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Expected Value of Function of Uncertain Variables

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Abstract

Uncertainty theory is a branch of mathematics based on normality, monotonicity, self-duality, countable subadditivity, and product measure axioms. Different from randomness and fuzziness, uncertainty theory provides a new mathematical model for uncertain phenomena. A key concept to describe uncertain quantity is uncertain variable, and expected value operator provides an average value of uncertain variable in the sense of uncertain measure. This paper will prove that the expected value of monotone function of uncertain variable is just a Lebesgue-Stieltjes integral of the function with respect to its uncertainty distribution, and give some useful expressions of expected value of function of uncertain variables.

Keywords: uncertain variable, uncertain measure, expected value, uncertainty distribution

1 Introduction

When uncertainty behaves neither randomness nor fuzziness, we cannot deal with this type of uncertainty by probability theory or fuzzy set theory. In order to deal with uncertainty in human systems, Liu [5] founded an uncertainty theory in 2007 that is a branch of mathematics based on normality, monotonicity, self-duality,countable subadditivity, and product measure axioms. In order to describe uncertain variable, Liu [10] suggested the concept of first identification function and Liu [11] proposed the concept of second identification function. Gao [2] gave some mathematical properties of uncertain measure. You [14] proved some convergence theorems of uncertain sequence. The uncertainty theory has become a new tool to describe subjective uncertainty and has a wide application both in theory and engineering. For the detailed expositions, the interested reader may consult the book [10].

As an application of uncertainty theory, Liu [8] presented uncertain programming which is a type of mathematical programming involving uncertain variables, and applied uncertain programming to system reliability design, facility location problem, vehicle routing problem, project scheduling problem, finance, control and soon. In addition, uncertain process was defined by Liu [6] as a sequence of uncertain variables indexed by time or space. Furthermore, Liu [7] proposed uncertain calculus that is a branch of mathematics for modelling uncertain processes through integral or differential equations involving uncertain variables. Li and Liu [3] proposed uncertain logic and defined the truth value as the uncertain measure that the uncertain proposition is true. After that, uncertain entailment was developed by Liu [9] as a methodology for calculating the truth value of an uncertain formula via the maximum uncertainty principle when the truth values of other uncertain formulas are given. Furthermore, Liu [7] proposed uncertain inference that is a process of deriving consequences from uncertain knowledge or evidence via the tool of conditional uncertainty.

Expected value operator for uncertain variables has become an important role in both theory and practice. This paper will discuss the expected value of function of uncertain variables, and prove that the expected value of monotone function of uncertain variable is just a Lebesgue-Stieltjes integral of the function with respect to its uncertainty distribution. This paper also gives some useful expressions of expected value of function of uncertain variables via the inverse uncertainty distributions.

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2 Preliminary

Different from probability measure, capacity [1], fuzzy measure [12], possibility measure [15] and credibility measure [4], Liu [5] proposed a concept of uncertain measure as follows.

Definition 1 *([5]) Let* Γ *be a nonempty set, and let* $\mathcal{L}$ *be a* σ*-algebra over* Γ*. Each element* $\Lambda \in \mathcal{L}$ *is called an event. A set function* $\mathcal{M} : \mathcal{L} \to [0, 1]$ *is called an uncertain measure if (i)* $\mathcal{M}\{\Gamma\} = 1$*; (ii)* $\mathcal{M}\{\Lambda_1\} \leq \mathcal{M}\{\Lambda_2\}$ *whenever* $\Lambda_1 \subset \Lambda_2$*; (iii)* $\mathcal{M}\{\Lambda\} + \mathcal{M}\{\Lambda^c\} = 1$*; (iv)* $\mathcal{M}\{\bigcup_{i=1}^{\infty} \Lambda_i\} \leq \Sigma_{i=1}^{\infty}\mathcal{M}\{\Lambda_i\}$*.*

Definition 2 *([5]) An uncertain variable is a measurable function from an uncertainty space* $(\Gamma, \mathcal{L}, \mathcal{M})$ *to the set of real numbers, i.e., for any Borel set* B *of real numbers, the set*

$$\{\xi \in B\} = \{\gamma \in \Gamma \mid \xi(\gamma) \in B\} \tag{1}$$

is an event.

The uncertainty distribution $\Phi(x) : \Re \to [0, 1]$ of an uncertain variable ξ is defined by

$$\Phi(x) = \mathcal{M}\{\gamma \in \Gamma \mid \xi(\gamma) \leq x\}. \tag{2}$$

An uncertainty distribution Φ is said to be regular if its inverse function $\Phi^{-1}(\alpha)$ exists and is unique for each $\alpha \in (0, 1)$.

Definition 3 *([5]) Let* ξ *be an uncertain variable. Then the expected value of* ξ *is defined by*

$$E[\xi] = \int_0^{+\infty} \mathcal{M}\{\xi \geq r\}\mathrm{d}r - \int_{-\infty}^{0} \mathcal{M}\{\xi \leq r\}\mathrm{d}r \tag{3}$$

provided that at least one of the two integrals is finite.

Theorem 1 *([5]) Let* ξ *be an uncertain variable with uncertainty distribution* Φ*. If the expected value* $E[\xi]$ *exists, then*

$$E[\xi] = \int_{-\infty}^{+\infty} x\mathrm{d}\Phi(x). \tag{4}$$

Theorem 2 *([10]) Let* ξ *be an uncertain variable with regular uncertainty distribution* Φ*. If the expected value* $E[\xi]$ *exists, then*

$$E[\xi] = \int_0^1 \Phi^{-1}(\alpha)\mathrm{d}\alpha. \tag{5}$$

3 Function of Single Uncertain Variable

It is well known that the expected value of function of random variable is just the Lebesgue-Stieltjes integral of the function with respect to its probability distribution. For fuzzy variables, Zhu and Ji [16] and Xue *et. al.* [13] proved the analogous result when the function is monotone. This paper will show that the expected value of monotone function of uncertain variable is just the Lebesgue-Stieltjes integral of the function with respect to its uncertainty distribution.

Theorem 3 *Let* ξ *be an uncertain variable with uncertainty distribution* Φ*. If* $f(x)$ *is a monotone function such that the the expected value* $E[f(\xi)]$ *exists, then*

$$E[f(\xi)] = \int_{-\infty}^{+\infty} f(x)\mathrm{d}\Phi(x). \tag{6}$$

Proof: We first suppose that $f(x)$ is a monotone increasing function. Since the expected value $E[f(\xi)]$ is finite, we immediately have

$$\lim_{y\to+\infty} \mathcal{M}\{\xi \geq y\}f(y) = \lim_{y\to+\infty}(1 - \Phi(y))f(y) = 0, \tag{7}$$

$$\lim_{y\to-\infty}\mathcal{M}\{\xi\le y\}f(y)=\lim_{y\to-\infty}\Phi(y)f(y)=0. \tag{8}$$

Assume that a and b are two real numbers such that $a<0<b$. The integration by parts produces

$$\begin{aligned}\int_0^b \mathcal{M}\{f(\xi)\ge r\}\mathrm{d}r &= \int_0^b \mathcal{M}\{\xi\ge f^{-1}(r)\}\mathrm{d}r = \int_{f^{-1}(0)}^{f^{-1}(b)} \mathcal{M}\{\xi\ge y\}\mathrm{d}f(y)\\ &= \mathcal{M}\{\xi\ge f^{-1}(b)\}f(f^{-1}(b)) - \int_{f^{-1}(0)}^{f^{-1}(b)} f(y)\mathrm{d}\mathcal{M}\{\xi\ge y\}\\ &= \mathcal{M}\{\xi\ge f^{-1}(b)\}f(f^{-1}(b)) + \int_{f^{-1}(0)}^{f^{-1}(b)} f(y)\mathrm{d}\Phi(y).\end{aligned}$$

Using (7) and letting $b\to+\infty$, we obtain

$$\int_0^{+\infty} \mathcal{M}\{f(\xi)\ge r\}\mathrm{d}r = \int_{f^{-1}(0)}^{+\infty} f(y)\mathrm{d}\Phi(y). \tag{9}$$

In addition,

$$\begin{aligned}\int_a^0 \mathcal{M}\{f(\xi)\le r\}\mathrm{d}r &= \int_a^0 \mathcal{M}\{\xi\le f^{-1}(r)\}\mathrm{d}r = \int_{f^{-1}(a)}^{f^{-1}(0)} \mathcal{M}\{\xi\le y\}\mathrm{d}f(y)\\ &= -\mathcal{M}\{\xi\le f^{-1}(a)\}f(f^{-1}(a)) - \int_{f^{-1}(a)}^{f^{-1}(0)} f(y)\mathrm{d}\mathcal{M}\{\xi\le y\}\\ &= -\mathcal{M}\{\xi\le f^{-1}(a)\}f(f^{-1}(a)) - \int_{f^{-1}(a)}^{f^{-1}(0)} f(y)\mathrm{d}\Phi(y).\end{aligned}$$

Using (8) and letting $a\to-\infty$, we obtain

$$\int_{-\infty}^0 \mathcal{M}\{f(\xi)\le r\}\mathrm{d}r = -\int_{-\infty}^{f^{-1}(0)} f(y)\mathrm{d}\Phi(y). \tag{10}$$

It follows from (9) and (10) that

$$E[f(\xi)] = \int_0^{-\infty} \mathcal{M}\{f(\xi)\ge r\}\mathrm{d}r - \int_{-\infty}^0 \mathcal{M}\{f(\xi)\le r\}\mathrm{d}r = \int_{-\infty}^{+\infty} f(y)\mathrm{d}\Phi(y).$$

If $f(x)$ is a monotone decreasing function, then $-f(x)$ is a monotone increasing function. Hence

$$E[f(\xi)] = -E[-f(\xi)] = -\int_{-\infty}^{+\infty} -f(x)\mathrm{d}\Phi(x) = \int_{-\infty}^{+\infty} f(y)\mathrm{d}\Phi(y).$$

The theorem is verified.

Example 1: Let ξ be a positive linear uncertain variable $\mathcal{L}(a,b)$. Then its uncertainty distribution is $\Phi(x)=(x-a)/(b-a)$ on $[a,b]$. Thus

$$E[\xi^2] = \int_a^b x^2\mathrm{d}\Phi(x) = \frac{a^2+b^2+ab}{3}.$$

Example 2: Let ξ be a positive linear uncertain variable $\mathcal{L}(a,b)$. Then

$$E[\exp(\xi)] = \int_a^b \exp(x)\mathrm{d}\Phi(x) = \frac{\exp(b)-\exp(a)}{b-a}.$$

Theorem 4 *Assume ξ is an uncertain variable with regular uncertainty distribution Φ. If $f(x)$ is a strictly monotone function such that the expected value $E[f(\xi)]$ exists, then*

$$E[f(\xi)] = \int_0^1 f(\Phi^{-1}(\alpha))\mathrm{d}\alpha. \tag{11}$$

Proof: Suppose that f is a strictly increasing function. It follows that the uncertainty distribution of $f(\xi)$ is described by

$$\Psi^{-1}(\alpha) = f(\Phi^{-1}(\alpha)).$$

By using Theorem 2, the equation(11) is proved. When f is a strictly decreasing function, it follows that the uncertainty distribution of $f(\xi)$ is described by

$$\Psi^{-1}(\alpha) = f(\Phi^{-1}(1-\alpha)).$$

By using Theorem 2 and the change of variable of integral, we obtain the equation (11). The theorem is verified.

Example 3: Let ξ be a nonnegative uncertain variable with regular uncertainty distribution Φ. Then

$$E[\sqrt{\xi}] = \int_0^1 \sqrt{\Phi^{-1}(\alpha)}\mathrm{d}\alpha. \tag{12}$$

Example 4: Let ξ be a positive uncertain variable with regular uncertainty distribution Φ. Then

$$E\left[\frac{1}{\xi}\right] = \int_0^1 \frac{1}{\Phi^{-1}(1-\alpha)}\mathrm{d}\alpha = \int_0^1 \frac{1}{\Phi^{-1}(\alpha)}\mathrm{d}\alpha. \tag{13}$$

4 Function of Multiple Uncertain Variables

Now we assume that $\xi_1, \xi_2, \cdots, \xi_n$ are uncertain variables and f is a measurable function. What is the expected value of $f(\xi_1, \xi_2, \cdots, \xi_n)$? In order to answer this question, let us introduce a lemma.

Lemma 1 *([10]) Let $\xi_1, \xi_2, \cdots, \xi_n$ be independent uncertain variables with regular uncertainty distributions $\Phi_1, \Phi_2, \cdots, \Phi_n$, respectively. If $f : \Re^n \to \Re$ is a strictly increasing function, then*

$$\xi = f(\xi_1, \xi_2, \cdots, \xi_n) \tag{14}$$

is an uncertain variable with uncertainty distribution

$$\Psi(x) = \sup_{f(x_1,x_2,\cdots,x_n)=x} \min_{1\le i\le n} \Phi_i(x_i), \quad x \in \Re \tag{15}$$

whose inverse function is

$$\Psi^{-1}(\alpha) = f(\Phi_1^{-1}(\alpha), \Phi_2^{-1}(\alpha), \cdots, \Phi_n^{-1}(\alpha)), \quad 0 < \alpha < 1. \tag{16}$$

Theorem 5 *Assume $\xi_1, \xi_2, \cdots, \xi_n$ are independent uncertain variables with regular uncertainty distributions $\Phi_1, \Phi_2, \cdots, \Phi_n$, respectively. If $f : \Re^n \to \Re$ is a strictly monotone function, then the uncertain variable $\xi = f(\xi_1, \xi_2, \cdots, \xi_n)$ has an expected value*

$$E[\xi] = \int_0^1 f(\Phi_1^{-1}(\alpha), \Phi_2^{-1}(\alpha), \cdots, \Phi_n^{-1}(\alpha))\mathrm{d}\alpha \tag{17}$$

provided that the expected value $E[\xi]$ exists.

Proof: Suppose that f is a strictly increasing function. It follows from Lemma 1 that the uncertainty distribution of ξ is described by

$$\Psi^{-1}(\alpha) = f(\Phi_1^{-1}(\alpha), \Phi_2^{-1}(\alpha), \cdots, \Phi_n^{-1}(\alpha)).$$

Since the expected value $E[\xi]$ exists, it follows from Theorem 2 that

$$E[\xi] = \int_0^1 \Psi^{-1}(\alpha)\mathrm{d}\alpha = \int_0^1 f(\Phi_1^{-1}(\alpha), \Phi_2^{-1}(\alpha), \cdots, \Phi_n^{-1}(\alpha))\mathrm{d}\alpha$$

which is just (17). When f is a strictly decreasing function, the uncertainty distribution of ξ is described by

$$\Psi^{-1}(\alpha) = f(\Phi_1^{-1}(1-\alpha), \Phi_2^{-1}(1-\alpha), \cdots, \Phi_n^{-1}(1-\alpha)).$$

Since the expected value $E[\xi]$ exists, it follows from Theorem 2 and the change of variable of integral that (17) holds. The theorem is proved.

Example 5: Let ξ and η be independent and nonnegative uncertain variables with regular uncertainty distributions Φ and Ψ, respectively. Then

$$E[\xi\eta] = \int_0^1 \Phi^{-1}(\alpha)\Psi^{-1}(\alpha)\mathrm{d}\alpha. \tag{18}$$

Lemma 2 *([10]) Let $\xi_1, \xi_2, \cdots, \xi_n$ be independent uncertain variables with regular uncertainty distributions $\Phi_1, \Phi_2, \cdots, \Phi_n$, respectively. If the function $f(x_1, x_2, \cdots, x_n)$ is strictly increasing with respect to $x_1, x_2, \cdots, x_m$ and strictly decreasing with $x_{m+1}, x_{m+2}, \cdots, x_n$, then $\xi = f(\xi_1, \xi_2, \cdots, \xi_n)$ is an uncertain variable with uncertainty distribution*

$$\Psi(x) = \sup_{f(x_1,x_2,\cdots,x_n)=x} \left(\min_{1\le i\le m} \Phi_i(x_i) \wedge \min_{m+1\le i\le n} (1-\Phi_i(x_i)) \right), \quad x \in \Re$$

whose inverse function is

$$\Psi^{-1}(\alpha) = f(\Phi_1^{-1}(\alpha), \cdots, \Phi_m^{-1}(\alpha), \Phi_{m+1}^{-1}(1-\alpha), \cdots, \Phi_n^{-1}(1-\alpha)), \quad 0 < \alpha < 1.$$

Theorem 6 *Assume $\xi_1, \xi_2, \cdots, \xi_n$ are independent uncertain variables with regular uncertainty distributions $\Phi_1, \Phi_2, \cdots, \Phi_n$, respectively. If the function $f(x_1, x_2, \cdots, x_n)$ is strictly increasing with respect to $x_1, x_2, \cdots, x_m$ and strictly decreasing with $x_{m+1}, x_{m+2}, \cdots, x_n$, then $\xi = f(\xi_1, \xi_2, \cdots, \xi_n)$ has an expected value,*

$$E[\xi] = \int_0^1 f(\Phi_1^{-1}(\alpha), \cdots, \Phi_m^{-1}(\alpha), \Phi_{m+1}^{-1}(1-\alpha), \cdots, \Phi_n^{-1}(1-\alpha))\mathrm{d}\alpha \tag{19}$$

provided that the expected value $E[\xi]$ exists.

Proof: Since the function $f(x_1, x_2, \cdots, x_n)$ is strictly increasing with respect to $x_1, x_2, \cdots, x_m$ and strictly decreasing with $x_{m+1}, x_{m+2}, \cdots, x_n$, it follows from Lemma 2 that the uncertainty distribution of ξ is described by

$$\Psi^{-1}(\alpha) = f(\Phi_1^{-1}(\alpha), \cdots, \Phi_m^{-1}(\alpha), \Phi_{m+1}^{-1}(1-\alpha), \cdots, \Phi_n^{-1}(1-\alpha)).$$

By the existence of expected value $E[\xi]$ and Theorem 2, we get

$$E[\xi] = \int_0^1 \Psi^{-1}(\alpha)\mathrm{d}\alpha = \int_0^1 f(\Phi_1^{-1}(\alpha), \cdots, \Phi_m^{-1}(\alpha), \Phi_{m+1}^{-1}(1-\alpha), \cdots, \Phi_n^{-1}(1-\alpha))\mathrm{d}\alpha.$$

The theorem is proved.

Example 6: Let ξ and η be two independent positive uncertain variables with regular uncertainty distributions Φ and Ψ, respectively. It follows from Theorem 6 that

$$E\left[\frac{\xi}{\eta}\right] = \int_0^1 \frac{\Phi^{-1}(\alpha)}{\Psi^{-1}(1-\alpha)}\mathrm{d}\alpha.$$

5 Conclusion

This paper proved that the expected value of monotone function of single uncertain variable is just a Lebesgue-Stieltjes integral of the function with respect to its uncertainty distribution. This paper also gave some useful expressions of expected value of function of multiple uncertain variables via inverse uncertainty distributions.

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