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From Abstract Natural Numbers to Physical Natural Numbers: A Probabilistic Approach

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Abstract

Natural numbers originated as a way to describe the result of counting procedures. In quantum physics, the results of counting are probabilistic, so, in general, real counting leads to a random natural number – a probability distribution on the set of all natural numbers. From the practical viewpoint, events with a very small probability (smaller than some threshold ε) cannot occur. Therefore, it is reasonable say that a random natural number represents an integer n if the probability of n is $> \varepsilon$, while the probability of every other number is $\le \varepsilon$. For thus defined physical natural numbers, we analyze how their properties differ from the properties of the standard mathematical natural numbers. Specifically, we analyze the following natural question: if a represents n and b represents m, what are the possible representations for $a + b$?

Keywords: integers, quantum physics, probabilistic uncertainty

1 Formulation of the Problem

Not all natural numbers are physically meaningful. Natural numbers originated from the need to count real objects. Reasonably small natural numbers can indeed be interpreted as the corresponding numbers of objects. However, very large abstract integers, integers like $10^{10^{10}}$ which are larger than the number of particles in the Universe, cannot be thus represented. A natural question is: what will happen if we only allow physically meaningful natural numbers? This question was analyzed in the past from the philosophical and logical viewpoint; see, e.g., [1, 2, 3, 9, 10].

In this paper, we analyze the same question from a more physical viewpoint; in this analysis, we follow ideas from [4, 5, 6, 7, 8].

2 Towards a Definition of a "Physical" Natural Number

Towards a definition of a physical natural number. It is reasonable to identify, e.g., number 1 with situations in which the result of a counting procedure can be 1 but cannot be anything else.

Real physical natural numbers are probabilistic. The formalization of the above idea is complicated by the fact that according to quantum physics, all predictions are probabilistic.

In particular, for every physical counting procedure applied to a physical state, the result is, in general, a *random* natural number – in other words, a probability distribution on the set of all natural numbers in which can get different values i with different probabilities $p(i) \ge 0$ $(\sum\limits_i p(i) = 1)$.

In these terms, how can we describe the idea that some values are possible and some are not?

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Events with very small probability are usually considered to be not physically possible. From the practical viewpoint, events with a very small probability (smaller than some very small threshold value $\varepsilon \ll 1$) cannot occur. For example, from the physical and engineering viewpoints, a cold kettle placed on a cold stove will never start boiling by itself.

From the traditional probabilistic viewpoint, there is a positive probability that it will start boiling, so a mathematician might say that this boiling event is rare but still possible, but a physicist will most definitely say that this event is simply impossible.

Similarly, if we toss a fair coin 100 times in a row, and get heads all the time, then a person who is knowledgeable in probability would say that it is possible – since the probability is still positive. On the other hand, a physicist (or any person who uses common sense reasoning) would say that the coin is not fair – because if it is was a fair coin, then this abnormal event would be impossible.

In view of this fact, we arrive at the following definition of a physical natural number.

The notion of a physical natural number: an informal description. As we have mentioned, it is reasonable to identify, e.g., number 1 with situations in which the counting result can be 1 but cannot be anything else.

In view of the above idea, for a random natural number with probabilities $p(i)$, the result i of counting is possible if $p(i) > \varepsilon$ and not possible if $p(i) \le \varepsilon$. Thus, we arrive at the following definition.

Definition of a physical natural number. We say that a random natural number with probabilities $p(i)$ *represents an integer* n if $p(n) > \varepsilon$ and $p(i) \le \varepsilon$ for all $i \ne n$.

Formulation of the problem. Our general objective is to analyze how the properties of the above-defined physical natural numbers differ from properties of the standard mathematical natural numbers.

In this paper, we analyze their behavior under addition. Once we have two independent random natural numbers a and b with probabilities $p_a(i)$ and $p_b(i)$, we can define their sum $c = a + b$. The corresponding probability distribution for c is described by the usual formula

$$p_c(i) = \sum_{i_a, i_b : i_a + i_b = i} p_a(i_a) \cdot p_b(i_b). \tag{1}$$

A natural question is: if a represents n and b represents m, what are the possible alternatives for $a + b$? Does this sum always represents some number? If yes, what are the possible numbers represented by this sum?

3 Main Results

Our main results.

1. For every n and m, there exist independent random natural numbers a and b for which a represent m, b represents n, but $a + b$ does not represent any number.

2. For every n, m, and s, the following two statements are equivalent to each other:

   - there exist independent random natural numbers a and b for which a represent m, b represents n, and $a + b$ represents s;

   - the value s satisfies the inequality $s \ge \max(m, n)$.

Discussion. So, for probabilistic natural numbers, in addition to the usual sum $m + n$, arbitrary values $ge \max(m, n)$ are also possible.

Proof of the first result. In the proof, we will use the values $\delta > 0$ and an integer N that will be specified later. Once these numbers are selected, we denote by k the the largest natural number for which $(\varepsilon+\delta)+k\cdot\varepsilon \le 1$ (and thus, $(\varepsilon+\delta)+(k+1)\cdot\varepsilon > 1$), i.e., the value

$$k = \left\lfloor \frac{1-\varepsilon-\delta}{\varepsilon} \right\rfloor. \tag{2}$$

Let us first define a random natural number a that represents the value m. For that, we take:

- $p_a(m) = \varepsilon + \delta$,
- $p_a(m+1) = p_a(m+2) = \ldots = p_a(m+k) = \varepsilon$,
- $p_a(m+(k+1)) = 1 - ((\varepsilon+\delta)+k\cdot\varepsilon)$, and
- $p_a(i) = 0$ for all other i.

Due to our choice of k, we have $p_a(m+(k+1)) < \varepsilon$, and thus, a indeed represents m.

Similarly, we define the following random natural number b that represents the value n:

- $p_b(n) = \varepsilon + \delta$,
- $p_b(n+N) = p_a(n+2N) = \ldots = p_a(n+k\cdot N) = \varepsilon$,
- $p_b(n+(k+1)\cdot N) = 1 - ((\varepsilon+\delta)+k\cdot\varepsilon)$, and
- $p_b(i) = 0$ for all other i.

Due to our choice of k, we have $p_b(n+(k+1)\cdot N) < \varepsilon$, and thus, b indeed represents n.

Let us select the value δ in such a way that $(\varepsilon+\delta)^2 \le \varepsilon$. Since $\varepsilon < 1$ and hence, $\varepsilon^2 < \varepsilon$, such a selection is always possible.

For the above-defined random natural number a, the only values i_a with $p_a(i_a) \ne 0$ are values $i_a = m + j_a$ for $j_a = 0, 1, \ldots, k+1$. Similarly, for the above-defined random natural number b, the only values i_b with $p_b(i_b) \ne 0$ are values $i_b = n + j_b \cdot N$ for $j_b = 0, 1, \ldots, k+1$. We want to select M and N in such a way that all the sums

$$i_a + i_b = m + n + j_a + j_b \cdot N \tag{3}$$

of such numbers are different. For that, we can take, e.g., $N = k+2$, for which $N > j_a$ for all $j_a \le k+1$. Then for $j_b < j_b'$ and arbitrary j_a and j_a', we have $j_b' \ge j_b + 1$, hence $j_b' \cdot N \ge j_b \cdot N + N$, and

$$i_a' + i_b' = m + n + j_a' + j_b' \cdot N \ge m + n + j_a' + j_b \cdot N + N > m + n + j_a' + j_b \cdot N + j_a \ge$$

$$m + n + j_a + j_b \cdot N = i_a + i_b. \tag{4}$$

Since all the sums $i_a + i_b$ are different, each probability value $p_c(i)$ in the expression (1) only contains one product $p_a(i_a) \cdot p_b(i_b)$. Here:

- All the probabilities $p_a(i_a)$ are smaller than or equal to $\varepsilon + \delta$.
- Similarly, all the probabilities $p_b(i_b)$ are smaller than or equal to $\varepsilon + \delta$.

Thus, the product $p_a(i_a) \cdot p_b(i_b)$ does not exceed $(\varepsilon+\delta)^2$, and we have selected the value δ in such a way that $(\varepsilon+\delta)^2 \le \varepsilon$. So, for c, we have $p_c(i) \le \varepsilon$ for all i. Hence, no integer value is possible, and c does not represent any number. The statement is proven.

Proof of the second result. Let us first prove that the sum $c = a + b$ cannot represent any number $s < \max(m, n)$. Without losing generality, let us assume that $m < n$. Then $\max(m, n) = n$ and $s < n$. In this case, according to (1), we have

$$p_c(s) = p_a(0) \cdot p_b(s) + p_a(1) \cdot p_b(s-1) + \ldots + p_a(s) \cdot p_b(0). \tag{5}$$

Since $s < n$, we have $p_b(s) \le \varepsilon$, $p_b(s-1) \le \varepsilon$, $\ldots$, $p_b(0) \le \varepsilon$. Thus, from (5), we can conclude that

$$p_c(s) \le (p_a(0) + p_a(1) + \ldots + p_a(s)) \cdot \varepsilon. \tag{6}$$

Here,

$$p_a(0) + p_a(1) + \ldots + p_a(s) \le \sum_i p_a(i) = 1, \tag{7}$$

and therefore,

$$p_c(s) \le \varepsilon. \tag{8}$$

So, c cannot represent s.

To complete the proof, let us show that every value $s \ge \max(m, n)$ can be represented by an appropriate sum $a + b$. Indeed, for $s = m + n$, we can simply take the standard natural numbers a and b for which

- $p_a(m) = 1$ and $p_a(i) = 0$ for all $i \ne m$; and
- $p_b(n) = 1$ and $p_b(i) = 0$ for all $i \ne n$.

Let us now consider the case when $s \ne m + n$ and thus, $s - n \ne m$ and $s - m \ne n$. For a, we take

- $p_a(m) = 1 - \varepsilon$,
- $p_a(s - n) = \varepsilon$, and
- $p_a(i) = 0$ for all other i.

To determine b, we select the parameters δ and $N > s$, determine k according to the formula (2), and take

- $p_b(n) = \varepsilon + \delta$;
- $p_b(s - m) = \varepsilon$;
- $p_b(N) = p_b(2N) = \ldots = p_b((k-1) \cdot N) = \varepsilon$;
- $p_b(k \cdot N) = 1 - \delta - k \cdot \varepsilon$; and
- $p_b(i) = 0$ for all other i.

Due to our choice of k, we have $p_b(k \cdot N) < \varepsilon$, and thus, b indeed represents n.

For $i = s$, we have

$$p_c(s) \ge p_a(m) \cdot p_b(s-m) + p_a(s-n) \cdot p_b(n) = (1-\varepsilon) \cdot \varepsilon + \varepsilon \cdot (\varepsilon + \delta) =$$

$$((1-\varepsilon) + (\varepsilon + \delta)) \cdot \varepsilon = (1 + \delta) \cdot \varepsilon > \varepsilon. \tag{9}$$

For $i = m + n$, we have

$$p_c(m+n) = p_a(m) \cdot p_b(n) = (1-\varepsilon) \cdot (\varepsilon + \delta). \tag{10}$$

We must select $\delta > 0$ in such a way that this value is $\le \varepsilon$. Since $(1-\varepsilon) \cdot (\varepsilon + 0) \le \varepsilon$, such a selection is always possible.

To make sure that al the other sums $i_a + i_b$ are different, we can take $N > 2s + 2n + 2m$. Then, for every other i, the value $p_c(i)$ is equal to the product of two values one of which is $\le \varepsilon$ and thus, we have $p_c(i) \le \varepsilon$.

So, the only value i for which $p_c(i) > \varepsilon$ is the value $i = s$. The statement is proven.

Comment about multiplication. Similarly to the sum, we can define a product of two independent random natural numbers:

$$p_c(i) = \sum_{i_a, i_b : i_a \cdot i_b = i} p_a(i_a) \cdot p_b(i_b). \tag{11}$$

For multiplication, we only have one result: that if a represent m, b represents n, and $a \cdot b$ represents s, then s must divide both m and n.

Indeed, if, e.g., s does not divide n, then the inequality $p_c(s) \le \varepsilon$ can be proven similarly to the above proof that the sum $c = a + b$ cannot represent any number $s < \max(m, n)$.

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