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Solving Fuzzy Nonlinear Equations by a General Iterative Method

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Abstract

The present paper aims to study solution of fuzzy nonlinear equations, whose some parameters are fuzzy numbers. The fuzzy numbers have been presented in parametric form and a general iterative method has been proposed for the numerical solution of a system of fuzzy nonlinear equations. The proposed method has also been illustrated by an example to show the efficiency of the developed algorithm.

Keywords: fuzzy number, parametric form, membership function, fuzzy nonlinear system, general iteration method

1 Introduction

The concept of fuzzy numbers and arithmetic operations on it was introduced by Zadeh [12, 30] which was further enriched by Mizumoto and Tanaka [21]. Later on Dubois and Prade [13] made a significant contribution by introducing the concept of LR fuzzy numbers and presented a computational formula for operations on fuzzy numbers. With the development in the theory of fuzzy numbers, one of the major area emerged for application of these fuzzy numbers, is the solution of equations whose parameters are fuzzy numbers. Solution of the system of fuzzy equations is required in various areas such as Physics, Chemistry, Economics and many financial systems. Buckley and Qu [9] studied the solution of linear and quadratic equations when parameters are either real or complex fuzzy numbers. The study used the extension principle approach and envisaged that many simple fuzzy equations have no solution. Further, Buckley and Qu [10, 11] addressed this problem and gave a new solution concept for such cases.

Friedman, Ming and Kandel [16] proposed a general model for solving a $n \times n$ fuzzy linear system whose coefficient matrix is crisp and right side column is an arbitrary fuzzy number vector. Wang, Zhong and Ha[27] also proposed an iterative algorithm for solving a system of fuzzy linear equations.

Allahviranloo [3] presented algorithm for numerical solution of fuzzy system of linear equations based on iterative Jacobi and Gauss Siedel methods. Asady, Abbasbandy and Alavi [2] considered the solution of a $m \times n$ fuzzy general linear system for a case when $m \leq n$. Different approaches to solve fuzzy linear systems have also been given by several authors, such as Abbasbandy and Jafarian [4], Abbasbandy, Ezzati and Jafarian [5], Muzzioli and Reynaerts [22], Dehghan, Hashemi and Ghatee [15] and Wang and Zheng [28].

In real life problem fuzzy system of equations also occur in nonlinear forms and are not solved by methods commonly used for linear systems, Abbasbandy and Asady [1] considered this problem and presented a Newton's method for solving fuzzy nonlinear equations, which was applied by Kajani, Asady and Venchen [19] for solving a dual fuzzy nonlinear systems. Selekwa and Collins [23] considered the numerical solution of systems of qualitative nonlinear algebraic equations by fuzzy logic. Ujevic [26] presented a method for solution of nonlinear equations based on derived quadrature rules. The solution of fuzzy nonlinear equations by steepest descent method was considered by Abbasbany and Jafarian [6], where as Jafari and Varsha [18] presented a revised adomian decomposition method. For the purpose, Abbasbandy [7] further proposed Newton's method for solving a system of fuzzy nonlinear equations, where as Saavedra, Mangueira, Cano and Flores [24] described solution of nonlinear fuzzy systems by decomposition of incremental fuzzy numbers. Recently, Shokri [25] proposed an approach of midpoint Newton's method for the solution of system of fuzzy nonlinear equations. Here, the motivation of the present work is

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to develop a general computational algorithm, which is based on the theory of fixed point iteration, for solutions of fuzzy nonlinear equations.

All the previous works of solving fuzzy system of nonlinear equations have used the Newton's approach. Here, in this paper, we have extended the theory of general iterative method for solving fuzzy nonlinear systems, where right hand side of equation was taken to be a fuzzy number. The algorithm for the method has been developed and illustrated by numerical example for finding the positive fuzzy real roots of a fuzzy nonlinear system of equations. The present work has been organized in the coming section; Section 2 describes and presents fuzzy numbers in parametric form and the arithmetic operations on it. Section 3 of the paper describes the method and algorithm to the general iteration method for solution fuzzy system of nonlinear equations followed by numerical illustration of the method in Section 4 and the conclusion is placed in Section 5.

2 Fuzzy Number and its Parametric Form

In this section, some basics of fuzzy numbers and its parametric representation as presented in [1,12,13,21,30] are being viewed and some of the needed are being reproduced to make the study self contained.

Definition 1 A fuzzy number is a fuzzy set, $\tilde{u}: \mathcal{R} \to I = [0,1]$, which satisfies

1) $\tilde{u}$ is upper semi-continuous;
2) There are real numbers a, b, c, d such that $c \le a \le b \le d$ and
   $\tilde{u}(x)$ is monotonically increasing on $[c, a]$,
   $\tilde{u}(x)$ is monotonically decreasing on $[b, d]$,
   $\tilde{u}(x) = 1, a \le x \le b$;
3) $\tilde{u}(x) = 0$, if x lies outside the interval $[c, d]$.

Definition 2 A fuzzy number $\tilde{u}$ in parametric form is a pair $\left(\underline{u}\ ,\overline{u}\right)$ of function $\underline{u}(r), \overline{u}(r)$ $0 \le r \le 1$, which satisfies the following conditions

1) $\underline{u}(r)$ is a bounded monotonic increasing left continuous function over $[0, 1]$;
2) $\overline{u}(r)$ is a bounded monotonic decreasing left continuous function over $[0, 1]$;
3) $\underline{u}(r) \le \overline{u}(r), 0 \le r \le 1$.

Trapezoidal Fuzzy Number

A fuzzy number $\tilde{u}$ is the trapezoidal fuzzy number defined as $\tilde{u} = (x_0, y_0, \sigma, \beta)$ with interval defuzzifier $[x_0, y_0]$ and left fuzziness σ and right fuzziness β, where the membership function is

$$\tilde{u}(x) = \begin{cases} \frac{1}{\sigma}\ (x - x_0 + \sigma) & x - x_0 \le x \le x_0 \\ 1 & x \in [x_0, y_0] \\ \frac{1}{\beta}\ (y_0 - x + \beta) & y_0 \le x \le y_0 + \beta \\ 0 & otherwise \end{cases}$$

and its parametric form is

$$\underline{u}(r) = x_0 - \sigma + \sigma r \ \text{ and } \ \overline{u}(r) = y_0 + \beta - \beta r.$$

Further, if $x_0 = y_0$, then $\tilde{u} = (x_0, \sigma, \beta)$ is called the triangular fuzzy number.

Arithmetic Operations

The addition and scalar multiplication of fuzzy numbers in parametric form are defined using the interval arithmetic and are presented as following.

For any two fuzzy numbers $\tilde{u} = \left(\underline{u}, \overline{u}\right)$ and $\tilde{v} = \left(\underline{v}, \overline{v}\right)$, we define addition $\tilde{u} + \tilde{v}$ and multiplication by scalar k as

Addition

$$\left(\underline{u+v}\right)(r) = \underline{u}(r) + \underline{v}(r), \qquad \left(\overline{u+v}\right)(r) = \overline{u}(r) + \overline{v}(r).$$

Scalar Multiplication

$$\begin{cases} \left(\underline{ku}\right)(r) = k\underline{u}(r), & \left(\overline{ku}\right)(r) = k\overline{u}(r) & \text{for } k > 0 \\ \left(\underline{ku}\right)(r) = k\overline{u}(r), & \left(\overline{ku}\right)(r) = k\underline{u}(r) & \text{for } k < 0. \end{cases}$$

Multiplication of Two Fuzzy Numbers

Multiplication of two fuzzy numbers $\tilde{u} = (\underline{u}, \overline{u})$ and $\tilde{v} = (\underline{v}, \overline{v})$ can be defined in parametric form using interval arithmetic as

$$\tilde{u} * \tilde{v} = [\underline{u}(r), \overline{u}(r)] * [\underline{v}(r), \overline{v}(r)] = [\min\{\underline{uv}, \underline{u}\overline{v}, \overline{u}\underline{v}, \overline{uv}\}, \max\{\underline{uv}, \underline{u}\overline{v}, \overline{u}\underline{v}, \overline{uv}\}].$$

1) If $\tilde{u} > 0$ and $\tilde{v} > 0$, then

$$[\underline{u}(r), \overline{u}(r)] * [\underline{v}(r), \overline{v}(r)] = [\underline{u}(r)\ \underline{v}(r), \overline{u}(r)\ \overline{v}(r)];$$

2) If $\tilde{u} < 0$ and $\tilde{v} > 0$, then

$$\tilde{u} = -[-\overline{u}(r), -\underline{u}(r)]$$

and

$$\tilde{u} * \tilde{v} = -\{[-\overline{u}(r), -\underline{u}(r)] * [\underline{v}(r), \overline{v}(r)]\} = [\underline{u}(r)\overline{v}(r), \underline{u}(r)\underline{v}(r)];$$

3) If $\tilde{u} < 0$ and $\tilde{v} < 0$, then

$$\tilde{u} = -[-\overline{u}(r), -\underline{u}(r)] \text{ and } \tilde{v} = -[-\overline{v}(r), -\underline{v}(r)]$$

and

$$\tilde{u} * \tilde{v} = \{[\overline{u}(r)\overline{v}(r), \underline{u}(r)\underline{v}(r)]\};$$

4) If $\tilde{u} > 0$ and $\tilde{v} < 0$, then

$$\tilde{v} = -[-\overline{v}(r), -\underline{v}(r)]$$

and

$$\tilde{u} * \tilde{v} = [\overline{u}(r)\underline{v}(r), \underline{u}(r)\overline{v}(r)].$$

Inverse operation

Inverse of a fuzzy number $\tilde{u}$ is defined considering different cases. Here it is to be noted that origin does not belong to the 0-cut of $\tilde{u}$, i.e. $\tilde{u}$ is either strictly positive or strictly negative.

$$\tilde{u} > 0 \rightarrow \underline{u}(r) > 0 \text{ and } \overline{u}(r) > 0,$$
$$\frac{1}{\tilde{u}} = \frac{1}{[\underline{u}(r), \overline{u}(r)]} = \left[\frac{1}{\overline{u}(r)}, \frac{1}{\underline{u}(r)}\right].$$

And if $\tilde{u} < 0 \rightarrow \underline{u}(r) < 0$ and $\overline{u}(r) < 0$, then

$$\tilde{u} = -[-\overline{u}(r), -\underline{u}(r)],$$
$$\frac{1}{\tilde{u}} = -\frac{1}{[-\overline{u}(r), -\underline{u}(r)]} = -\left[-\frac{1}{\underline{u}(r)}, -\frac{1}{\overline{u}(r)}\right] = \left[\frac{1}{\overline{u}(r)}, \frac{1}{\underline{u}(r)}\right].$$

Division

Division of a fuzzy number $\tilde{u}$ by a fuzzy number $\tilde{v}$ can be defined by multiplication of $\tilde{u}$ with inverse of $\tilde{v}$ as discussed above. Here again, origin does not lie in the 0-cut of $\tilde{v}$.

$$\frac{\tilde{u}}{\tilde{v}} = \tilde{u} * \frac{1}{\tilde{v}}$$

and then considering different possible cases, value of division of $\tilde{u}$ by $\tilde{v}$, i.e., $\frac{\tilde{u}}{\tilde{v}}$ can be calculated.

3 A General Iteration Method

In this section we consider the solution of fuzzy nonlinear system

$$\begin{cases} f(\tilde{x}, \tilde{y}) = \tilde{c_1}, \\ g(\tilde{x}, \tilde{y}) = \tilde{c_2}. \end{cases} \quad (1)$$

First, we rewrite system of equation (1) using parametric representation of fuzzy numbers $\tilde{x}$, $\tilde{y}$, $\tilde{c_1}$ and $\tilde{c_2}$ as

$$\begin{cases} f_1(\underline{x}, \overline{x}, \underline{y}, \overline{y}, r) = \underline{c_1}, \\ f_2(\underline{x}, \overline{x}, \underline{y}, \overline{y}, r) = \overline{c_1}, \\ g_1(\underline{x}, \overline{x}, \underline{y}, \overline{y}, r) = \underline{c_2}, \\ g_2(\underline{x}, \overline{x}, \underline{y}, \overline{y}, r) = \overline{c_2}. \end{cases} \quad (2)$$

Then the system (2) can be equivalently written as

$$\begin{cases} \underline{x} = F_1\left(\underline{x}, \overline{x}, \underline{y}, \overline{y}, r\right), \\ \underline{y} = G_1\left(\underline{x}, \overline{x}, \underline{y}, \overline{y}, r\right), \\ \overline{x} = F_2\left(\underline{x}, \overline{x}, \underline{y}, \overline{y}, r\right), \\ \overline{y} = G_2\left(\underline{x}, \overline{x}, \underline{y}, \overline{y}, r\right) \end{cases} \tag{3}$$

where

$$\begin{cases} F_1\left(\underline{x}, \overline{x}, \underline{y}, \overline{y}, r\right) = \underline{x} + \alpha\left(f_1\left(\underline{x}, \overline{x}, \underline{y}, \overline{y}\right) - \underline{c_1}\right), \\ G_1\left(\underline{x}, \overline{x}, \underline{y}, \overline{y}, r\right) = \underline{y} + \beta\left(g_1\left(\underline{x}, \overline{x}, \underline{y}, \overline{y}\right) - \underline{c_2}\right), \\ F_2\left(\underline{x}, \overline{x}, \underline{y}, \overline{y}, r\right) = \overline{x} + \gamma\left(f_2\left(\underline{x}, \overline{x}, \underline{y}, \overline{y}\right) - \overline{c_1}\right), \\ G_2\left(\underline{x}, \overline{x}, \underline{y}, \overline{y}, r\right) = \overline{y} + \delta\left(g_2\left(\underline{x}, \overline{x}, \underline{y}, \overline{y}\right) - \overline{c_2}\right). \end{cases} \tag{4}$$

Let $\left(\tilde{\zeta}, \tilde{\eta}\right)$ be the exact solution of the above system, where $\tilde{\zeta} = \left(\underline{\zeta}, \overline{\zeta}\right)$ and $\tilde{\eta} = \left(\underline{\eta}, \overline{\eta}\right)$, then it will satisfy the given equations $\forall\, r \in [0,1]$,

$$\begin{cases} \underline{\zeta} = F_1\left(\underline{\zeta}, \overline{\zeta}, \underline{\eta}, \overline{\eta}, r\right), \\ \underline{\eta} = G_1\left(\underline{\zeta}, \overline{\zeta}, \underline{\eta}, \overline{\eta}, r\right), \\ \overline{\zeta} = F_2\left(\underline{\zeta}, \overline{\zeta}, \underline{\eta}, \overline{\eta}, r\right), \\ \overline{\eta} = G_2\left(\underline{\zeta}, \overline{\zeta}, \underline{\eta}, \overline{\eta}, r\right), \end{cases} \tag{5}$$

and its solution vector can be represented as $\begin{bmatrix} \underline{\zeta} \\ \underline{\eta} \\ \overline{\zeta} \\ \overline{\eta} \end{bmatrix}$.

Now, if $(\widetilde{x_0}, \widetilde{y_0})$ be a suitable initial approximation to $\left(\tilde{\zeta}, \tilde{\eta}\right)$, then we study the fixed point iteration as if k^{th} iterate is known, then $(k+1)^{th}$ iterate can be found as

$$\begin{cases} \underline{x_{k+1}} = F_1\left(\underline{x_k}, \overline{x_k}, \underline{y_k}, \overline{y_k}, r\right), \\ \underline{y_{k+1}} = G_1\left(\underline{x_k}, \overline{x_k}, \underline{y_k}, \overline{y_k}, r\right), \\ \overline{x_{k+1}} = F_2\left(\underline{x_k}, \overline{x_k}, \underline{y_k}, \overline{y_k}, r\right), \\ \overline{y_{k+1}} = G_2\left(\underline{x_k}, \overline{x_k}, \underline{y_k}, \overline{y_k}, r\right). \end{cases} \tag{6}$$

Using vector notation $X_{k+1} = g(X_k)$ where

$$X_k = \begin{bmatrix} \underline{x_k} \\ \underline{y_k} \\ \overline{x_k} \\ \overline{y_k} \end{bmatrix} \text{ and } g(X) = \begin{bmatrix} F_1\left(\underline{x}, \overline{x}, \underline{y}, \overline{y}, r\right) \\ G_1\left(\underline{x}, \overline{x}, \underline{y}, \overline{y}, r\right) \\ F_2\left(\underline{x}, \overline{x}, \underline{y}, \overline{y}, r\right) \\ G_2\left(\underline{x}, \overline{x}, \underline{y}, \overline{y}, r\right) \end{bmatrix}.$$

If above iterative method converges, then

$$\lim_{k\to\infty} \widetilde{x_k} = \tilde{\zeta},$$
$$\lim_{k\to\infty} \widetilde{y_k} = \tilde{\eta}.$$

Now subtracting system of equations (6) from system of equations (5) respectively, we get

$$\begin{cases} \underline{\zeta} - \underline{x_{k+1}} = F_1\left(\underline{\zeta}, \overline{\zeta}, \underline{\eta}, \overline{\eta}, r\right) - F_1\left(\underline{x_k}, \overline{x_k}, \underline{y_k}, \overline{y_k}, r\right), \\ \underline{\eta} - \underline{y_{k+1}} = G_1\left(\underline{\zeta}, \overline{\zeta}, \underline{\eta}, \overline{\eta}, r\right) - G_1\left(\underline{x_k}, \overline{x_k}, \underline{y_k}, \overline{y_k}, r\right), \\ \overline{\zeta} - \overline{x_{k+1}} = F_2\left(\underline{\zeta}, \overline{\zeta}, \underline{\eta}, \overline{\eta}, r\right) - F_2\left(\underline{x_k}, \overline{x_k}, \underline{y_k}, \overline{y_k}, r\right), \\ \overline{\eta} - \overline{y_{k+1}} = G_2\left(\underline{\zeta}, \overline{\zeta}, \underline{\eta}, \overline{\eta}, r\right) - G_2\left(\underline{x_k}, \overline{x_k}, \underline{y_k}, \overline{y_k}, r\right). \end{cases} \tag{7}$$

Now, let $\underline{\varepsilon_k} = \underline{\zeta} - \underline{x_k}$, $\overline{\varepsilon_k} = \overline{\zeta} - \overline{x_k}$, $\underline{\delta_k} = \underline{\eta} - \underline{y_k}$ and $\overline{\delta_k} = \overline{\eta} - \overline{y_k}$ be errors in the k^{th} iterate. Then,

$$\begin{cases} \underline{\varepsilon_{k+1}} = F_1\left(\underline{x_k} + \underline{\varepsilon_k}, \overline{x_k} + \overline{\varepsilon_k}, \underline{y_k} + \underline{\delta_k}, \overline{y_k} + \overline{\delta_k}, r\right) - F_1\left(\underline{x_k}, \overline{x_k}, \underline{y_k}, \overline{y_k}, r\right), \\ \underline{\delta_{k+1}} = G_1\left(\underline{x_k} + \underline{\varepsilon_k}, \overline{x_k} + \overline{\varepsilon_k}, \underline{y_k} + \underline{\delta_k}, \overline{y_k} + \overline{\delta_k}, r\right) - G_1\left(\underline{x_k}, \overline{x_k}, \underline{y_k}, \overline{y_k}, r\right), \\ \overline{\varepsilon_{k+1}} = F_2\left(\underline{x_k} + \underline{\varepsilon_k}, \overline{x_k} + \overline{\varepsilon_k}, \underline{y_k} + \underline{\delta_k}, \overline{y_k} + \overline{\delta_k}, r\right) - F_2\left(\underline{x_k}, \overline{x_k}, \underline{y_k}, \overline{y_k}, r\right), \\ \overline{\delta_{k+1}} = G_2\left(\underline{x_k} + \underline{\varepsilon_k}, \overline{x_k} + \overline{\varepsilon_k}, \underline{y_k} + \underline{\delta_k}, \overline{y_k} + \overline{\delta_k}, r\right) - G_2\left(\underline{x_k}, \overline{x_k}, \underline{y_k}, \overline{y_k}, r\right). \end{cases} \tag{8}$$

Further, using Taylor series expansion of F_1, F_2, G_1 and G_2 about $\left(\underline{x_k}, \overline{x_k}, \underline{y_k}, \overline{y_k}, r\right)$ and neglecting the second and higher powers of $\underline{\varepsilon_k}$, $\overline{\varepsilon_k}$, $\underline{\delta_k}$ and $\overline{\delta_k}$, we get, $\forall\, r \in [0,1]$,

$$\begin{cases} \underline{\varepsilon_{k+1}} = \underline{\varepsilon_k} F_{1_{\underline{x}}}\left(\underline{x_k}, \overline{x_k}, \underline{y_k}, \overline{y_k}, r\right) + \underline{\delta_k} F_{1_{\underline{y}}}\left(\underline{x_k}, \overline{x_k}, \underline{y_k}, \overline{y_k}, r\right) + \overline{\varepsilon_k} F_{1_{\overline{x}}}\left(\underline{x_k}, \overline{x_k}, \underline{y_k}, \overline{y_k}, r\right) + \overline{\delta_k} F_{1_{\overline{y}}}\left(\underline{x_k}, \overline{x_k}, \underline{y_k}, \overline{y_k}, r\right), \\ \underline{\delta_{k+1}} = \underline{\varepsilon_k} G_{1_{\underline{x}}}\left(\underline{x_k}, \overline{x_k}, \underline{y_k}, \overline{y_k}, r\right) + \underline{\delta_k} G_{1_{\underline{y}}}\left(\underline{x_k}, \overline{x_k}, \underline{y_k}, \overline{y_k}, r\right) + \overline{\varepsilon_k} G_{1_{\overline{x}}}\left(\underline{x_k}, \overline{x_k}, \underline{y_k}, \overline{y_k}, r\right) + \overline{\delta_k} G_{1_{\overline{y}}}\left(\underline{x_k}, \overline{x_k}, \underline{y_k}, \overline{y_k}, r\right), \\ \overline{\varepsilon_{k+1}} = \underline{\varepsilon_k} F_{2_{\underline{x}}}\left(\underline{x_k}, \overline{x_k}, \underline{y_k}, \overline{y_k}, r\right) + \underline{\delta_k} F_{2_{\underline{y}}}\left(\underline{x_k}, \overline{x_k}, \underline{y_k}, \overline{y_k}, r\right) + \overline{\varepsilon_k} F_{2_{\overline{x}}}\left(\underline{x_k}, \overline{x_k}, \underline{y_k}, \overline{y_k}, r\right) + \overline{\delta_k} F_{2_{\overline{y}}}\left(\underline{x_k}, \overline{x_k}, \underline{y_k}, \overline{y_k}, r\right), \\ \overline{\delta_{k+1}} = \underline{\varepsilon_k} G_{2_{\underline{x}}}\left(\underline{x_k}, \overline{x_k}, \underline{y_k}, \overline{y_k}, r\right) + \underline{\delta_k} G_{2_{\underline{y}}}\left(\underline{x_k}, \overline{x_k}, \underline{y_k}, \overline{y_k}, r\right) + \overline{\varepsilon_k} G_{2_{\overline{x}}}\left(\underline{x_k}, \overline{x_k}, \underline{y_k}, \overline{y_k}, r\right) + \overline{\delta_k} G_{2_{\overline{y}}}\left(\underline{x_k}, \overline{x_k}, \underline{y_k}, \overline{y_k}, r\right), \end{cases} \tag{9}$$

which can be written in matrix form as

$$\begin{bmatrix} \underline{\varepsilon_{k+1}} \\ \underline{\delta_{k+1}} \\ \overline{\varepsilon_{k+1}} \\ \overline{\delta_{k+1}} \end{bmatrix} = \begin{bmatrix} F_{1_{\underline{x}}} & F_{1_{\underline{y}}} & F_{1_{\overline{x}}} & F_{1_{\overline{y}}} \\ G_{1_{\underline{x}}} & G_{1_{\underline{y}}} & G_{1_{\overline{x}}} & G_{1_{\overline{y}}} \\ F_{2_{\underline{x}}} & F_{2_{\underline{y}}} & F_{2_{\overline{x}}} & F_{2_{\overline{y}}} \\ G_{2_{\underline{x}}} & G_{2_{\underline{y}}} & G_{2_{\overline{x}}} & G_{2_{\overline{y}}} \end{bmatrix} \begin{bmatrix} \underline{\varepsilon_k} \\ \underline{\delta_k} \\ \overline{\varepsilon_k} \\ \overline{\delta_k} \end{bmatrix} \tag{10}$$

where the partial derivatives $F_{1_{\underline{x}}}, F_{1_{\underline{y}}}, F_{1_{\overline{x}}}, F_{1_{\overline{y}}}$,…. are evaluated at k^{th} iterate. It can also be written as

$$e^{(k+1)} = A_k\, e^{(k)}$$

where $e^{(k)} = \begin{bmatrix} \underline{\varepsilon_k} \\ \underline{\delta_k} \\ \overline{\varepsilon_k} \\ \overline{\delta_k} \end{bmatrix}$ and the Jacobian matrix is given by

$$A(x) = \begin{bmatrix} F_{1_{\underline{x}}} & F_{1_{\underline{y}}} & F_{1_{\overline{x}}} & F_{1_{\overline{y}}} \\ G_{1_{\underline{x}}} & G_{1_{\underline{y}}} & G_{1_{\overline{x}}} & G_{1_{\overline{y}}} \\ F_{2_{\underline{x}}} & F_{2_{\underline{y}}} & F_{2_{\overline{x}}} & F_{2_{\overline{y}}} \\ G_{2_{\underline{x}}} & G_{2_{\underline{y}}} & G_{2_{\overline{x}}} & G_{2_{\overline{y}}} \end{bmatrix}. \tag{11}$$

A_k is the Jacobian matrix of the iteration functions $\tilde{F}$ and $\tilde{G}$ evaluated at $(\widetilde{x_k}, \widetilde{y_k})$.

Now a sufficient condition for convergence of the solution for $\forall\, k$ and $\forall\, r \in [0,1]$ is $\|A_k\| < 1$ where $\|.\|$ is some suitable norm.

If we use the maximum absolute row sum norm, we get the conditions

$$\begin{cases} \left|F_{1_{\underline{x}}}\right| + \left|F_{1_{\underline{y}}}\right| + \left|F_{1_{\overline{x}}}\right| + \left|F_{1_{\overline{y}}}\right| < 1, \\ \left|G_{1_{\underline{x}}}\right| + \left|G_{1_{\underline{y}}}\right| + \left|G_{1_{\overline{x}}}\right| + \left|G_{1_{\overline{y}}}\right| < 1, \\ \left|F_{2_{\underline{x}}}\right| + \left|F_{2_{\underline{y}}}\right| + \left|F_{2_{\overline{x}}}\right| + \left|F_{2_{\overline{y}}}\right| < 1, \\ \left|G_{2_{\underline{x}}}\right| + \left|G_{2_{\underline{y}}}\right| + \left|G_{2_{\overline{x}}}\right| + \left|G_{2_{\overline{y}}}\right| < 1 \end{cases} \tag{12}$$

and the necessary and sufficient condition for convergence is that for each k, $\rho(A_k) < 1$, where $\rho(A_k)$ is the spectral radius of the matrix A_k.

Further, if $(\widetilde{x_0}, \widetilde{y_0})$ is the close approximation of $(\tilde{\zeta}, \tilde{\eta})$, then we usually test the conditions at the initial approximation $(\widetilde{x_0}, \widetilde{y_0})$ only.

Remark 1: Sequence $\left\{\left(\underline{x_n}, \overline{x_n}\right)\right\}_{n=0}^{\infty}$ and $\left\{\left(\underline{y_n}, \overline{y_n}\right)\right\}_{n=0}^{\infty}$ converge to $\left(\underline{\zeta}, \overline{\zeta}\right)$ and $\left(\underline{\eta}, \overline{\eta}\right)$ respectively if and only if $\forall r \in [0,1]$,

$$\lim_{n\to\infty} \underline{x_n}(r) = \underline{\zeta}(r),$$
$$\lim_{n\to\infty} \overline{x_n}(r) = \overline{\zeta}(r),$$
$$\lim_{n\to\infty} \underline{y_n}(r) = \underline{\eta}(r),$$
$$\lim_{n\to\infty} \overline{y_n}(r) = \overline{\eta}(r).$$

Lemma 1 Let $\begin{cases} f(\tilde{\zeta}, \tilde{\eta}) = 0 \\ g(\tilde{\zeta}, \tilde{\eta}) = 0. \end{cases}$ If the sequence of $\left\{\left(\underline{x_n}, \overline{x_n}\right)\right\}_{n=0}^{\infty}$ and $\left\{\left(\underline{y_n}, \overline{y_n}\right)\right\}_{n=0}^{\infty}$ converge to $\left(\underline{\zeta}, \overline{\zeta}\right)$ and $\left(\underline{\eta}, \overline{\eta}\right)$ respectively, then

$$\lim_{n\to\infty} P_n = 0$$

where $P_n = \sup \max_{0 \le r \le 1} \left\{\underline{\varepsilon_n}(r), \overline{\varepsilon_n}(r), \underline{\delta_n}(r), \overline{\delta_n}(r)\right\}$.

Proof: Proof is obvious, because for all $r \in [0,1]$ in the convergent case

$$\lim_{n\to\infty} \underline{\varepsilon_n}(r) = \lim_{n\to\infty} \overline{\varepsilon_n}(r) = \lim_{n\to\infty} \underline{\delta_n}(r) = \lim_{n\to\infty} \overline{\delta_n}(r) = 0.$$

Theorem Let D be a closed, bounded and convex set in the plane. Assume that the components of F and G, i.e. F_1, F_2, G_1 *and* G_2, are continuously differentiable with respect to $\underline{x}, \overline{x}, \underline{y}$ and $\overline{y}$ and further assume that

1) $g(D) \subset D$;
2) $\lambda \equiv \max\|A(x)\|_\infty < 1$.

Then

a) $X = g(X)$ has a unique solution $\alpha \in D$;
b) for any initial point X_0, the iteration will converge in D to α;
c) $\|\alpha - X_{n+1}\|_\infty \le (\|G(\alpha)\|_\infty + \varepsilon_n) \|\alpha - X_n\|_\infty$ with $\varepsilon_n \to 0$ *as* $n \to \infty$.

Proof: See [8].

4 Numerical Illustration

In this section, we implement the developed algorithm for solution of fuzzy nonlinear equations for positive real roots. Consider a fuzzy nonlinear system as

$$\begin{cases} \widetilde{x^2} + 3\tilde{x} + \tilde{y} = \tilde{5}, \\ \widetilde{x^2} + 3\widetilde{y^2} = \tilde{4}. \end{cases} \tag{13}$$

Now assuming the fuzzy numbers $\tilde{x}$ and $\tilde{y}$ be positive fuzzy numbers and are being represented as $\tilde{x} = \left(\underline{x}, \overline{x}\right)$ and $\tilde{y} = \left(\underline{y}, \overline{y}\right)$ and let fuzzy numbers $\tilde{4}$ and $\tilde{5}$ be written in parametric form as

$$\tilde{4} = (3.5 + 0.5r, 4.5 - 0.5r),$$
$$\tilde{5} = (4.5 + 0.5r, 5.5 - 0.5r).$$

Thus the fuzzy system (13) in fuzzy parametric form be written as

$$\begin{cases} \underline{x^2} + 3\underline{x} + \underline{y} = 4.5 + 0.5r, \\ \underline{x^2} + 3\underline{y^2} = 3.5 + 0.5r, \\ \overline{x^2} + 3\overline{x} + \overline{y} = 5.5 - 0.5r, \\ \overline{x^2} + 3\overline{y^2} = 4.5 - 0.5r. \end{cases} \tag{14}$$

It can be suitably rewritten as

$$\begin{cases} \underline{x} = F_1\left(\underline{x}, \overline{x}, \underline{y}, \overline{y}, r\right), \\ \underline{y} = G_1\left(\underline{x}, \overline{x}, \underline{y}, \overline{y}, r\right), \\ \overline{x} = F_2\left(\underline{x}, \overline{x}, \underline{y}, \overline{y}, r\right), \\ \overline{y} = G_2\left(\underline{x}, \overline{x}, \underline{y}, \overline{y}, r\right) \end{cases} \tag{15}$$

where

$$\begin{aligned} F_1\left(\underline{x}, \overline{x}, \underline{y}, \overline{y}, r\right) &= \underline{x} + \alpha\left(\underline{x^2} + 3\underline{x} + \underline{y} - 4.5 - 0.5r\right), \\ G_1\left(\underline{x}, \overline{x}, \underline{y}, \overline{y}, r\right) &= \underline{y} + \beta\left(\underline{x^2} + 3\underline{y^2} - 3.5 - 0.5r\right), \\ F_2\left(\underline{x}, \overline{x}, \underline{y}, \overline{y}, r\right) &= \overline{x} + \gamma\left(\overline{x^2} + 3\overline{x} + \overline{y} - 5.5 + 0.5r\right), \\ G_2\left(\underline{x}, \overline{x}, \underline{y}, \overline{y}, r\right) &= \overline{y} + \delta\left(\overline{x^2} + 3\overline{y^2} - 4.5 + 0.5r\right), \end{aligned} \tag{16}$$

and the Jacobian matrix is

$$A_k = \begin{bmatrix} F_{1_{\underline{x}}} & F_{1_{\underline{y}}} & F_{1_{\overline{x}}} & F_{1_{\overline{y}}} \\ G_{1_{\underline{x}}} & G_{1_{\underline{y}}} & G_{1_{\overline{x}}} & G_{1_{\overline{y}}} \\ F_{2_{\underline{x}}} & F_{2_{\underline{y}}} & F_{2_{\overline{x}}} & F_{2_{\overline{y}}} \\ G_{2_{\underline{x}}} & G_{2_{\underline{y}}} & G_{2_{\overline{x}}} & G_{2_{\overline{y}}} \end{bmatrix}_{at\ k^{th} iterate\ .} = \begin{bmatrix} 1 + \alpha\left(2\underline{x_k} + 3\right) & \alpha & 0 & 0 \\ \beta\left(2\underline{x_k}\right) & 1 + \beta\left(6\underline{y_k}\right) & 0 & 0 \\ 0 & 0 & 1 + \gamma(2\overline{x_k} + 3) & \gamma \\ 0 & 0 & \delta(2\overline{x_k}) & 1 + \delta(6\overline{y_k}) \end{bmatrix}. \tag{17}$$

By the necessary condition of convergence of solution, we have

$$\begin{cases} \left|1 + \alpha\left(2\underline{x_k} + 3\right)\right| + |\alpha| < 1, \\ \left|\beta\left(2\underline{x_k}\right)\right| + \left|1 + \beta\left(6\underline{y_k}\right)\right| < 1, \\ |1 + \gamma(2\overline{x_k} + 3)| + |\gamma| < 1, \\ |\delta(2\overline{x_k})| + |1 + \delta(6\overline{y_k})| < 1. \end{cases} \tag{18}$$

Now if we choose the initial guess as

$$\begin{cases} \widetilde{x_0} = \widetilde{0.5} = (0.25 + 0.25r, 0.75 - 0.25r), \\ \widetilde{y_0} = \widetilde{0.5} = (0.25 + 0.25r, 0.75 - 0.25r), \end{cases} \tag{19}$$

then at $r = 0$, it reduces to

$$\underline{x_0} = 0.25, \qquad \overline{x_0} = 0.75, \qquad \underline{y_0} = 0.25, \qquad \overline{y_0} = 0.75,$$

and the above conditions (18) reduces to

$$\begin{cases} |1 + 3.5\alpha| + |\alpha| < 1, \\ |0.5\beta| + |1 + 1.5\beta| < 1, \\ |1 + 4.5\gamma| + |\gamma| < 1, \\ |1.5\delta| + |1 + 4.5\delta| < 1. \end{cases} \tag{20}$$

Now, a choice of $\alpha = \gamma = -1/4$ and $\beta = \delta = -1/6$, satisfy those conditions in (20). Hence the equations for finding successive iterates become

$$\begin{cases} \underline{x_{k+1}} = -\frac{1}{4}\left(\underline{x_k}^2 - \underline{x} + \underline{y_k} - 4.5 - 0.5r\right), \\ \underline{y_{k+1}} = -\frac{1}{6}\left(\underline{x_k^2} + 3\underline{y_k^2} - 6\underline{y_k} - 3.5 - 0.5r\right), \\ \overline{x_{k+1}} = -\frac{1}{4}\left(\overline{x_k^2} - \overline{x_k} + \overline{y_k} - 5.5 + 0.5r\right), \\ \overline{y_{k+1}} = -\frac{1}{6}\left(\overline{x_k^2} + 3\overline{y_k^2} - 6\overline{y_k} - 4.5 + 0.5r\right). \end{cases} \tag{21}$$

Computing the solutions only for three iterations by the proposed general iterative method, we obtain the solution of $\tilde{x}$ and $\tilde{y}$ with the maximum error of the order 10^{-2} and are placed in Table 1. Computations have also been carried out by Newton's iterative method and are placed in Table 1 for comparison with the proposed method.

Table 1: The solution of $\tilde{x}$ and $\tilde{y}$

	Solution by proposed method (3rd iteration)		Solution by Newton's method (3rd iteration)	
r	x	y	x	y
0	0.9340	0.9390	0.8972	1.0080
0.1	0.9434	0.9476	0.9103	0.9935
0.2	0.9529	0.9560	0.9222	0.9853
0.3	0.9624	0.9641	0.9331	0.9814
0.4	0.9720	0.9721	0.9435	0.9805
0.5	0.9817	0.9798	0.9534	0.9817
0.6	0.9914	0.9873	0.9630	0.9842
0.7	1.0011	0.9946	0.9724	0.9877
0.8	1.0108	1.0018	0.9816	0.9918
0.9	1.0206	1.0088	0.9907	0.9963
1	1.0302	1.0157	0.9998	1.0011
0.9	1.0399	1.0224	1.0088	1.0060
0.8	1.0494	1.0290	1.0177	1.0110
0.7	1.0590	1.0354	1.0266	1.0161
0.6	1.0684	1.0417	1.0355	1.0212
0.5	1.0777	1.0478	1.0443	1.0263
0.4	1.0869	1.0537	1.0532	1.0314
0.3	1.0960	1.0596	1.0620	1.0364
0.2	1.1050	1.0652	1.0707	1.0414
0.1	1.1138	1.0707	1.0795	1.0464
0	1.1225	1.0760	1.0882	1.0513

The efficiency of the proposed algorithm in comparison with the existing Newton's algorithm [1, 7] can be further visualized in Figure 1 to Figure 4.

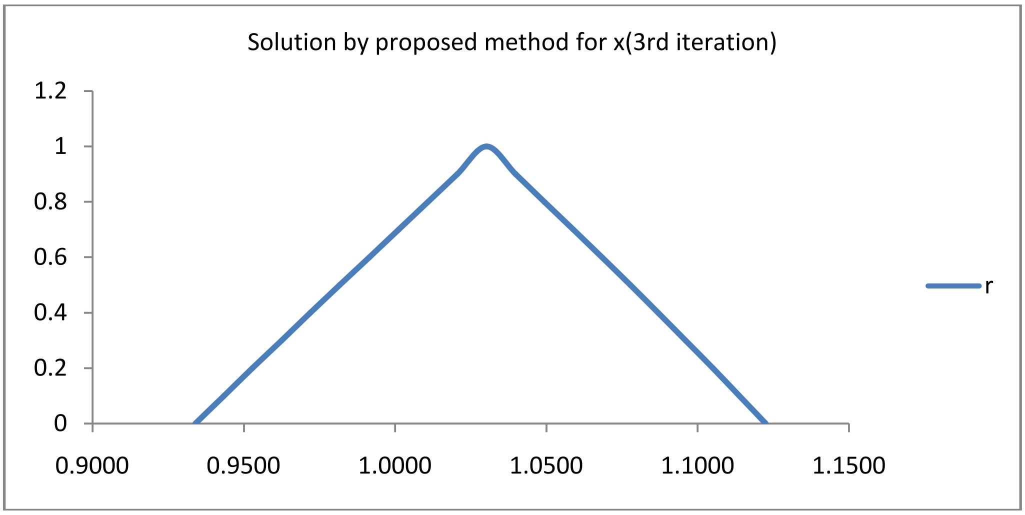

Figure 1: Solution for x by proposed method

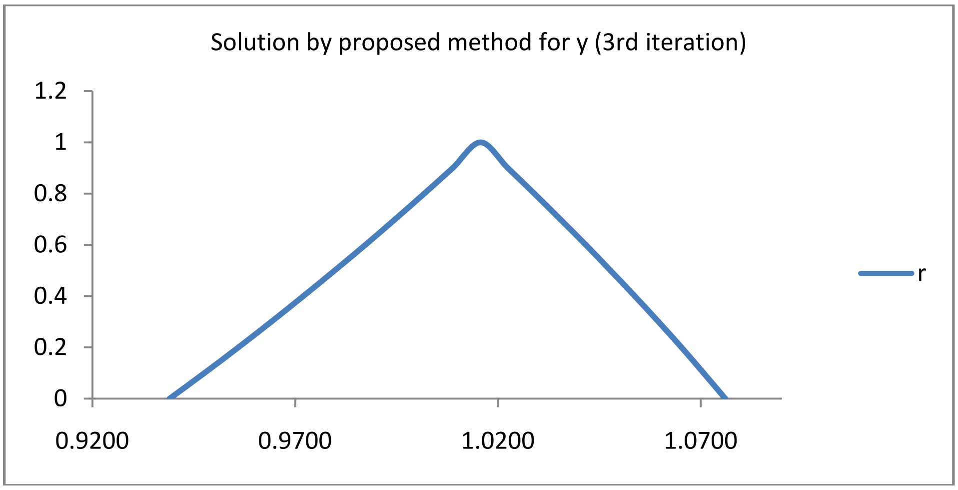

Figure 2: Solution for y by proposed method

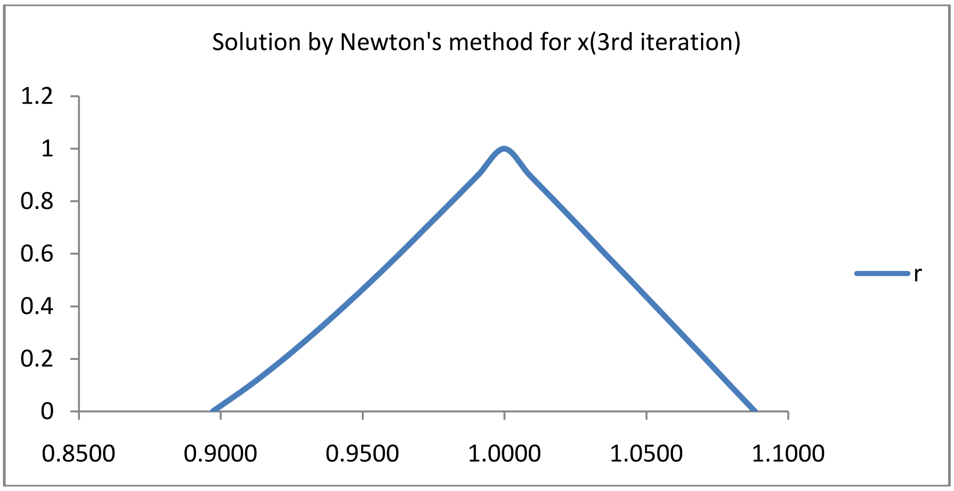

Figure 3: Solution for x by Newton's method

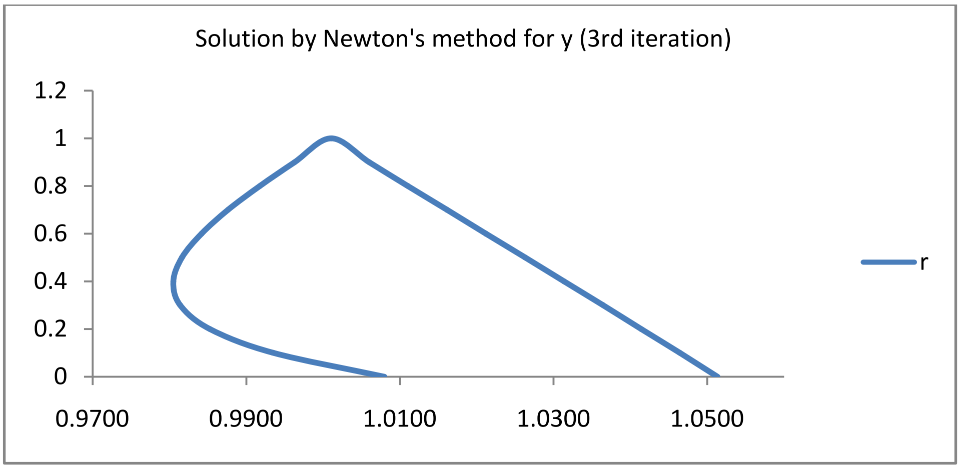

Figure 4: Solution for y by Newton's method

5 Conclusions

Thus this paper describes a procedure for solution of fuzzy nonlinear equations by a general iteration method based on fixed point theory. The method may be useful in modeling of nonlinear systems arising in Economics and Financial problems, where the parameters have vagueness and some degree of fuzziness. The developed algorithm has been implemented for numerical solution of a fuzzy nonlinear system of equations as an example to illustrate the efficiency of the developed algorithm.

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