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A Minkowski Type Inequality for Fuzzy Integrals

Hamzeh Agahi, M.A. Yaghoobi*

Department of Statistics, Faculty of Mathematics and Computer,
Shahid Bahonar University of Kerman, Kerman, Iran

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Abstract

One of the famous mathematical inequalities is Minkowski's inequality. It is an important inequality from both mathematical and application points of view. In this paper, a Minkowski type inequality for fuzzy integrals is studied. The established results are based on the classical Minkowski inequality for integrals. Moreover, a generalized Minkowski type inequality for fuzzy integrals is suggested. To illustrate the proposed inequalities some examples are given.

Keywords: Sugeno integral, fuzzy measure, Monotone function

1 Introduction

Fuzzy measure and fuzzy integrals can be used for modelling problems in non deterministic environment. Since Sugeno [22] initiated research on fuzzy measure and fuzzy integral, these area have been widely developed and a wide variety of topics have been investigated (see, e.g., [7-19] and references therein). Fuzzy integrals or Sugeno integrals have very interesting properties from a mathematical point of view which have been studied by many authors, including Ralescu and Adams [11], Roman-Flores *et al.* [12-19], Wang and Klir [24], among others. Ralescu and Adams [11] studied several equivalent definitions of fuzzy integrals. Román-Flores *et al.* [12, 13] developed the aspects of level-continuity and H-continuity of fuzzy integrals. The fuzzy integral for monotone functions was presented in [15]. A general overview on fuzzy integral measurement and fuzzy integration theory was presented by Wang and Klir [24]. In fact, fuzzy measures and fuzzy integrals are versatile operators which can be used in different areas. They have a broad use in information fusion, electronic auctions, decision making, and etc. Chen *et al.* [1] employed fuzzy integral and fuzzy measure to establish a public attitude analysis model. They applied their model to the gas taxi policy in Taipei City. Chen *et al.* [2] used fuzzy integral for face recognition. Narukawa and Torra [7] explored the use of fuzzy measures and fuzzy integrals to evaluate strategies in games. Fuzzy integral and fuzzy measure were applied to the problem of classifying highly confusable human non-speech sounds by Temko *et al.* [23].

The integral inequalities are useful results in several theoretical and applied fields. For instance, integral inequalities play a major role in the development of a time scales calculus. Özkan *et al.* [10] obtained Hölder's inequality, Minkowski's inequality and Jensen's inequality on time scales. Some famous inequalities have been generalized to fuzzy integral. Román-Flores and Chalco-Cano [14] analyzed an interesting type of geometric inequalities for fuzzy integral with some applications to convex geometry. Román-Flores *et al.* [16, 18] studied a Jensen type inequality and a convolution type inequality for fuzzy integrals. Also, they have investigated a Chebyshev type inequality and a Stolarsky type inequality for fuzzy integrals [3, 17]. In [3], a fuzzy Chebyshev inequality for a special case was obtained which has been generalized by Ouyang *et al.* [9]. Furthermore, Chybyshev type inequalities for fuzzy integral were proposed in a rather general form by Mesiar and Ouyang [5]. Recently, Román-Flores *et al.* [19] proved a Hardy type inequality for fuzzy integrals.

This paper intends to present a Minkowski type inequality for fuzzy integrals. The rest of this paper is organized as follows: in Section 2 some preliminaries and summarization of some previous known results are given. Section 3 proposes a Minkowski type inequality for fuzzy integrals. Section 4 deals with a generalized Minkowski type inequality. Finally, Section 5 contains a short conclusion.

*Corresponding author. Email: yaghoobi@mail.uk.ac.ir (M.A. Yaghoobi); Tel & Fax: +98 341 3220057.

2 Preliminaries

In this section, some definitions and basic properties of the Sugeno integral which will be used in the next sections are presented.

Definition 2.1 *([3, 9]) Let $\sum$ be a $\sigma-$algebra of subsets of R and let $\mu : \sum \to [0,\infty]$ be a nonnegative, extended real valued set function. μ is a fuzzy measure on R if*
$(FM1)$ $\mu(\varnothing) = 0$;
$(FM2)$ $E, F \in \sum$ *and* $E \subseteq F$ *imply* $\mu(E) \leq \mu(F)$;
$(FM3)$ $\{E_p\} \subseteq \sum, E_1 \subseteq E_2 \subseteq ...,$ *imply* $lim_{p\to\infty}\mu(E_p) = \mu\left(\bigcup_{p=1}^{\infty} E_p\right)$;
$(FM4)$ $\{E_p\} \subseteq \sum, E_1 \supseteq E_2 \supseteq ...,$ *imply* $lim_{p\to\infty}\mu(E_p) = \mu\left(\bigcap_{p=1}^{\infty} E_p\right)$.

When μ is a fuzzy measure, the triple $(X, \sum, \mu)$ is called a fuzzy measure space. Let f be a nonnegative real valued function defined on R. An $\alpha - level$ of f (for $\alpha > 0$) is denoted by $L_\alpha f = \{x \in R \mid f(x) \geq \alpha\} = \{f \geq \alpha\}$ and $L_0 f = \overline{\{x \in R \mid f(x) \geq 0\}}$. $L_0 f$ is called the support of f. Note that $\alpha \leq \beta$ implies $L_\beta f = \{f \geq \beta\} \subseteq L_\alpha f = \{f \geq \alpha\}$. If μ is a fuzzy measure on R, then $\aleph^\mu(R)$ is defined as follows

$$\aleph^\mu(R) = \{f : R \to [0,\infty) \mid f \text{ is } \mu - \text{measurable}\}.$$

Definition 2.2 *([11, 22]) Let μ be a fuzzy measure on R, $f \in \aleph^\mu(R)$, and $A \in \sum$, then the Sugeno integral (or fuzzy integral) of f on A, with respect to the fuzzy measure μ, is defined as*

$$(S)\int_A f d\mu = \bigvee_{\alpha\geq 0} [\alpha \wedge \mu(A \cap \{f \geq \alpha\})] \ , A \in \sum \tag{1}$$

where $\vee, \wedge$ denote the operations sup and inf on $[0,\infty)$, respectively. In particular, if $A = R$, then

$$(S)\int_R f d\mu = \bigvee_{\alpha\geq 0} [\alpha \wedge \mu\{f \geq \alpha\}].$$

The following properties of the Sugeno integral are well known [11].

Proposition 2.3 *([11]) If μ is a fuzzy measure on R and $f, g \in \aleph^\mu(R)$, then*
(i) $(S)\int_A f d\mu \leq \mu(A)$;
(ii) If $f \leq g$ *on* A, *then* $(S)\int_A f d\mu \leq (S)\int_A g d\mu$;
(iii) $(S)\int_A k d\mu = k \wedge \mu(A)$, *where* k *is a nonnegative constant.*

Román-Flores *et al.* have studied several fuzzy integral inequalities [14-19]. In particular, the following optimal fuzzy integral inequalities for monotone functions are proved in [15].

Theorem 2.4 *Let μ be the Lebesgue measure on R and let $g : [0,\infty] \to [0,\infty]$ be a continuous and strictly increasing function. If $(S)\int_0^a g d\mu = p$, then*

$$g(a-p) \geq (S)\int_0^a g d\mu = p, \ \forall \ a \geq 0. \tag{2}$$

Moreover, if $0 < p < a$, then $g(a-p) = p$.

An analogous result is obtained for the decreasing case.

Theorem 2.5 *Let μ be the Lebesgue measure on R and let $g : [0,\infty] \to [0,\infty]$ be a continuous and strictly decreasing function. If $(S)\int_0^a g d\mu = p$, then*

$$g(p) \geq (S)\int_0^a g d\mu = p, \quad \forall \ a \geq 0. \tag{3}$$

Moreover, if $0 < p < a$, then $g(p) = p$.

Ouyang *et al.* [8] proved the following two theorems which generalized the corresponding results in [15].

Theorem 2.6 *Let m be the Lebesgue measure on R and let $g : [0,\infty] \to [0,\infty]$ be a non-decreasing function. If $(S)\int_0^a gdm = p$, then*

$$g((a-p)+) \geq (S)\int_0^a gdm = p, \quad \forall\ a \geq 0 \tag{4}$$

where $g(x+) = \lim_{\varepsilon \longrightarrow 0^+} g(x+\varepsilon)$. Moreover, if $p < a$, and g is continuous at $a-p$, then

$$g((a-p)+) = g(a-p) = p.$$

Notice that if m is the Lebesgue measure and g is a non-decreasing function, then

$$g((a-p)+) \geq p \Longrightarrow (S)\int_0^a gdm \geq p.$$

Theorem 2.7 *Let m be the Lebesgue measure on R and let $g : [0,\infty] \to [0,\infty]$ be a non-increasing function. If $(S)\int_0^a gdm = p$, then*

$$g(p-) \geq (S)\int_0^a gdm = p, \quad \forall\ a \geq 0 \tag{5}$$

where $g(x-) = \lim_{\varepsilon \longrightarrow 0^+} g(x-\varepsilon)$. Moreover, if $p < a$, and g is continuous at p, then

$$g(p-) = g(p) = p.$$

Notice that if m is the Lebesgue measure and g is a non-increasing function, then

$$g(p-) \geq p \Longrightarrow (S)\int_0^a gdm \geq p.$$

3 Minkowski's Inequality for Fuzzy Integrals

The classical Minkowski inequality was published by Minkowski [6] in his famous book 'Geometrie der Zahlen'. A proof of Minkowski's inequality as well as several extensions, related results, and interesting geometrical interpretations can be found in [20, 21]. An extension of Minkowski's inequality, which is based on Hölder's inequality, is given in [24, 31-32]. Applications of Minkowski's inequality have been studied by many authors. For example Özkan *et al.* [10] applied Minkowski's inequality, Hölders inequality and Jensen's inequality on time scales. Lu *et al.* [4] used Minkowski's inequality for fast full search in motion estimation.

The classical Minkowski inequality [6] is as follows

$$\left(\int_0^1 (f+g)^s\, d\mu\right)^{\frac{1}{s}} \leq \left(\int_0^1 f^s d\mu\right)^{\frac{1}{s}} + \left(\int_0^1 g^s d\mu\right)^{\frac{1}{s}} \tag{6}$$

where $1 \leq s < \infty$, and $f, g : [0,1] \to [0,\infty)$ are two nonnegative functions. The aim of this section is to show the Minkowski inequality for the Sugeno integral.

Theorem 3.1 *Let $f, g : [0,1] \to [0,\infty)$ be two real valued functions and let μ be the Lebesgue measure on R. If f, g are both continuous and strictly decreasing functions, then the inequality*

$$\left((S)\int_0^1 (f+g)^s\, d\mu\right)^{\frac{1}{s}} \leq \left((S)\int_0^1 f^s d\mu\right)^{\frac{1}{s}} + \left((S)\int_0^1 g^s d\mu\right)^{\frac{1}{s}} \tag{7}$$

holds for all $1 \leq s < \infty$.

Proof: Let $(S)\int_0^1 (f+g)^s\, d\mu = r$, $(S)\int_0^1 f^s d\mu = p$ and $(S)\int_0^1 g^s d\mu = q$, where $1 \leq s < \infty$. The proof is trivial for $p = 1$ or $p = 0$ ($q = 1$ or $q = 0$) due to Proposition 2.3(i). Let $p, q \in (0,1)$. Then, Theorem 2.5 implies that

$$(f+g)^s(r) \geq r; \quad f^s(p) = p; \quad g^s(q) = q. \tag{8}$$

Then,

$$(f+g)(r) \geq r^{\frac{1}{s}}; \quad f(p) = p^{\frac{1}{s}}; \quad g(q) = q^{\frac{1}{s}}. \tag{9}$$

Now, on the contrary suppose that

$$r^{\frac{1}{s}} > p^{\frac{1}{s}} + q^{\frac{1}{s}}. \tag{10}$$

$$r^{\frac{1}{s}} > p^{\frac{1}{s}} + q^{\frac{1}{s}} \Longrightarrow \begin{cases} r^{\frac{1}{s}} > p^{\frac{1}{s}} & \Longrightarrow \quad r > p, \\ \\ r^{\frac{1}{s}} > q^{\frac{1}{s}} & \Longrightarrow \quad r > q. \end{cases} \tag{11}$$

Since f and g are decreasing functions, (9) and (11) imply that

$$f(r) < f(p) = p^{\frac{1}{s}} \tag{12}$$

and

$$g(r) < g(q) = q^{\frac{1}{s}}. \tag{13}$$

(9), (12) and (13) imply that

$$r^{\frac{1}{s}} \leq (f+g)(r) = f(r) + g(r) < p^{\frac{1}{s}} + q^{\frac{1}{s}},$$

which is a contradiction to (10). Hence $r^{\frac{1}{s}} \leq p^{\frac{1}{s}} + q^{\frac{1}{s}}$ and the proof is completed.

Example 1: Let f and g be two real valued functions defined as $f(x) = 1 - x$ and $g(x) = 1 - x^2$ where $x \in [0,1]$. Both f and g are strictly decreasing functions. In (7), let $s = 1$. A straightforward calculus shows that

$$(i)\ \ (S)\int_0^1 f(x)d\mu = \bigvee_{\alpha\in[0,1]} [\alpha \wedge \mu(\{1 - x \geq \alpha\})] = \bigvee_{\alpha\in[0,1]} [\alpha \wedge (1-\alpha)] = 0.5,$$

$$(ii)\ \ (S)\int_0^1 g(x)d\mu = \bigvee_{\alpha\in[0,1]} [\alpha \wedge \mu(\{1 - x^2 \geq \alpha\})] = \bigvee_{\alpha\in[0,1]} \left[\alpha \wedge \left(\sqrt{1-\alpha}\right)\right] = 0.618\,03,$$

$$\begin{aligned}(iii)\ \ (S)\int_0^1 (f+g)(x)d\mu &= \bigvee_{\alpha\in[0,2]} [\alpha \wedge \mu(\{-x^2 - x + 2 \geq \alpha\})] \\ &= \bigvee_{\alpha\in[0,2]} \left[\alpha \wedge \left(-\frac{1}{2} + \frac{1}{2}\sqrt{(9-4\alpha)}\right)\right] = 0.\,732\,05.\end{aligned}$$

Therefore,

$$0.\,732\,05 = (S)\int_0^1 (f+g)\,d\mu \leq \left((S)\int_0^1 f d\mu\right) + \left((S)\int_0^1 g d\mu\right) = 0.5 + 0.618\,03 = 1.\,118.$$

Theorem 3.2 *Let $f, g : [0,1] \to [0,\infty)$ be two real valued functions and let μ be the Lebesgue measure on R. If f, g are both continuous and strictly increasing functions, then the inequality*

$$\left((S)\int_0^1 (f+g)^s\,d\mu\right)^{\frac{1}{s}} \leq \left((S)\int_0^1 f^s d\mu\right)^{\frac{1}{s}} + \left((S)\int_0^1 g^s d\mu\right)^{\frac{1}{s}} \tag{14}$$

holds for all $1 \leq s < \infty$.

Proof: The proof is similar to that of Theorem 3.1.

Remark 1: Note that the inequalities (7) and (14) do not hold when f and g have different monotony. This matter is illustrated by the next example.

Example 2: Let $X = [0,1]$, $f(x) = x^2$, $g(x) = 1 - x^2$, $s = 1$ and $\mu(X) = m^2(X)$ where m is the Lebesgue measure on R. A straightforward calculus shows that

$$(S)\int (f+g)\,d\mu = 1,$$

$$(S)\int f d\mu = \bigvee_{\alpha\in[0,1]} \left[\alpha \wedge \left(1-\sqrt{\alpha}\right)^2\right] = \frac{1}{4},\ (S)\int g d\mu = \bigvee_{\alpha\in[0,1]} [\alpha \wedge (1-\alpha)] = \frac{1}{2},$$

but

$$1 = \left((S)\int (f+g)\,dm\right) > \left((S)\int f dm\right) + \left((S)\int g dm\right) = \frac{3}{4},$$

which violates (7) and (14).

4 A Generalized Minkowski's Inequality for Fuzzy Integrals

In this section, the inequalities (7) and (14) are generalized. In fact, the restrictions of continuous and strictly increasing (decreasing) are changed to a more general case as non-decreasing (non-increasing). Furthermore, $(S)\int_0^1 (.)d\mu$ is changed to the general form of $(S)\int_0^a (.)d\mu$, where $a \geq 1$. To prove the generalized inequalities the following lemma is needed.

Lemma 4.1 *([9]) Let $(S)\int_A f d\mu = p < \infty$. Then $\forall r \geq p$, $(S)\int_A f d\mu = (S)\int_0^r \mu(A \cap \{f \geq \alpha\})dm$, where m is the Lebesgue measure.*

Theorem 4.2 *Let μ be an arbitrary fuzzy measure on $[0,a]$ and let $f, g : [0,a] \to R^+$ be two real valued measurable functions such that $(S)\int_0^a (f+g)^s d\mu \leq 1$. If f, g are both non-decreasing functions, then the inequality*

$$\left((S)\int_0^a (f+g)^s d\mu\right)^{\frac{1}{s}} \leq \left((S)\int_0^a f^s d\mu\right)^{\frac{1}{s}} + \left((S)\int_0^a g^s d\mu\right)^{\frac{1}{s}} \tag{15}$$

holds for all $1 \leq s < \infty$.

Proof: Denote $A(\alpha) = \mu([0,a] \cap \{f^s \geq \alpha\})$, $B(\alpha) = \mu([0,a] \cap \{g^s \geq \alpha\})$, and $C(\alpha) = \mu([0,a] \cap \{(f+g)^s \geq \alpha\})$. By Lemma 4.1, $(S)\int_0^a (f+g)^s d\mu = (S)\int_0^1 C(\alpha)dm$. Therefore, it suffices to prove

$$\left((S)\int_0^1 C(\alpha)dm\right)^{\frac{1}{s}} \leq \left((S)\int_0^1 A(\alpha)dm\right)^{\frac{1}{s}} + \left((S)\int_0^1 B(\alpha)dm\right)^{\frac{1}{s}}.$$

Let $(S)\int_0^1 C(\alpha)dm = r$, $p = (S)\int_0^1 A(\alpha)dm$, and $q = (S)\int_0^1 B(\alpha)dm$. The proof is trivial for $p = 1$ or $p = 0$ ($q = 1$ or $q = 0$) due to Proposition 2.3(i). Let $p, q \in (0,1)$. Since $A(\alpha)$, $B(\alpha)$ and $C(\alpha)$ are non-increasing with respect to α and m is a Lebesgue measure, Theorem 2.7 implies that $A(p-) = p$, $B(q-) = q$, and $C(r-) \geq r$. Now, on the contrary suppose that

$$r^{\frac{1}{s}} > p^{\frac{1}{s}} + q^{\frac{1}{s}}. \tag{16}$$

Since $(p+q)^{\frac{1}{s}} \leq p^{\frac{1}{s}} + q^{\frac{1}{s}}$ where $s \geq 1$, then

$$r^{\frac{1}{s}} > (p+q)^{\frac{1}{s}} \quad \Longrightarrow \quad r > (p+q),$$

and

$$r^{\frac{1}{s}} > p^{\frac{1}{s}} + q^{\frac{1}{s}} \quad \Longrightarrow \quad r > \left(p^{\frac{1}{s}} + q^{\frac{1}{s}}\right)^s.$$

Thus

$$\begin{aligned} \{(f+g)^s \geq r\} &\subset \left\{(f+g)^s \geq \left(p^{\frac{1}{s}} + q^{\frac{1}{s}}\right)^s\right\} = \left\{(f+g) \geq \left(p^{\frac{1}{s}} + q^{\frac{1}{s}}\right)\right\} \\ &\subset \left\{f \geq p^{\frac{1}{s}}\right\} \cup \left\{g \geq q^{\frac{1}{s}}\right\} = \{f^s \geq p\} \cup \{g^s \geq q\}. \end{aligned}$$

Also

$$\begin{aligned} r &\leq C(r-) = \lim_{\varepsilon \longrightarrow 0^+} \mu([0,a] \cap \{(f+g)^s \geq r - \varepsilon\}) \\ &\leq \lim_{\varepsilon \longrightarrow 0^+} \mu([0,a] \cap [\{f^s \geq p - \varepsilon\} \cup \{g^s \geq q - \varepsilon\}]) \\ &\leq \lim_{\varepsilon \longrightarrow 0^+} \mu([0,a] \cap \{f^s \geq p - \varepsilon\}) + \lim_{\varepsilon \longrightarrow 0^+} \mu([0,a] \cap \{g^s \geq q - \varepsilon\}) = A(p-) + B(q-) = p + q, \end{aligned}$$

which is a contradiction to (16). Hence $r^{\frac{1}{s}} \leq p^{\frac{1}{s}} + q^{\frac{1}{s}}$ and the proof is completed.

Example 3: Let $A = [0,5]$ and m be the Lebesgue measure. Let f and g be two real valued functions defined as $f(x) = \begin{cases} \frac{x}{16} & x \in [0, \frac{9}{2}) \\ \frac{2}{5} & x \in [\frac{9}{2}, 5] \end{cases}$ and $g(x) = \frac{\sqrt{x}}{4}$. Both f and g are non-decreasing functions. In (15), let

$s=2$. A straightforward calculus shows that

$$
\begin{aligned}
(i)\ (S)\int_0^5 f^2 dm &= \left(\bigvee_{\alpha\in[0,7.9102\times10^{-2})}\left[\alpha\wedge\mu\left(\left\{\left(\frac{x}{16}\right)^2\geq\alpha\right\}\right)\right]\right)\\
&\quad\bigvee\left(\bigvee_{\alpha\in[7.9102\times10^{-2},.16]}\left[\alpha\wedge\mu\left(\left\{\left(\frac{2}{5}\right)^2\geq\alpha\right\}\right)\right]\right)\\
&= \left(\bigvee_{\alpha\in[0,7.9102\times10^{-2})}[\alpha\wedge(5-16\sqrt{\alpha})]\right)\bigvee\left(\bigvee_{\alpha\in[7.9102\times10^{-2},.16]}[\alpha\wedge0.5]\right)\\
&= 7.9102\times10^{-2}\vee0.16=0.16,
\end{aligned}
$$

$$
(ii)\ (S)\int_0^5 g^2 dm = \bigvee_{\alpha\in[0,\frac{5}{16}]}\left[\alpha\wedge\mu\left(\left\{\left(\frac{\sqrt{x}}{4}\right)^2\geq\alpha\right\}\right)\right]=\bigvee_{\alpha\in[0,\frac{5}{16}]}[\alpha\wedge(5-16\alpha)]=0.29412,
$$

$$
\begin{aligned}
(iii)\ (S)\int_0^5 (f+g)^2 dm &= \left(\bigvee_{\alpha\in[0,.65866)}\left[\alpha\wedge\mu\left(\left\{\left(\frac{x}{16}+\frac{\sqrt{x}}{4}\right)^2\geq\alpha\right\}\right)\right]\right)\\
&\quad\bigvee\left(\bigvee_{\alpha\in[.65866,.91971]}\left[\alpha\wedge\mu\left(\left\{\left(\frac{2}{5}+\frac{\sqrt{x}}{4}\right)^2\geq\alpha\right\}\right)\right]\right)\\
&= \bigvee_{\alpha\in[0,.65866]}\left[\alpha\wedge\left(5-\left(8-8\sqrt{(1+4\sqrt{\alpha})}+16\sqrt{\alpha}\right)\right)\right]\\
&\quad\bigvee\left(\bigvee_{\alpha\in[.65866,.91971]}\left[\alpha\wedge\left(5-\left(\frac{64}{25}-\frac{64}{5}\sqrt{\alpha}+16\alpha\right)\right)\right]\right)\\
&= 0.63268\vee0.82913=0.82913.
\end{aligned}
$$

Therefore,

$$
0.91057=\left((S)\int_0^5 (f+g)^2 dm\right)^{\frac{1}{2}}\leq\left((S)\int_0^5 f^2 dm\right)^{\frac{1}{2}}+\left((S)\int_0^5 g^2 dm\right)^{\frac{1}{2}}=0.94233.
$$

Theorem 4.3 *Let μ be an arbitrary fuzzy measure on $[0,a]$ and let $f,g:[0,a]\to R^+$ be two real valued measurable functions such that $(S)\int_0^a (f+g)^s d\mu\leq1$. If f,g are both non-increasing functions, then the inequality*

$$
\left((S)\int_0^a (f+g)^s d\mu\right)^{\frac{1}{s}}\leq\left((S)\int_0^a f^s d\mu\right)^{\frac{1}{s}}+\left((S)\int_0^a g^s d\mu\right)^{\frac{1}{s}} \tag{17}
$$

holds for all $1\leq s<\infty$.

Proof: The proof is similar to that of Theorem 4.2.

Example 4: Let $A=[0,2]$ and m be the Lebesgue measure. Let f and g be two real valued functions defined as $f(x)=\frac{1}{x+3}$ and $g(x)=\begin{cases}1-x & x\in[0,\frac{1}{2})\\ \frac{1}{10} & x\in[\frac{1}{2},2].\end{cases}$ Both f and g are non-increasing functions. In (17), let $s=1$. A straightforward calculus shows that

$$
\begin{aligned}
(i)\ (S)\int_0^2 f dm &= \bigvee_{\alpha\in[0,0.33333]}\left[\alpha\wedge\mu\left(\left\{\frac{1}{x+3}\geq\alpha\right\}\right)\right]=\bigvee_{\alpha\in[0,0.33333]}\left[\alpha\wedge\left(\frac{1-3\alpha}{\alpha}\right)\right]\\
&= 0.30278,
\end{aligned}
$$

$$(ii)\ \ (S)\int_0^2 gdm = \left(\bigvee_{\alpha\in[0,0.1]}\left[\alpha\wedge\mu\left(\left\{\frac{1}{10}\geq\alpha\right\}\right)\right]\right)\bigvee\left(\bigvee_{\alpha\in(0.1,1]}\left[\alpha\wedge\mu\left(\{1-x\geq\alpha\}\right)\right]\right)$$

$$\left(\bigvee_{\alpha\in[0,0.1)}[\alpha\wedge 1.5]\right)\bigvee\left(\bigvee_{\alpha\in[0.1,1]}[\alpha\wedge(1-\alpha)]\right)$$

$$= 0.1\vee 0.5 = 0.5,$$

$$(iii)\ \ (S)\int_0^2 (f+g)\,dm = \left(\bigvee_{\alpha\in[0,0.\,385\,71]}\left[\alpha\wedge\mu\left(\left\{\left(\frac{1}{x+3}+\frac{1}{10}\right)\geq\alpha\right\}\right)\right]\right)$$

$$\bigvee\left(\bigvee_{\alpha\in(0.\,385\,71,\,1.\,333\,3]}\left[\alpha\wedge\mu\left(\left\{\left(\frac{1}{x+3}+1-x\right)\geq\alpha\right\}\right)\right]\right)$$

$$= \left(\bigvee_{\alpha\in[0,0.\,385\,71]}\left[\alpha\wedge\left(\frac{3-2\alpha}{2\alpha-1}\right)\right]\right)$$

$$\bigvee\left(\bigvee_{\alpha\in(0.\,385\,71,\,1.\,333\,3]}\left[\alpha\wedge\left(-1-\frac{1}{2}\alpha+\frac{1}{2}\sqrt{(20-8\alpha+\alpha^2)}\right)\right]\right)$$

$$= 0.\,394\,59\vee 0.\,637\,46 = 0.\,637\,46.$$

Therefore,

$$0.\,637\,46 = \left((S)\int_0^2 (f+g)\,dm\right)\leq\left((S)\int_0^2 fdm\right)+\left((S)\int_0^2 gdm\right) = 0.\,802\,78.$$

5 Conclusion

The classical Minkowski inequality is an important result in theoretical and applied fields. This paper proposed a Minkowski type inequality for fuzzy integrals based on the classical one. Moreover, a generalized Minkowski's inequality for fuzzy integrals is introduced. To illustrate the proposed inequalities some examples are solved.

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