Journal of Uncertain Systems
Vol.4, No.4, pp.306-311, 2010
Online at: www.jus.org.uk

Random Utility Functions are Uniquely Determined by User Preferences

Sthaporn Thepsumritporn[1], Nuanpan Nangsue[1], Olga Kosheleva[2,*]

[1] *Department of Applied Statistics, Faculty of Applied Science, King Mongkut's University of Technology North Bangkok, 1518 Pibulsongkram Rd., Bangsue, Bangkok 10800 Thailand*

[2] *Department of Teacher Education, University of Texas at El Paso, 500 W. University, El Paso, TX 79968, USA*

Received 9 January, 2009; Revised 30 January, 2010

Abstract

In decision theory, a traditional way to describe the user's preferences is to use a utility function. Preferences determine the utility function uniquely – modulo linear transformations (re-scaling). In practice, user's preferences are often probabilistic: when presented with the choice between the same two alternatives A and A', the user may sometimes select A and sometimes select A'. Since deterministic preferences are described by utilities, it is natural to describe probabilistic preferences by random utilities. In this paper, we show that, similar to the deterministic case, random utilities are also uniquely determined by the user preferences.

Keywords: utility functions, probabilistic preferences

1 Formulation of the Problem

Utility: a standard way to describe preferences. To describe user's preferences, traditionally, the notion of a utility is used; see, e.g., [1, 3].

To assign numerical values to different alternatives, we must fix two alternatives:

- a very negative alternative A_- (such as losing a large amount of money) and
- a very positive alternative A_+ (such as winning a large amount of money).

For every real number $p \in [0, 1]$, we can now define a lottery $L(p)$ in which the outcome is A_+ with probability p and A_- with the remaining probability $1 - p$.

- When $p = 1$, the corresponding lottery $L(1)$ coincides with the positive alternative A_+.
- When $p = 0$, the corresponding lottery $L(0)$ coincides with the negative alternative A_-.

In general, the larger the value p, the larger the probability of the positive outcome A_+ and thus, the more preferable the lottery $A(p)$. If we denote the fact that to the user, an alternative A is better than the alternative A' by $A' < A$, we can then conclude that if $p < p'$, then $L(p) < L(p')$.

The lotteries $L(p)$ corresponding to different values $p \in [0, 1]$ thus form a continuous scale going from the negative alternative $L(0) = A_-$ to the positive alternative $L(1) = A_+$. Since we selected A_- as a very negative alternative and A_+ as a very positive alternative, most alternatives A are in between A_- and A_+. Since $A_- = L(0) < A < L(1) = A_+$, it is reasonable to assume that there exists an intermediate value $p \in [0, 1]$ for which the alternative A is equivalent to the lottery $L(p)$: $A \equiv L(p)$. This value p is called the *utility* $u(A)$ of the alternative A.

In particular, according to this definition, the original alternatives A_- and A_+ have utilities $u(A_-) = 0$ and $u(A_+) = 1$.

*Corresponding author. Email: olgak@utep.edu (O. Kosheleva).

Expected utility: description and justification. One of the main objectives of decision theory is to help the user select the best actions. To make this selection, we must characterize the quality of different actions.

Let us assume that the action a can lead to n possible outcomes $A_1, \ldots, A_n$. For each outcome A_i, we know its probability p_i (so that $p_1 + \ldots + p_n = 1$) and its utility u_i. How can we describe the quality of this action?

The action a is, in effect, a lottery in which we get each alternative A_i with the corresponding probability p_i. By definition of utility, to the user, each alternative A_i is, in each turn, equivalent to a lottery $L(u_i)$ in which we get A_+ with probability u_i and A_- with the remaining probability $1 - u_i$. Thus, the action a is equivalent to the compound lottery in which:

- with probability p_1, we launch the lottery $L(u_1)$;
- with probability p_2, we launch the lottery $L(u_2)$;
- $\ldots$; and
- with probability p_n, we launch the lottery $L(u_n)$.

Each lottery $L(u_i)$ results either in A_+ or in A_-. Thus, the ultimate outcome of the above compound lottery is either A_+ or A_-. The probability to get A_+ in this compound lottery can be computed by using the formula of the full probability:

$$\text{Prob}(A_+) = \text{Prob}(\text{launching } L(u_1)) \cdot \text{Prob}(A_+ | \text{launching } L(u_1)) + \ldots +$$

$$\text{Prob}(\text{launching } L(u_n)) \cdot \text{Prob}(A_+ | \text{launching } L(u_n)) = p_1 \cdot u_1 + \ldots + p_n \cdot u_n. \quad (1)$$

Thus, the action a is equivalent to a lottery $L(u)$, with

$$u = p_1 \cdot u_1 + \ldots + p_n \cdot u_n. \quad (2)$$

By definition of the utility, this means that the utility u of an action is equal to the expression (2). In other words, the utility u of the action is equal to the expected value $p_1 \cdot u_1 + \ldots + p_n \cdot u_n$ of the utilities of different outcomes.

Each action a is thus equivalent to the lottery $L(u)$, where u is the utility of this action. So, comparing different actions a, a', $\ldots$, is equivalent to comparing the corresponding lotteries $L(u)$, $L(u')$, $\ldots$ The higher the value u, the more preferable the lottery. Thus, we must select the action with the largest value of expected utility.

How unique are utility values? The numerical values of the utilities u_i depend on the selection of the alternatives A_- and A_+. One can show that if we select different alternatives A'_- and A'_+, then the resulting utilities u'_i can be obtained from the original ones by an appropriate linear transformation: $u'_i = a \cdot u_i + b$ for some $a > 0$ and b.

Thus, in general, the utility function is defined uniquely modulo linear transformation. If we restrict ourselves to *normalized* utilities, i.e., utilities for which, for given two alternatives A_- and A_+ we have $u_- \stackrel{\text{def}}{=} u(A_-) = 0$ and $u_+ \stackrel{\text{def}}{=} u(A_+) = 1$, then preferences uniquely determine utility values.

In practice, preferences are often probabilistic. The traditional decision making theory is based on the assumption that the user's preferences are deterministic, i.e., that when presented with a choice between two alternatives A and A', the user will always choose the same one.

In practice, preferences are often probabilistic: when the alternatives A and A' are close to each other, a user often selects one of the them at random, so that when presented with the same choice, the user can make different choices at different times.

Random utilities: a reasonable way to describe probabilistic preferences. Probabilistic preferences mean that a user is not sure about his or her own preferences. For example, when

- sometimes A is preferred to A' and
- sometimes, A' is preferred to A,

this means that the user cannot decide between two different models:

- one in which A is better and
- one in which A' is better.

Different probabilities of different decisions mean that the user selects different models with different probabilities. Each of these models can be described by certain values of the utilities $u_1 = u(A_1), \ldots, u_n = u(A_n)$. Thus, random preferences mean that instead of a fixed vector $u \stackrel{\text{def}}{=} (u_1, \ldots, u_n)$, we can have different vectors u with different probabilities.

A situation in which we have different vectors with different probabilities is called a *random vector* – just like a situation in which we have different numbers with different probabilities is called a *random number*. In these terms, instead of a *deterministic* utility vector, we now have a *random* utility vector.

A natural question: are random utilities uniquely determined by the user preferences? We have mentioned that for deterministic preferences, the utility function is uniquely determined by the user preferences – provided that we normalize it by setting $u(A_-) = 0$ and $u(A_+) = 1$. We thus arrive at a natural question: Is the random utility also uniquely determined by the user preferences?

What we do in this paper. In this paper, we show that indeed, uniquely of utilities can be extended to the probabilistic case. This paper expands on the preliminary results published in [2].

2 Formulation of the Problem in Precise Terms

Let us describe the problem in precise terms.

Definition 1. *Let an integer n be given.*

- *By a* utility vector*, we mean an n-dimensional real-valued vector*
$$u = (u_1, \ldots, u_n).$$
- *By a* lottery*, we mean a collection $p = (p_-, p_1, \ldots, p_n, p_+)$ of $n+2$ non-negative numbers $p_i \geq 0$ whose sum is equal to 1:*
$$p_- + p_1 + \ldots + p_n + p_+ = 1.$$
- *For each utility vector $u = (u_1, \ldots, u_n)$ and each lottery*
$$p = (p_-, p_1, \ldots, p_n, p_+),$$
the utility of a lottery *is defined as the value*
$$p \cdot u \stackrel{\text{def}}{=} p^- \cdot 0 + p_1 \cdot u_1 + \ldots + p_n \cdot u_n + p^+ \cdot 1. \tag{3}$$

Discussion. In other words, $p \cdot u$ is the expected utility provided that we take $u_- = 0$ and $u_+ = 1$.

For each utility vector u,

- we select a lottery p over a lottery p' if $u \cdot p > u \cdot p'$, and
- we consider the lotteries p and p' to be of equal value if $u \cdot p = u \cdot p'$.

For a *random* utility vector u, it is reasonable to define the probability of preferring p to p' as the probability that $u \cdot p > u \cdot p'$. Thus, the definition of the random utility vector must guarantee that such probabilities exist:

Definition 2. *By a random utility vector, we mean a probability measure P on the set of all utility vectors for which for every two lotteries p and p', the set $\{u : u \cdot p > u \cdot p'\}$ is measurable.*

Discussion. We would like to also require that the set $\{u : u \cdot p = u \cdot p'\}$ is also measurable. However, this requirements follows from Definition 2. Indeed, the condition $u \cdot p = u \cdot p'$ simply means that we do not have $u \cdot p > u \cdot p'$ and we do not have $u \cdot p' > u \cdot p$. Thus, the set $\{u : u \cdot p = u \cdot p'\}$ is a complement to the union of two (disjoint) measurable sets $\{u : u \cdot p > u \cdot p'\}$ and $\{u : u \cdot p' > u \cdot p\}$ and is, thus, itself measurable – with the probability

$$P(u \cdot p = u \cdot p') = 1 - P(u \cdot p > u \cdot p') - P(u \cdot p' > u \cdot p). \tag{4}$$

3 Main Result

Now, we are ready to formulate and prove our main uniqueness result.

Definition 3. *Let P be a random utility vector, and let p and p' be lotteries. By the* probability $P(p > p')$ that p is preferable to p' *mean a value*

$$P(p > p') = P(u \cdot p > u \cdot p'). \tag{5}$$

Main uniqueness result. *Let P_1 and P_2 be random utility vectors which lead to the same preference probabilities for every two lotteries p and p', i.e., for which*

$$P_1(p > p') = P_2(p > p') \tag{6}$$

for all p and p'. Then, $P_1 = P_2$.

Discussion. In other words, a random utility vector is uniquely determined by the user preferences.

Proof: first comment. As we have mentioned, from the fact that

$$P_1(p > p') = P_2(p > p') \tag{7}$$

for all p and p', we can also conclude:

- that $P_1(p = p') = P_2(p = p')$ for all p and p', and thus,
- that $P_1(p \geq p') = P_2(p \geq p')$ for all p and p'.

Proof: reduction to characteristic functions. It is known that a probability measure can be uniquely reconstructed from its characteristic function $\chi(\omega) \stackrel{\text{def}}{=} E[\exp(\mathrm{i} \cdot \omega \cdot u)]$, where $\mathrm{i} \stackrel{\text{def}}{=} \sqrt{-1}$, $\omega = (\omega_1, \ldots, \omega_n)$ and

$$\omega \cdot u \stackrel{\text{def}}{=} \omega_1 \cdot u_1 + \ldots + \omega_n \cdot u_n. \tag{8}$$

The characteristic function is, in effect, the Fourier transform of the probability measure.

Thus, to show that the probability measures P_1 and P_2 coincide, it is sufficient to prove that their characteristic functions are equal.

Proof: reduction to cdf of linear combinations of u_i. For each vector ω, the corresponding value of the characteristic function is uniquely determined by probability distribution of the corresponding random variable $U \stackrel{\text{def}}{=} \omega_1 \cdot u_1 + \ldots + \omega_n \cdot u_n$. Thus, to prove that the characteristic functions of P_1 and P_2 are equal, it is sufficient to prove that for every ω, both probability measures P_1 and P_2 lead to the same probability distribution for U, i.e., to the same cumulative distribution function

$$F(t) \stackrel{\text{def}}{=} P(U \leq t) = P(\omega_1 \cdot u_1 + \ldots + \omega_n \cdot u_n \leq t). \tag{9}$$

So, to prove that $P_1 = P_2$, it is sufficient to prove that for all possible values of $\omega_1, \ldots, \omega_n$, and t, we have

$$P_1(\omega_1 \cdot u_1 + \ldots + \omega_n \cdot u_n \leq t) = P_2(\omega_1 \cdot u_1 + \ldots + \omega_n \cdot u_n \leq t). \tag{10}$$

Proof: case of positive $t > 0$. Let us start with the case of positive t. For $t > 0$, the inequality $\omega_1 \cdot u_1 + \ldots + \omega_n \cdot u_n \le t$ is equivalent to

$$a_1 \cdot u_1 + \ldots + a_n \cdot u_n \le 1, \tag{11}$$

where we denoted $a_i \stackrel{\text{def}}{=} \omega_i / t$. Thus, to prove the equality (10) for positive $t > 0$, it is sufficient to prove that for all possible real numbers $a_1, \ldots, a_n$, we have

$$P_1(a_1 \cdot u_1 + \ldots + a_n \cdot u_n \le 1) = P_2(a_1 \cdot u_1 + \ldots + a_n \cdot u_n \le 1). \tag{12}$$

We know that for every two probability vectors p and p', the inequality $p \cdot u \ge p' \cdot u$ has the same probability under the probability measures P_1 and P_2. This inequality has the form

$$p_1 \cdot u_1 + \ldots + p_n \cdot u_n + p_+ \ge p'_1 \cdot u_1 + \ldots + p'_n \cdot u_n + p'_+. \tag{13}$$

Thus, to prove the equality (12), it is sufficient to find the vectors p and p' for which the inequality (13) is equivalent to (11). By moving all the terms of (13) which are proportional to u_i into the right-hand side and all other terms to the left-hand side, we can reduce (13) to the following inequality

$$(p'_1 - p_1) \cdot u_1 + \ldots + (p'_n - p_n) \cdot u_n \le p_+ - p'_+. \tag{14}$$

When $p_+ > p'_+$, we can divide both sides of (14) by the positive difference $p_+ - p'_+$ and get the equivalent inequality

$$\frac{p'_1 - p_1}{p_+ - p'_+} \cdot u_1 + \ldots + \frac{p'_n - p_n}{p_+ - p'_+} \cdot u_n \le 1. \tag{15}$$

For this inequality to be equivalent to (8), we must make sure that for all i from 1 to n, we have

$$a_i = \frac{p'_i - p_i}{p_+ - p'_+}. \tag{16}$$

Let us denote $\varepsilon \stackrel{\text{def}}{=} p_+ - p'_+$, then (16) means that

$$a_i = \frac{p'_i - p_i}{\varepsilon}, \tag{17}$$

i.e., that

$$p'_i = p_i + \varepsilon \cdot a_i. \tag{18}$$

Let us choose $p_- = p_1 = \ldots = p_n = p_+ = \dfrac{1}{n+2}$. By definition of ε as $\varepsilon = p_+ - p'_+$, we have

$$p'_+ = \frac{1}{n+2} - \varepsilon. \tag{19}$$

From (18), we get

$$p'_i = \frac{1}{n+2} + \varepsilon \cdot a_i. \tag{20}$$

The value p'_- can be determined from the condition that the sum of all the probabilities is 1, as

$$p'_- = 1 - p'_1 - \ldots - p'_n - p'_+. \tag{21}$$

Substituting the formulas (19) and (20) into this expression, we conclude that

$$p'_- = \frac{1}{n+2} + \varepsilon \cdot \left(1 - \sum_{i=1}^{n} a_i\right). \tag{22}$$

For the values (19), (20), and (22) to represent a lottery, we must make sure that all the values p'_-, p'_i, and p'_+ are non-negative. When $\varepsilon \to 0$, these values all tend to a positive number $\dfrac{1}{n+2}$. Thus, for a sufficiently small $\varepsilon > 0$, they are indeed all positive. For this ε and for the corresponding values p and p', the inequality (13) is equivalent to (11).

Thus, from the fact that the preference probabilities coincide we deduce the desired equality (10).

Proof: case of $t = 0$. For $t = 0$, the desired inequality has the form

$$\omega_1 \cdot u_1 + \ldots + \omega_n \cdot u_n \le 0. \tag{23}$$

We want to find the values p and p' for which this inequality is equivalent to

$$(p'_1 - p_1) \cdot u_1 + \ldots + (p'_n - p_n) \cdot u_n \le p_+ - p'_+. \tag{24}$$

For this, we take $p_- = p_1 = \ldots = p_n = p_+ = \dfrac{1}{n+2}$, $p'_+ = p_+ = \dfrac{1}{n+2}$, $p'_i = p_i + \varepsilon \cdot \omega_i$ for some small $\varepsilon > 0$, and $p'_- = 1 - p'_1 - \ldots - p'_n - p'_+$.

Similarly to the case $t > 0$, for sufficiently small ε, the resulting values p'_-, p'_i, and p'_+ are all non-negative and thus, form a lottery. For this lottery, the desired inequality (23) is equivalent to the inequality (24) corresponding to comparing the lotteries p and p'. Thus, from the fact that the preference probabilities coincide, we conclude that the probabilities of satisfying the inequality (23) also coincide. So, we get the desired equality (10) for $t = 0$ as well.

Proof: auxiliary result for $t > 0$. Similarly to the inequality (10), we can similarly prove that

$$P_1(\omega_1 \cdot u_1 + \ldots + \omega_n \cdot u_n < t) = P_2(\omega_1 \cdot u_1 + \ldots + \omega_n \cdot u_n < t) \tag{25}$$

for all $\omega_1, \ldots, \omega_n$, and $t > 0$.

Proof: case of $t < 0$. If we change the signs of all the values $\omega_1, \ldots, \omega_n$, and t, then the inequality (10) for $t < 0$ takes the equivalent form

$$P_1(\omega'_1 \cdot u_1 + \ldots + \omega'_n \cdot u_n \ge t') = P_2(\omega'_1 \cdot u_1 + \ldots + \omega'_n \cdot u_n \ge t'), \tag{26}$$

where we denoted $\omega'_i \stackrel{\text{def}}{=} -\omega_i$ and $t' \stackrel{\text{def}}{=} -t$. Here, $t' > 0$. The probability that $U' \stackrel{\text{def}}{=} \omega'_1 \cdot u_1 + \ldots + \omega'_n \cdot u_n \ge t'$ is equal to 1 minus the probability that $U' < t'$:

$$P_1(\omega'_1 \cdot u_1 + \ldots + \omega'_n \cdot u_n \ge t') = 1 - P_1(\omega'_1 \cdot u_1 + \ldots + \omega'_n \cdot u_n < t'); \tag{27}$$

$$P_2(\omega'_1 \cdot u_1 + \ldots + \omega'_n \cdot u_n \ge t') = 1 - P_2(\omega'_1 \cdot u_1 + \ldots + \omega'_n \cdot u_n < t'). \tag{28}$$

We already know, from the formula (25), that

$$P_1(\omega'_1 \cdot u_1 + \ldots + \omega'_n \cdot u_n < t') = P_2(\omega'_1 \cdot u_1 + \ldots + \omega'_n \cdot u_n < t'). \tag{29}$$

Thus, we have

$$\begin{gathered} P_1(\omega'_1 \cdot u_1 + \ldots + \omega'_n \cdot u_n \ge t') = 1 - P_1(\omega'_1 \cdot u_1 + \ldots + \omega'_n \cdot u_n < t') = \\ 1 - P_2(\omega'_1 \cdot u_1 + \ldots + \omega'_n \cdot u_n < t') = P_2(\omega'_1 \cdot u_1 + \ldots + \omega'_n \cdot u_n \ge t'), \end{gathered} \tag{30}$$

i.e., the desired equality (26).

We can considered all possible cases; thus, our main statement is proved.

Acknowledgments

The authors are thankful to Sa-aat Niwitpong and Vladik Kreinovich for their help.

References

[1] Fishburn, P.C., *Utility Theory for Decision Making*, Wiley, New York, 1970.

[2] Kosheleva, O.M., and V. Kreinovich, *Uniqueness Theorem for Random Utility Functions*, Technical Report, Informatika Research Center, St. Petersburg, Russia, 1989 (in Russian).

[3] Luce, R.D., and H. Raiffa, *Games and Decisions: Introduction and Critical Survey*, Dover, New York, 1989.