ is increasing on $(-\infty, x_0)$ and decreasing on $(x_0, +\infty)$. A membership function $\mu(x)$ is called normalized if there is a point x_0 such that $\mu(x_0) = 1$.

Theorem 3 *Let ξ and η be independent uncertain sets with membership functions μ and ν, respectively. Then their union $\xi \cup \eta$ has a membership function*

$$\lambda(x) = \mu(x) \vee \nu(x). \tag{6}$$

Proof: It follows from the definition of membership function and independence of ξ and η that

$$\begin{aligned} \lambda(x) &= \mathcal{M}\{x \in \xi \cup \eta\} = \mathcal{M}\{(x \in \xi) \cup (x \in \eta)\} \\ &= \mathcal{M}\{x \in \xi\} \vee \mathcal{M}\{x \in \eta\} = \mu(x) \vee \nu(x). \end{aligned}$$

Theorem 4 *Let ξ and η be independent uncertain sets with membership functions μ and ν, respectively. Then their intersection $\xi \cap \eta$ has a membership function*

$$\lambda(x) = \mu(x) \wedge \nu(x). \tag{7}$$

Proof: It follows from the definition of membership function and independence of ξ and η that

$$\begin{aligned} \lambda(x) &= \mathcal{M}\{x \in \xi \cap \eta\} = \mathcal{M}\{(x \in \xi) \cap (x \in \eta)\} \\ &= \mathcal{M}\{x \in \xi\} \wedge \mathcal{M}\{x \in \eta\} = \mu(x) \wedge \nu(x). \end{aligned}$$

Theorem 5 *Let ξ be an uncertain set with membership function μ. Then its complement ξ^c has a membership function*

$$\lambda(x) = 1 - \mu(x). \tag{8}$$

Proof: It follows from the definition of membership function and the self-duality of uncertain measure that

$$\lambda(x) = \mathcal{M}\{x \in \xi^c\} = \mathcal{M}\{x \notin \xi\} = 1 - \mathcal{M}\{x \in \xi\} = 1 - \mu(x).$$

Theorem 6 *Let ξ be an uncertain set with membership function μ, and let f be a strictly monotone function. Then $f(\xi)$ has a membership function*

$$\lambda(x) = \mu(f^{-1}(x)). \tag{9}$$

Proof: Since f is a strictly monotone function, its inverse function f^{-1} exists and is unique. It follows from the definition of membership function that $f(\xi)$ has a membership function

$$\lambda(x) = \mathcal{M}\{x \in f(\xi)\} = \mathcal{M}\{f^{-1}(x) \in \xi\} = \mu(f^{-1}(x)).$$

The theorem is verified.

Definition 3 *Let ξ be an uncertain set with finite expected value e. Then the variance of ξ is defined by $V[\xi] = E[(\xi - e)^2]$.*

If an uncertain set is given by a membership function, how do we calculate its variance? In order to answer this question, we have to determine the value of $\mathcal{M}\{(\xi - e)^2 \succeq x\}$ via the information of membership function μ. Unfortunately, it is almost impossible! However, it is reasonable to accept the following measure inversion formula,

$$\mathcal{M}\{(\xi - e)^2 \succeq x\} = \frac{1}{2}\left(\sup_{(y-e)^2 \ge x} \mu(y) + 1 - \sup_{(y-e)^2 < x} \mu(y)\right). \tag{10}$$

Then the variance of ξ is

$$V[\xi] = \frac{1}{2}\int_0^{+\infty} \left(\sup_{(y-e)^2 \ge x} \mu(y) + 1 - \sup_{(y-e)^2 < x} \mu(y)\right) \mathrm{d}x. \tag{11}$$

Definition 4 *Suppose that ξ is an uncertain set with membership function μ. Then its entropy is defined by*

$$H[\xi] = \int_{-\infty}^{+\infty} S(\mu(x))\mathrm{d}x \tag{12}$$

where $S(t) = -t\ln t - (1-t)\ln(1-t)$.

If ξ is a discrete uncertain set taking values in $\{x_1, x_2, \cdots\}$, then the entropy becomes

$$H[\xi] = \sum_{i=1}^{\infty} S(\mu(x_i)). \tag{13}$$

Definition 5 *The distance between uncertain sets ξ and η is defined as $d(\xi, \eta) = E[|\xi - \eta|]$.*

Let an uncertain set ξ be given by a membership function μ, and let b be a real number. If we accept the following measure inversion formula,

$$\mathcal{M}\{|\xi - b| \succeq x\} = \frac{1}{2}\left(\sup_{|y-b| \ge x} \mu(y) + 1 - \sup_{|y-b| < x} \mu(y)\right), \tag{14}$$

then the distance between ξ and b is

$$d(\xi, b) = \frac{1}{2}\int_0^{+\infty} \left(\sup_{|y-b| \ge x} \mu(y) + 1 - \sup_{|y-b| < x} \mu(y)\right) \mathrm{d}x. \tag{15}$$

One problem is how to determine the membership function of an uncertain set via uncertain statistics. The first step is to ask the domain expert to choose a possible point x that the uncertain set ξ may contain, and then quiz him

$$\text{“How likely does } x \text{ belong to } \xi\text{?”} \tag{16}$$

Assume the expert's belief degree is α in uncertain measure. Note that the expert's belief degree of x not belonging to ξ must be $1-\alpha$ due to the self-duality of uncertain measure. An expert's experimental data (x, α) is thus acquired from the domain expert. Repeating the above process, the following expert's experimental data are obtained by the questionnaire,

$$(x_1, \alpha_1),\ (x_2, \alpha_2),\ \cdots,\ (x_n, \alpha_n). \tag{17}$$

How do we determine the membership function for an uncertain set? Assume that we have obtained a set of expert's experimental data

$$(x_1, \alpha_1), (x_2, \alpha_2), \cdots, (x_n, \alpha_n) \tag{18}$$

that meet the following consistence condition (perhaps after a rearrangement)

$$x_1 < x_2 < \cdots < x_n. \tag{19}$$

Based on those expert's experimental data, an empirical membership function is determined as follows,

$$\mu(x) = \begin{cases} \alpha_i + \dfrac{(\alpha_{i+1} - \alpha_i)(x - x_i)}{x_{i+1} - x_i}, & \text{if } x_i \le x \le x_{i+1},\ 1 \le i < n \\ 0, & \text{otherwise.} \end{cases}$$

Assume that a membership function to be determined has a known functional form $\mu(x|\theta)$ with an unknown parameter θ. In order to estimate the parameter θ, we may employ the principle of least squares that minimizes the sum of the squares of the distance of the expert's experimental data to the membership function. If the expert's experimental data

$$(x_1, \alpha_1),\ (x_2, \alpha_2),\ \cdots,\ (x_n, \alpha_n) \tag{20}$$

are obtained, then we have

$$\min_\theta \sum_{i=1}^n (\mu(x_i|\theta) - \alpha_i)^2. \tag{21}$$

The optimal solution $\widehat{\theta}$ of (21) is called the least squares estimate of θ, and then the least squares membership function is $\mu(x|\widehat{\theta})$.

3 Individual Feature Data

In order to design a system of uncertain logic, we should have a universe $\mathbb{A}$ of individuals we are talking about. Without loss of generality, we may assume that $\mathbb{A}$ consists of n individuals and is represented by

$$\mathbb{A} = \{a_1, a_2, \cdots, a_n\}. \tag{22}$$

When we talk about the universe $\mathbb{A}$, we should have feature data of all individuals. When we talk about "the days are warm", we should know the individual feature data of all days, for example,

$$\mathbb{A} = \{22,\ 23,\ 25,\ 28,\ 30,\ 32,\ 36\} \tag{23}$$

whose elements are temperatures in centigrades. When we talk about "the students are young", we should know the individual feature data of all students, for example,

$$\mathbb{A} = \{20,\ 20,\ 21,\ 22,\ 24,\ 24,\ 25,\ 25,\ 26,\ 27,\ 28,\ 28,\ 30,\ 33,\ 38\} \tag{24}$$

whose elements are ages in years. When we talk about "the sportsmen are tall", we should know the individual feature data of all sportsmen, for example,

$$\mathbb{A} = \left\{ \begin{array}{l} 165,\ 168,\ 168,\ 170,\ 178,\ 183,\ 185,\ 186 \\ 188,\ 190,\ 192,\ 192,\ 193,\ 194,\ 195,\ 198 \end{array} \right\} \tag{25}$$

whose elements are heights in centimeters. Sometimes the individual feature data are represented by vectors rather a scalar number. When we talk about "the young teachers are tall", we should know the individual feature data of all teachers, for example,

$$\mathbb{A} = \left\{ \begin{array}{l} (21,185),\ (22,190),\ (22,184),\ (23,170),\ (24,187),\ (24,188) \\ (25,160),\ (25,190),\ (26,185),\ (26,176),\ (27,185),\ (27,188) \\ (30,164),\ (34,178),\ (40,182),\ (45,186),\ (52,165),\ (60,170) \end{array} \right\} \tag{26}$$

whose elements are ages and heights in years and centimeters, respectively.

4 Uncertain Quantifier

If we want to represent all individuals in the universe $\mathbb{A}$, we use the universal quantifier ($\forall$) and

$$\forall = \text{"for all"}. \tag{27}$$

If we want to represent some (at least one) individuals, we use the existential quantifier ($\exists$) and

$$\exists = \text{"there exists at least one"}. \tag{28}$$

In addition to the two quantifiers, there are numerous imprecise quantifiers in human language, for example, *almost all*, *almost none*, *about 10*, *many*, *several*, *some*, *most*, *a few*, *about 70%*. This section will model them by the concept of uncertain quantifier.

Definition 6 *Uncertain quantifier is an uncertain set representing the number of individuals.*

Example 1: The universal quantifier ($\forall$) on the universe $\mathbb{A}$ is a special uncertain quantifier $\forall \equiv \{n\}$ whose membership function is

$$\lambda(x) = \begin{cases} 1, & \text{if } x = n \\ 0, & \text{otherwise.} \end{cases} \tag{29}$$

Example 2: The existential quantifier ($\exists$) on the universe $\mathbb{A}$ is a special uncertain quantifier $\exists \equiv \{1, 2, \cdots, n\}$ whose membership function is

$$\lambda(x) = \begin{cases} 0, & \text{if } x = 0 \\ 1, & \text{otherwise.} \end{cases} \tag{30}$$

Example 3: The uncertain quantifier $\mathcal{Q}$ of "*almost all*" on the universe $\mathbb{A}$ may have a membership function

$$\lambda(x) = \begin{cases} 0, & \text{if } 0 \le x \le n-5 \\ (x-n+5)/3, & \text{if } n-5 \le x \le n-2 \\ 1, & \text{if } n-2 \le x \le n. \end{cases} \tag{31}$$

Example 4: The uncertain quantifier $\mathcal{Q}$ of "*almost none*" on the universe $\mathbb{A}$ may have a membership function

$$\lambda(x) = \begin{cases} 1, & \text{if } 0 \le x \le 2 \\ (5-x)/3, & \text{if } 2 \le x \le 5 \\ 0, & \text{if } 5 \le x \le n. \end{cases} \tag{32}$$

Example 5: The uncertain quantifier $\mathcal{Q}$ of "*about 10*" on the universe $\mathbb{A}$ may have a membership function

$$\lambda(x) = \begin{cases} 0, & \text{if } 0 \le x \le 7 \\ (x-7)/2, & \text{if } 7 \le x \le 9 \\ 1, & \text{if } 9 \le x \le 11 \\ (13-x)/2, & \text{if } 11 \le x \le 13 \\ 0, & \text{if } 13 \le x \le n. \end{cases} \tag{33}$$

Example 6: In many cases, it is more convenient for us to use a percentage than an absolute quantity. For example, we may use the uncertain quantifier $\mathcal{Q}$ of "*about 70%*". For this case, a possible membership function of $\mathcal{Q}$ is

$$\lambda(x)=\begin{cases} 0, & \text{if } 0\le x\le 0.6 \\ 20(x-0.6), & \text{if } 0.6\le x\le 0.65 \\ 1, & \text{if } 0.65\le x\le 0.75 \\ 20(0.8-x), & \text{if } 0.75\le x\le 0.8 \\ 0, & \text{if } 0.8\le x\le 1. \end{cases} \tag{34}$$

Definition 7 *An uncertain quantifier is said to be unimodal if its membership function is unimodal.*

The uncertain quantifiers "almost all", "almost none", "about 10" and "about 70%" are unimodal.

Definition 8 *An uncertain quantifier is said to be monotone if its membership function is monotone. Especially, an uncertain quantifier is said to be increasing if its membership function is increasing; and an uncertain quantifier is said to be decreasing if its membership function is decreasing.*

The uncertain quantifiers "almost all" and "almost none" are monotone, but "about 10" and "about 70%" are not monotone. Note that both increasing uncertain quantifiers and decreasing uncertain quantifiers are monotone. In addition, any monotone uncertain quantifiers are unimodal.

Definition 9 *Uncertain quantifiers are said to be independent if they are independent uncertain sets.*

Theorem 7 *Let $\mathcal{Q}_1$ and $\mathcal{Q}_2$ be independent uncertain quantifiers with membership functions λ_1 and λ_2, respectively. Then their union $\mathcal{Q}_1\cup\mathcal{Q}_2$ has a membership function*

$$\nu(x)=\lambda_1(x)\vee\lambda_2(x), \tag{35}$$

and their intersection $\mathcal{Q}_1\cap\mathcal{Q}_2$ has a membership function

$$\nu(x)=\lambda_1(x)\wedge\lambda_2(x). \tag{36}$$

Proof: It follows from the operational law of uncertain set immediately.

Negated Quantifier

What is the negation of an uncertain quantifier? The following definition gives a formal answer.

Definition 10 *Let $\mathcal{Q}$ be an uncertain quantifier. Then the negated quantifier $\neg\mathcal{Q}$ is the complement of $\mathcal{Q}$ in the sense of uncertain set, i.e.,*

$$\neg\mathcal{Q}=\mathcal{Q}^c. \tag{37}$$

Example 7: Let $\forall=\{n\}$ be the universal quantifier. Then its negated quantifier

$$\neg\forall\equiv\{0,1,2,\cdots,n-1\}. \tag{38}$$

Example 8: Let $\exists=\{1,2,\cdots,n\}$ be the existential quantifier. Then its negated quantifier is

$$\neg\exists\equiv\{0\}. \tag{39}$$

Example 9: The negated quantifier of "*there exist exactly m*" (i.e., $\mathcal{Q}\equiv\{m\}$) is

$$\neg\mathcal{Q}\equiv\{0,1,\cdots,m-1,m+1,\cdots,n\}. \tag{40}$$

Theorem 8 *Let $\mathcal{Q}$ be an uncertain quantifier whose membership function is λ. Then the negated quantifier $\neg\mathcal{Q}$ has a membership function*

$$\neg\lambda(x)=1-\lambda(x). \tag{41}$$

Proof: This theorem follows from the operational law of uncertain set immediately.

Example 10: Let $\mathcal{Q}$ be the uncertain quantifier "*almost all*" defined by (31). Then its negated quantifier $\neg\mathcal{Q}$ has a membership function

$$\neg\lambda(x) = \begin{cases} 1, & \text{if } 0 \le x \le n-5 \\ (n-x-2)/3, & \text{if } n-5 \le x \le n-2 \\ 0, & \text{if } n-2 \le x \le n. \end{cases} \tag{42}$$

Example 11: Let $\mathcal{Q}$ be the uncertain quantifier "*about 70%*" defined by (34). Then its negated quantifier $\neg\mathcal{Q}$ has a membership function

$$\neg\lambda(x) = \begin{cases} 1, & \text{if } 0 \le x \le 0.6 \\ 20(0.65-x), & \text{if } 0.6 \le x \le 0.65 \\ 0, & \text{if } 0.65 \le x \le 0.75 \\ 20(x-0.75), & \text{if } 0.75 \le x \le 0.8 \\ 1, & \text{if } 0.8 \le x \le 1. \end{cases} \tag{43}$$

Theorem 9 *Let $\mathcal{Q}$ be an uncertain quantifier. Then we have $\neg\neg\mathcal{Q} = \mathcal{Q}$.*

Proof: This theorem follows from $\neg\neg\mathcal{Q} = \neg\mathcal{Q}^c = (\mathcal{Q}^c)^c = \mathcal{Q}$.

Theorem 10 *If $\mathcal{Q}$ is a monotone uncertain quantifier, then $\neg\mathcal{Q}$ is also monotone. Especially, if $\mathcal{Q}$ is increasing, then $\neg\mathcal{Q}$ is decreasing; if $\mathcal{Q}$ is decreasing, then $\neg\mathcal{Q}$ is increasing.*

Proof: Assume λ is the membership function of $\mathcal{Q}$. Then $\neg\mathcal{Q}$ has a membership function $\nu(x) = 1 - \lambda(x)$. The theorem follows from this fact immediately.

Dual Quantifier

Definition 11 *Let $\mathcal{Q}$ be an uncertain quantifier. Then the dual quantifier of $\mathcal{Q}$ is*

$$\mathcal{Q}^* = \forall - \mathcal{Q}. \tag{44}$$

Remark 1: Note that $\mathcal{Q}$ and $\mathcal{Q}^*$ are dependent uncertain sets such that $\mathcal{Q} + \mathcal{Q}^* \equiv \forall$. Since the cardinality of the universe $\mathbb{A}$ is n, we also have

$$\mathcal{Q}^* = n - \mathcal{Q}. \tag{45}$$

Example 12: Since $\forall \equiv \{n\}$, we immediately have $\forall^* = \{0\} = \neg\exists$. That is

$$\forall^* \equiv \neg\exists. \tag{46}$$

Example 13: Since $\neg\forall = \{0, 1, 2, \cdots, n-1\}$, we immediately have $(\neg\forall)^* = \{1, 2, \cdots, n\} = \exists$. That is,

$$(\neg\forall)^* \equiv \exists. \tag{47}$$

Example 14: Since $\exists \equiv \{1, 2, \cdots, n\}$, we have $\exists^* = \{0, 1, 2, \cdots, n-1\} = \neg\forall$. That is,

$$\exists^* \equiv \neg\forall. \tag{48}$$

Example 15: Since $\neg\exists = \{0\}$, we immediately have $(\neg\exists)^* = \{n\} = \forall$. That is,

$$(\neg\exists)^* = \forall. \tag{49}$$

Example 16: The dual quantifier of "*there exist exactly m*" (i.e., $\mathcal{Q} \equiv \{m\}$) is

$$\mathcal{Q}^* \equiv \{n - m\}. \tag{50}$$

Theorem 11 *Let $\mathcal{Q}$ be an uncertain quantifier whose membership function is λ. Then the dual quantifier $\mathcal{Q}^*$ has a membership function*

$$\lambda^*(x) = \lambda(n - x) \tag{51}$$

where n is the cardinality of the universe $\mathbb{A}$.

Proof: This theorem follows from the operational law of uncertain set immediately.

Example 17: Let $\mathcal{Q}$ be the uncertain quantifier "*almost all*" defined by (31). Then its dual quantifier $\mathcal{Q}^*$ has a membership function

$$\lambda^*(x) = \begin{cases} 1, & \text{if } 0 \le x \le 2 \\ (5-x)/3, & \text{if } 2 \le x \le 5 \\ 0, & \text{if } 5 \le x \le n. \end{cases} \tag{52}$$

Example 18: Let $\mathcal{Q}$ be the uncertain quantifier "*about 70%*" defined by (34). Then its dual quantifier $\mathcal{Q}^*$ has a membership function

$$\lambda^*(x) = \begin{cases} 0, & \text{if } 0 \le x \le 0.2 \\ 20(x-0.2), & \text{if } 0.2 \le x \le 0.25 \\ 1, & \text{if } 0.25 \le x \le 0.35 \\ 20(0.4-x), & \text{if } 0.35 \le x \le 0.4 \\ 0, & \text{if } 0.4 \le x \le 1. \end{cases} \tag{53}$$

Theorem 12 *Let $\mathcal{Q}$ be an uncertain quantifier. Then we have $\mathcal{Q}^{**} = \mathcal{Q}$.*

Proof: The theorem follows from $\mathcal{Q}^{**} = \forall - \mathcal{Q}^* = \forall - (\forall - \mathcal{Q}) = \mathcal{Q}$.

Theorem 13 *If $\mathcal{Q}$ is a unimodal uncertain quantifier, then $\mathcal{Q}^*$ is also unimodal. Especially, if $\mathcal{Q}$ is a monotone, then $\mathcal{Q}^*$ is monotone; if $\mathcal{Q}$ is increasing, then $\mathcal{Q}^*$ is decreasing; if $\mathcal{Q}$ is decreasing, then $\mathcal{Q}^*$ is increasing.*

Proof: Assume λ is the membership function of $\mathcal{Q}$. Then $\mathcal{Q}^*$ has a membership function $\nu(x) = \lambda(n - x)$. The theorem follows from this fact immediately.

5 Uncertain Subject

Definition 12 *Uncertain subject is an uncertain set containing some specified individuals in the universe.*

Example 19: "Young teachers" in the sentence "most young teachers are tall" is an uncertain subject that is an uncertain set on the universe of "all teachers", whose membership function may be defined by

$$\nu(x) = \begin{cases} 1, & \text{if } 0 \le x \le 25 \\ (45-x)/20, & \text{if } 25 \le x \le 45 \\ 0, & \text{if } x \ge 45. \end{cases} \tag{54}$$

Example 20: "Tall students" in the sentence "about 80% of tall students are heavy" is an uncertain subject that is an uncertain set on the universe of "all students", whose membership function may be defined by

$$\nu(x) = \begin{cases} 0, & \text{if } 0 \le x \le 175 \\ (x-175)/10, & \text{if } 175 \le x \le 185 \\ 1, & \text{if } x \ge 185. \end{cases} \tag{55}$$

Let S be an uncertain subject with membership function ν on the universe $\mathbb{A} = \{a_1, a_2, \cdots, a_n\}$ of individuals. Then S is an uncertain set of $\mathbb{A}$ such that

$$\mathcal{M}\{a_i \in S\} = \nu(a_i), \quad i = 1, 2, \cdots, n. \tag{56}$$

In many cases, we are interested in some individuals a's with $\nu(a) \geq \omega$, where ω is a confidence level. Thus we have a subset of individuals,

$$S_\omega = \{a \in \mathbb{A} \,|\, \nu(a) \geq \omega\}. \tag{57}$$

The set S_ω will play a new universe we are talking about, and the individuals out of S_ω will be ignored at the confidence level ω.

6 Uncertain Predicate

There are numerous imprecise predicates in human language, for example, *warm*, *cold*, *hot*, *young*, *old*, *tall*, *small*, and *big*. This section will model them by the concept of uncertain predicate.

Definition 13 *Uncertain predicate is an uncertain set representing a property that the individuals have in common.*

Example 21: "Today is warm" is a statement in which "today" is a subject and "warm" is an uncertain predicate that may be represented by a membership function

$$\mu(x) = \begin{cases} 0, & \text{if } x \leq 15 \\ (x-15)/3, & \text{if } 15 \leq x \leq 18 \\ 1, & \text{if } 18 \leq x \leq 24 \\ (28-x)/4, & \text{if } 24 \leq x \leq 28 \\ 0, & \text{if } 28 \leq x. \end{cases} \tag{58}$$

Example 22: "John is young" is a statement in which "John" is a subject and "young" is an uncertain predicate that may be represented by a membership function

$$\mu(x) = \begin{cases} 1, & \text{if } 0 \leq x \leq 25 \\ (45-x)/20, & \text{if } 25 \leq x \leq 45 \\ 0, & \text{if } x \geq 45. \end{cases} \tag{59}$$

Example 23: "Tom is tall" is a statement in which "Tom" is a subject and "tall" is an uncertain predicate that may be represented by a membership function

$$\mu(x) = \begin{cases} 0, & \text{if } 0 \leq x \leq 175 \\ (x-175)/10, & \text{if } 175 \leq x \leq 185 \\ 1, & \text{if } x \geq 185. \end{cases} \tag{60}$$

Definition 14 *Uncertain predicates are said to be independent if they are independent uncertain sets.*

Theorem 14 *Let ξ_1 and ξ_2 be independent uncertain predicates with membership functions μ_1 and μ_2, respectively. Then their union $\xi_1 \cup \xi_2$ has a membership function*

$$\nu(x) = \mu_1(x) \vee \mu_2(x), \tag{61}$$

and their intersection $\xi_1 \cap \xi_2$ has a membership function

$$\nu(x) = \mu_1(x) \wedge \mu_2(x). \tag{62}$$

Proof: The theorem follows from the operational law of uncertain set immediately.

Negated Predicate

Definition 15 *Let ξ be an uncertain predicate. Then its negated predicate $\neg\xi$ is the complement of ξ in the sense of uncertain set, i.e.,*

$$\neg\xi = \xi^c. \tag{63}$$

Theorem 15 *Let ξ be an uncertain predicate with membership function μ. Then its negated predicate $\neg\xi$ has a membership function*

$$\neg\mu(x) = 1 - \mu(x). \tag{64}$$

Proof: The theorem follows from the definition of negated predicate and the operational law of uncertain set immediately.

Example 24: Let ξ be the uncertain predicate "warm" defined by (58). Then its negated predicate $\neg\xi$ has a membership function

$$\neg\mu(x) = \begin{cases} 1, & \text{if } x \le 15 \\ (18-x)/3, & \text{if } 15 \le x \le 18 \\ 0, & \text{if } 18 \le x \le 24 \\ (x-24)/4, & \text{if } 24 \le x \le 28 \\ 1, & \text{if } 28 \le x. \end{cases} \tag{65}$$

Example 25: Let ξ be the negated predicate "*young*" defined by (59). Then its negated predicate $\neg\xi$ has a membership function

$$\neg\mu(x) = \begin{cases} 0, & \text{if } 0 \le x \le 25 \\ (x-25)/20, & \text{if } 25 \le x \le 45 \\ 1, & \text{if } x \ge 45. \end{cases} \tag{66}$$

Example 26: Let ξ be the uncertain predicate "*tall*" defined by (60). Then its negated predicate $\neg\xi$ has a membership function

$$\neg\mu(x) = \begin{cases} 1, & \text{if } 0 \le x \le 175 \\ (185-x)/10, & \text{if } 175 \le x \le 185 \\ 0, & \text{if } x \ge 185. \end{cases} \tag{67}$$

7 Uncertain Proposition

Definition 16 *Assume that $\mathcal{Q}$ is an uncertain quantifier, S is an uncertain subject, and ξ is an uncertain predicate. Then the triplet $(\mathcal{Q}, S, \xi) =$ "$\mathcal{Q}$ of S are ξ" is an uncertain proposition.*

Remark 2: Let $\mathbb{A}$ be the universe of individuals. Then $(\mathcal{Q}, \mathbb{A}, \xi)$ is a special uncertain proposition because $\mathbb{A}$ itself is an uncertain subject.

Remark 3: Let $\forall$ be the universal quantifier. Then $(\forall, \mathbb{A}, \xi)$ is an uncertain proposition representing "all of $\mathbb{A}$ are ξ".

Remark 4: Let $\exists$ be the existential quantifier. Then $(\exists, \mathbb{A}, \xi)$ is an uncertain proposition representing "at least one of $\mathbb{A}$ is ξ".

Example 27: "Almost all students are young" is an uncertain proposition in which the uncertain quantifier $\mathcal{Q}$ is "almost all", the uncertain subject S is "students" (the universe itself) and the uncertain predicate ξ is "young".

Example 28: "Most young teachers are tall" is an uncertain proposition in which the uncertain quantifier $\mathcal{Q}$ is "most", the uncertain subject S is "young teachers" and the uncertain predicate ξ is "tall".

Example 29: "About 80% of tall workers are heavy" is an uncertain proposition in which the uncertain quantifier $\mathcal{Q}$ is "80%", the uncertain subject S is "tall workers" and the uncertain predicate ξ is "heavy".

Theorem 16 *Let $(\mathcal{Q}, S, \xi)$ be an uncertain proposition. Then*

$$(\mathcal{Q}^*, S, \xi) = (\mathcal{Q}, S, \neg\xi). \tag{68}$$

Proof: Note that $(\mathcal{Q}^*, S, \xi)$ represents "$\mathcal{Q}^*$ of S are ξ". In fact, the statement "$\mathcal{Q}^*$ of S are ξ" implies "$\mathcal{Q}^{**}$ of S are not ξ". Since $\mathcal{Q}^{**} = \mathcal{Q}$, we obtain $(\mathcal{Q}, S, \neg\xi)$. Conversely, the statement "$\mathcal{Q}$ of S are not ξ" implies "$\mathcal{Q}^*$ of S are ξ", i.e., $(\mathcal{Q}^*, S, \xi)$. Thus (68) is verified.

Example 30: When $\mathcal{Q}^* = \neg\forall$, we have $\mathcal{Q} = \exists$. If $S = \mathbb{A}$, then (68) becomes the classical equivalence $(\neg\forall, \mathbb{A}, \xi) = (\exists, \mathbb{A}, \neg\xi)$.

Example 31: When $\mathcal{Q}^* = \neg\exists$, we have $\mathcal{Q} = \forall$. If $S = \mathbb{A}$, then (68) becomes the classical equivalence $(\neg\exists, \mathbb{A}, \xi) = (\forall, \mathbb{A}, \neg\xi)$.

8 Truth Value

Let $(\mathcal{Q}, S, \xi)$ be an uncertain proposition. The truth value of $(\mathcal{Q}, S, \xi)$ should be the uncertain measure that "$\mathcal{Q}$ of S are ξ". That is,

$$T(\mathcal{Q}, S, \xi) = \mathcal{M}\{\mathcal{Q} \text{ of } S \text{ are } \xi\}. \tag{69}$$

However, it is impossible for us to deduce the value of $\mathcal{M}\{\mathcal{Q} \text{ of } S \text{ are } \xi\}$ from the information of $\mathcal{Q}$, S and ξ within the framework of uncertain set theory. Thus we need an additional formula to compose $\mathcal{Q}$, S and ξ.

Definition 17 *Let $(\mathcal{Q}, S, \xi)$ be an uncertain proposition in which $\mathcal{Q}$ is a unimodal uncertain quantifier with membership function λ, S is an uncertain subject with membership function ν, and ξ is an uncertain predicate with membership function μ. Then the truth value of $(\mathcal{Q}, S, \xi)$ with respect to the universe $\mathbb{A}$ is*

$$T(\mathcal{Q}, S, \xi) = \sup_{0\le\omega\le 1}\left(\omega \wedge \sup_{K\in\mathbb{K}_\omega}\inf_{a\in K}\mu(a) \wedge \sup_{K\in\mathbb{K}^*_\omega}\inf_{a\in K}\neg\mu(a)\right) \tag{70}$$

where

$$\mathbb{K}_\omega = \left\{K \subset S_\omega \,|\, \lambda(|K|) \ge \omega\right\}, \tag{71}$$

$$\mathbb{K}^*_\omega = \left\{K \subset S_\omega \,|\, \lambda(|S_\omega| - |K|) \ge \omega\right\}, \tag{72}$$

$$S_\omega = \left\{a \in \mathbb{A} \,|\, \nu(a) \ge \omega\right\}. \tag{73}$$

Remark 5: Keep in mind that the truth value formula (70) is vacuous if the individual feature data $\mathbb{A}$ are not available.

Remark 6: Note that $\neg\mu$ is the membership function of the negated predicate of ξ, and $\neg\mu(a) = 1 - \mu(a)$.

Remark 7: When the subset K becomes an empty set $\emptyset$, we will define $\inf_{a\in\emptyset}\mu(a) = \inf_{a\in\emptyset}\neg\mu(a) = 1$.

Remark 8: If $\mathcal{Q}$ is an uncertain percentage rather than an absolute quantity, then $\mathbb{K}_\omega$ and $\mathbb{K}^*_\omega$ are replaced with

$$\mathbb{K}_\omega = \left\{K \subset S_\omega \,|\, \lambda(|K|/|S_\omega|) \ge \omega\right\}, \tag{74}$$

$$\mathbb{K}^*_\omega = \left\{K \subset S_\omega \,|\, \lambda(1 - |K|/|S_\omega|) \ge \omega\right\}. \tag{75}$$

Theorem 17 *Let $(\mathcal{Q}, S, \xi)$ be an uncertain proposition in which $\mathcal{Q}$ is a unimodal uncertain quantifier with membership function λ, S is an uncertain subject with membership function ν, and ξ is an uncertain predicate with membership function μ. Then the truth value of $(\mathcal{Q}, S, \xi)$ is*

$$T(\mathcal{Q}, S, \xi) = \sup_{0\le\omega\le 1}\left(\omega \wedge \Delta(k_\omega) \wedge \Delta^*(k^*_\omega)\right) \tag{76}$$

where

$$k_\omega = \min\left\{x \,|\, \lambda(x) \ge \omega\right\}, \tag{77}$$

$$\Delta(k_\omega) = \textit{the } k_\omega\textit{-th largest value of } \{\mu(a_i) \mid a_i \in S_\omega\}, \tag{78}$$

$$k^*_\omega = |S_\omega| - \max\{x \mid \lambda(x) \geq \omega\}, \tag{79}$$

$$\Delta^*(k^*_\omega) = \textit{the } k^*_\omega\textit{-th largest value of } \{1 - \mu(a_i) \mid a_i \in S_\omega\}. \tag{80}$$

Proof: Since the supremum is achieved at the subset with minimum cardinality, we have

$$\sup_{K \in \mathbb{K}_\omega} \inf_{a \in K} \mu(a) = \sup_{K \subset S_\omega, |K| = k_\omega} \inf_{a \in K} \mu(a) = \Delta(k_\omega),$$

$$\sup_{K \in \mathbb{K}^*_\omega} \inf_{a \in K} \neg\mu(a) = \sup_{K \subset S_\omega, |K| = k^*_\omega} \inf_{a \in K} \neg\mu(a) = \Delta^*(k^*_\omega).$$

The theorem is thus verified. Please note that $\Delta(0) = \Delta^*(0) = 1$.

Remark 9: If $\mathbb{Q}$ is an uncertain percentage, then k_ω and k^*_ω are determined by

$$k_\omega = \min\left\{x \mid \lambda(x/|S_\omega|) \geq \omega\right\}, \tag{81}$$

$$k^*_\omega = |S_\omega| - \max\{x \mid \lambda(x/|S_\omega|) \geq \omega\}. \tag{82}$$

Example 32: If the uncertain quantifier $\mathbb{Q} = \forall$, then the dual quantifier is $\mathbb{Q}^* = \{0\}$ and

$$T(\forall, \mathbb{A}, \xi) = \inf_{a \in \mathbb{A}} \mu(a). \tag{83}$$

Example 33: If the uncertain quantifier $\mathbb{Q} = \exists$, then the dual quantifier is $\mathbb{Q}^* = \{0, 1, 2, \cdots, n-1\}$ and

$$T(\exists, \mathbb{A}, \xi) = \sup_{a \in \mathbb{A}} \mu(a). \tag{84}$$

Example 34: If the uncertain quantifier $\mathbb{Q} = \neg\forall$, then the dual quantifier is $\mathbb{Q}^* = \exists$ and

$$T(\neg\forall, \mathbb{A}, \xi) = 1 - \inf_{a \in \mathbb{A}} \mu(a). \tag{85}$$

Example 35: If the uncertain quantifier $\mathbb{Q} = \neg\exists$, then the dual quantifier is $\mathbb{Q}^* = \forall$ and

$$T(\neg\exists, \mathbb{A}, \xi) = 1 - \sup_{a \in \mathbb{A}} \mu(a). \tag{86}$$

Truth Value Algorithm

In order to calculate $T(\mathbb{Q}, S, \xi)$ based on the truth value formula (70), a truth value algorithm is designed as follows:

Step 1: Calculate $S_1 = \{a \in \mathbb{A} \mid \nu(a) = 1\}$ and $k = \min\{x \mid \lambda(x) = 1\}$ and $k^* = |S_1| - \max\{x \mid \lambda(x) = 1\}$. If $\Delta(k) \wedge \Delta^*(k^*) = 1$, then $T = 1$ and stop.

Step 2: Calculate $S_0 = \{a \in \mathbb{A} \mid \nu(a) > 0\}$ and $k = \min\{x \mid \lambda(x) > 0\}$ and $k^* = |S_0| - \max\{x \mid \lambda(x) > 0\}$. If $\Delta(k) \wedge \Delta^*(k^*) = 0$, then $T = 0$ and stop.

Step 3: Set $b = 0$ and $t = 1$.

Step 4: Set $c = (b + t)/2$.

Step 5: Calculate $S_c = \{a \in \mathbb{A} \mid \nu(a) \geq c\}$ and $k = \min\{x \mid \lambda(x) \geq c\}$ and $k^* = |S_c| - \max\{x \mid \lambda(x) \geq c\}$. If $\Delta(k) \wedge \Delta^*(k^*) > c$, then $b = c$; otherwise $t = c$.

Step 6: If $|b - t| > \varepsilon$ (a predetermined precision), then go to Step 4; otherwise $T = (b + t)/2$ and stop.

Remark 10: If $\mathcal{Q}$ is an uncertain percentage, then k_ω and k^*_ω in the truth value algorithm are replaced with (81) and (82), respectively.

Example 36: Assume that the daily temperatures of some week from Monday to Sunday are

$$22, 23, 25, 28, 30, 32, 36 \tag{87}$$

in centigrades, respectively. Consider an uncertain proposition

$$(\mathcal{Q}, \mathbb{A}, \xi) = \text{"two or three days are warm"}. \tag{88}$$

The uncertain quantifier is $\mathcal{Q} = \{2, 3\}$ and its dual quantifier is $\mathcal{Q}^* = \{5, 6\}$. Suppose the uncertain predicate ξ = "warm" has a membership function

$$\mu(x) = \begin{cases} 0, & \text{if } x \le 15 \\ (x-15)/3, & \text{if } 15 \le x \le 18 \\ 1, & \text{if } 18 \le x \le 24 \\ (28-x)/4, & \text{if } 24 \le x \le 28 \\ 0, & \text{if } 28 \le x. \end{cases} \tag{89}$$

It is clear that Monday and Tuesday are warm with truth value 1, and Wednesday is warm with truth value 0.75. But Thursday to Sunday are not "warm" at all (in fact, they are "hot"). Intuitively, the uncertain proposition "two or three days are warm" should be completely true. The truth value algorithm yields that the truth value is

$$T(\text{"two or three days are warm"}) = 1. \tag{90}$$

Example 37: Assume that in a class there are 15 students whose ages are

$$20, 20, 21, 22, 24, 24, 25, 25, 26, 27, 28, 28, 30, 33, 38 \tag{91}$$

in years. Consider an uncertain proposition

$$(\mathcal{Q}, \mathbb{A}, \xi) = \text{"almost all students are young"}. \tag{92}$$

Suppose the uncertain quantifier $\mathcal{Q}$ = "almost all" has a membership function

$$\lambda(x) = \begin{cases} 0, & \text{if } 0 \le x \le 10 \\ (x-10)/3, & \text{if } 10 \le x \le 13 \\ 1, & \text{if } 13 \le x \le 15, \end{cases} \tag{93}$$

and the uncertain predicate ξ = "young" has a membership function

$$\mu(x) = \begin{cases} 1, & \text{if } 0 \le x \le 25 \\ (45-x)/20, & \text{if } 25 \le x \le 45 \\ 0, & \text{if } x \ge 45. \end{cases} \tag{94}$$

The truth value algorithm yields that the uncertain proposition has a truth value

$$T(\text{"almost all students are young"}) = 0.75. \tag{95}$$

Example 38: Assume that in a team there are 16 sportsmen whose heights are

$$\begin{array}{c} 165, 168, 168, 170, 178, 183, 185, 186 \\ 188, 190, 192, 192, 193, 194, 195, 198 \end{array} \tag{96}$$

in centimeters. Consider an uncertain proposition

$$(\mathcal{Q}, \mathbb{A}, \xi) = \text{"about 70\% of sportsmen are tall"}. \tag{97}$$

Suppose the uncertain quantifier $\mathcal{Q}$ = "about 70%" has a membership function

$$\lambda(x) = \begin{cases} 0, & \text{if } 0 \le x \le 0.6 \\ 20(x-0.6), & \text{if } 0.6 \le x \le 0.65 \\ 1, & \text{if } 0.65 \le x \le 0.75 \\ 20(0.8-x), & \text{if } 0.75 \le x \le 0.8 \\ 0, & \text{if } 0.8 \le x \le 1. \end{cases} \tag{98}$$

For this case, the argument of $\lambda(x)$ is $|K|/|\mathbb{A}|$. We also suppose the uncertain predicate ξ = "tall" has a membership function

$$\mu(x) = \begin{cases} 0, & \text{if } 0 \le x \le 175 \\ (x-175)/10, & \text{if } 175 \le x \le 185 \\ 1, & \text{if } x \ge 185. \end{cases} \tag{99}$$

The truth value algorithm yields that the uncertain proposition has a truth value

$$T(\text{"about 70\% of sportsmen are tall"}) = 0.6875. \tag{100}$$

Example 39: Assume that in a school there are 24 teachers whose ages and heights are

$$\begin{array}{c} (21,185),\ (22,190),\ (22,184),\ (23,170),\ (24,187),\ (24,188) \\ (25,160),\ (25,190),\ (26,185),\ (26,176),\ (27,185),\ (27,188) \\ (30,164),\ (34,178),\ (40,182),\ (45,186),\ (52,165),\ (60,170) \end{array} \tag{101}$$

in years and centimeters. Consider an uncertain proposition

$$(\mathcal{Q}, S, \xi) = \text{"most young teachers are tall"}. \tag{102}$$

Suppose the uncertain quantifier $\mathcal{Q}$ = "most" has a membership function

$$\lambda(x) = \begin{cases} 0, & \text{if } 0 \le x \le 0.7 \\ 20(x-0.7), & \text{if } 0.7 \le x \le 0.75 \\ 1, & \text{if } 0.75 \le x \le 0.85 \\ 20(0.9-x), & \text{if } 0.85 \le x \le 0.9 \\ 0, & \text{if } 0.9 \le x \le 1. \end{cases} \tag{103}$$

Note that each individual a is described by a feature data (y, z). For this case, the uncertain subject S = "young teachers" has a membership function

$$\nu(a) = \nu(y,z) = \begin{cases} 1, & \text{if } 0 \le y \le 25 \\ (45-y)/20, & \text{if } 25 \le y \le 45 \\ 0, & \text{if } y \ge 45 \end{cases} \tag{104}$$

and the uncertain predicate ξ = "tall" has a membership function

$$\mu(a) = \mu(y,z) = \begin{cases} 0, & \text{if } 0 \le z \le 175 \\ (z-175)/10, & \text{if } 175 \le z \le 185 \\ 1, & \text{if } z \ge 185. \end{cases} \tag{105}$$

The truth value algorithm yields that the uncertain proposition has a truth value

$$T(\text{"most young teachers are tall"}) = 0.9. \tag{106}$$

9 Data Mining and Linguistic Summary

Linguistic summary is a natural language statement that is concise and easy-to-understand by humans. For example, "most young teachers are tall" is a linguistic summary of teachers' ages and heights. Uncertain logic provides a flexible means that is capable of extracting linguistic summary from a collection of raw data.

First, we should have some raw data that are just the individual feature data,

$$\mathbb{A} = \{a_1, a_2, \cdots, a_n\}. \tag{107}$$

Next, we should have some linguistic terms to represent quantifiers, for example, "most" and "all". Denote them by a collection of uncertain quantifiers,

$$\mathbb{Q} = \{\mathcal{Q}_1, \mathcal{Q}_2, \cdots, \mathcal{Q}_m\}. \tag{108}$$

Then, we should have some linguistic terms to represent subjects, for example, "young teachers" and "old teachers". Denote them by a collection of uncertain subjects,

$$\mathbb{S} = \{S_1, S_2, \cdots, S_n\}. \tag{109}$$

Last, we should have some linguistic terms to represent predicates, for example, "short" and "tall". Denote them by a collection of uncertain predicates,

$$\mathbb{P} = \{\xi_1, \xi_2, \cdots, \xi_k\}. \tag{110}$$

One problem of data mining is to choose an uncertain quantifier $\mathcal{Q} \in \mathbb{Q}$, an uncertain subject $S \in \mathbb{S}$ and an uncertain predicate $\xi \in \mathbb{P}$ such that

$$T(\mathcal{Q}, S, \xi) \ge \beta \tag{111}$$

for the universe $\mathbb{A} = \{a_1, a_2, \cdots, a_n\}$, where β is a confidence level. Thus we have the following uncertain logic summarizer,

$$\begin{cases} \text{Find } \mathcal{Q}, S \text{ and } \xi \\ \text{subject to:} \\ \qquad \mathcal{Q} \in \mathbb{Q} \\ \qquad S \in \mathbb{S} \\ \qquad \xi \in \mathbb{P} \\ \qquad T(\mathcal{Q}, S, \xi) \ge \beta. \end{cases} \tag{112}$$

Each solution $(\overline{\mathcal{Q}}, \overline{S}, \overline{\xi})$ of the uncertain logic summarizer produces a linguistic summary "$\overline{\mathcal{Q}}$ of $\overline{S}$ are $\overline{\xi}$".

Example 40: Assume $\mathbb{P}$ consists of a single uncertain predicate ξ with membership function μ, and $\mathbb{Q}$ consists of all rectangular uncertain quantifiers. We would like to extract an uncertain quantifier $\mathcal{Q} \in \mathbb{Q}$ such that $T(\mathcal{Q}, \mathbb{A}, \xi) \ge \beta$. Denote

$$i = |\{a \in \mathbb{A} \mid \mu(a) \ge \beta\}|, \tag{113}$$

$$j = n - |\{a \in \mathbb{A} \mid \neg\mu(a) \ge \beta\}|. \tag{114}$$

When $\beta > 0.5$, we always have $i \le j$. It is easy to verify that the uncertain quantifier

$$\overline{\mathcal{Q}} = \{i, i+1, \cdots, j\} \tag{115}$$

ensures $T(\overline{\mathcal{Q}}, \mathbb{A}, \xi) \ge \beta$. This implies that the linguistic summary "$\overline{\mathcal{Q}}$ of $\mathbb{A}$ are ξ" has truth value β in uncertain measure. For example, the daily temperatures of some week from Monday to Sunday are assumed to be

$$22,\ 23,\ 25,\ 28,\ 30,\ 32,\ 36 \tag{116}$$

in centigrades, respectively. Suppose the uncertain predicate ξ = "warm" has a membership function

$$\mu(x) = \begin{cases} 0, & \text{if } x \le 15 \\ (x-15)/3, & \text{if } 15 \le x \le 18 \\ 1, & \text{if } 18 \le x \le 24 \\ (28-x)/4, & \text{if } 24 \le x \le 28 \\ 0, & \text{if } 28 \le x. \end{cases} \tag{117}$$

If the confidence level $\beta = 0.9$, then we have

$$i = |\{a \in \mathbb{A} \,|\, \mu(a) \ge 0.9\}| = |\{22, 23\}| = 2,$$

$$j = 7 - |\{a \in \mathbb{A} \,|\, \neg\mu(a) \ge 0.9\}| = 7 - |\{28, 30, 32, 36\}| = 3.$$

Thus the uncertain quantifier $\overline{\mathcal{Q}} = \{2, 3\}$ solves the uncertain logic summarizer. In other words, the uncertain logic summarizer extracts a linguistic summary "two or three days are warm".

Example 41: Assume that in a school there are 24 teachers whose ages and heights are

$$\begin{array}{c}(21,185),\ (22,190),\ (22,184),\ (23,170),\ (24,187),\ (24,188)\\ (25,160),\ (25,190),\ (26,185),\ (26,176),\ (27,185),\ (27,188)\\ (30,164),\ (34,178),\ (40,182),\ (45,186),\ (52,165),\ (60,170)\end{array} \tag{118}$$

in years and centimeters. Suppose we have three linguistic terms "about half", "most" and "all" as uncertain quantifiers whose membership functions are

$$\lambda_{half}(x) = \begin{cases} 0, & \text{if } 0 \le x \le 0.4 \\ 20(x-0.4), & \text{if } 0.4 \le x \le 0.45 \\ 1, & \text{if } 0.45 \le x \le 0.55 \\ 20(0.6-x), & \text{if } 0.55 \le x \le 0.6 \\ 0, & \text{if } 0.6 \le x \le 1, \end{cases} \tag{119}$$

$$\lambda_{most}(x) = \begin{cases} 0, & \text{if } 0 \le x \le 0.7 \\ 20(x-0.7), & \text{if } 0.7 \le x \le 0.75 \\ 1, & \text{if } 0.75 \le x \le 0.85 \\ 20(0.9-x), & \text{if } 0.85 \le x \le 0.9 \\ 0, & \text{if } 0.9 \le x \le 1, \end{cases} \tag{120}$$

$$\lambda_{all}(x) = \begin{cases} 1, & \text{if } x = 1 \\ 0, & \text{if } 0 \le x < 1, \end{cases} \tag{121}$$

respectively. Denote the collection of uncertain quantifiers by $\mathbb{Q} = \{$"about half", "most","all"$\}$. We also have three linguistic terms "young teachers", "middle-aged teachers" and "old teachers" as uncertain subjects whose membership functions are

$$\nu_{young}(a) = \nu_{young}(y,z) = \begin{cases} 1, & \text{if } 0 \le y \le 25 \\ (45-y)/20, & \text{if } 25 \le y \le 45 \\ 0, & \text{if } y \ge 45, \end{cases} \tag{122}$$

$$\nu_{middle}(a) = \nu_{middle}(y,z) = \begin{cases} 0, & \text{if } 0 \le y \le 30 \\ (y-30)/10, & \text{if } 30 \le y \le 40 \\ 1, & \text{if } 40 \le y \le 50 \\ (60-y)/10, & \text{if } 50 \le y \le 60 \\ 0, & \text{if } y \ge 60, \end{cases} \tag{123}$$

$$\nu_{old}(a) = \nu_{old}(y,z) = \begin{cases} 0, & \text{if } 0 \le y \le 50 \\ (y-50)/10, & \text{if } 50 \le y \le 60 \\ 1, & \text{if } y \ge 60, \end{cases} \tag{124}$$

respectively. Denote the collection of uncertain subjects by

$$\mathbb{S} = \{\text{"young teachers", "middle-aged teachers", "old teachers"}\}.$$

Finally, we suppose that there are two linguistic terms "short" and "tall" as uncertain predicates whose membership functions are

$$\mu_{short}(a) = \mu_{short}(y,z) = \begin{cases} 1, & \text{if } z \le 150 \\ (160-z)/10, & \text{if } 150 \le z \le 160 \\ 0, & \text{if } z \ge 160, \end{cases} \tag{125}$$

$$\mu_{tall}(a) = \mu_{tall}(y,z) = \begin{cases} 0, & \text{if } 0 \le z \le 175 \\ (z-175)/10, & \text{if } 175 \le z \le 185 \\ 1, & \text{if } z \ge 185, \end{cases} \tag{126}$$

respectively. Denote the collection of uncertain predicates by $\mathbb{P} = \{\text{"short"}, \text{"tall"}\}$. Assume the confidence level $\beta = 0.9$. The uncertain logic summarizer yields

$$\overline{\mathcal{Q}} = \text{"most"}, \quad \overline{S} = \text{"young teachers"}, \quad \overline{\xi} = \text{"tall"}$$

and then extracts a linguistic summary "most young teachers are tall".

10 Fuzzy Logic

When $(\mathcal{Q}, S, \xi)$ is a fuzzy proposition, where $\mathcal{Q}$ is a unimodal fuzzy quantifier with membership function λ, S is a fuzzy subject with membership function ν, and ξ is a fuzzy predicate with membership function μ, the truth value of fuzzy proposition $(\mathcal{Q}, S, \xi)$ has been defined in different ways by Zadeh [23], Yager [20], Bosc and Lietard [1], Delgado, Sánchez and Vila [3][4]. Based on fuzzy logic, Yager [20] and Kacprzyk and Yager [7] developed some methods to extract linguistic summaries from raw data. If $(\mathcal{Q}, S, \xi)$ is indeed a fuzzy proposition, different from the above methods, this paper suggests that the truth value of $(\mathcal{Q}, S, \xi)$ with respect to the universe $\mathbb{A}$ is

$$T(\mathcal{Q}, S, \xi) = \sup_{0 \le \omega \le 1} \left(\omega \wedge \sup_{K \in \mathbb{K}_\omega} \inf_{a \in K} \mu(a) \wedge \sup_{K \in \mathbb{K}_\omega^*} \inf_{a \in K} \neg\mu(a) \right) \tag{127}$$

where $\mathbb{K}_\omega = \{K \subset S_\omega \mid \lambda(|K|) \ge \omega\}$, $\mathbb{K}_\omega^* = \{K \subset S_\omega \mid \lambda(|S_\omega| - |K|) \ge \omega\}$, and $S_\omega = \{a \in \mathbb{A} \mid \nu(a) \ge \omega\}$. The difference between uncertain logic and fuzzy logic is that the former uses uncertain measure and the latter uses possibility measure. Perhaps some readers would like to argue that there is no measure in (127). Please keep in mind that membership functions cannot be generated if the underlying measure is not available.

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