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Uncertain Risk Analysis and Uncertain Reliability Analysis

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Abstract

Uncertainty theory is a branch of axiomatic mathematics for modeling human uncertainty. This paper assumes that a system contains uncertain elements in the sense of uncertainty theory, and defines the risk as the “accidental loss” plus “uncertain measure of such loss”. Uncertain risk analysis is also presented as a tool to quantify risk via uncertainty theory. Finally, this paper discusses the tool of uncertain reliability analysis.

Keywords: uncertainty theory, risk analysis, risk index, hazard distribution, reliability analysis

1 Introduction

Some information and knowledge are usually represented by human language like “about 100km”, “approximately 39 °C”, “roughly 80kg”, “low speed”, “middle age”, and “big size”. How do we understand them? How do we model them? Perhaps some people think that they are subjective probability or they are fuzzy concepts. However, a lot of surveys showed that those imprecise quantities behave neither like randomness nor like fuzziness. This fact provides a motivation to invent uncertainty theory that was founded by Liu [6] in 2007 and refined by Liu [12] in 2010. In addition, Liu [8], Gao [3], You [15], Liu and Ha [13], and Peng and Iwamura [14] also made significant contributions to this research area. Nowadays uncertainty theory has become a branch of mathematics based on normality, monotonicity, self-duality, countable subadditivity, and product measure axioms.

As an application of uncertainty theory, uncertain programming was proposed by Liu [10] in 2009. Some scholars have applied uncertain programming to operations research, industrial engineering and management science successfully.

An uncertain process is essentially a sequence of uncertain variables indexed by time or space. The study of uncertain process was started by Liu [7] in 2008. Elementary renewal theorem and renewal reward theorem were proved by Liu [12] in 2010. Canonical process, proposed by Liu [8] in 2009, is a Lipschitz continuous uncertain process that has stationary and independent increments and every increment is a normal uncertain variable. Uncertain calculus was then developed as a branch of mathematics that deals with differentiation and integration of function of uncertain processes. Uncertain differential equation was proposed by Liu [7] in 2008 as a type of differential equation driven by canonical process. After that, an existence and uniqueness theorem of solution of uncertain differential equation was proved by Chen and Liu [2] in 2010. Uncertainty differential equations were also applied to finance, including uncertain stock models and uncertain insurance models. In addition, Zhu [16] derived an optimal control policy when the system is characterized by an uncertain differential equation.

Uncertain logic is a generalization of mathematical logic for dealing with uncertain knowledge via uncertainty theory. The basic model is uncertain propositional logic designed by Li and Liu [5] in which the truth value of an uncertain proposition is defined as the uncertain measure that the proposition is true. In addition, uncertain entailment, developed by Liu [9] in 2009, is a methodology for calculating the truth value of an uncertain formula via the maximum uncertainty principle when the truth values of other uncertain formulas are given.

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Uncertain set theory was proposed by Liu [11] in 2010 as a generalization of uncertainty theory to the domain of uncertain sets. Uncertain inference is a process of deriving consequences from uncertain knowledge or evidence via the tool of conditional uncertain set. The first inference rule was proposed by Liu [11] in 2010. Then Gao, Gao and Ralescu [4] extended the inference rule to the case with multiple antecedents and with multiple if-then rules.

Uncertain statistics is a methodology for collecting and interpreting expert's experimental data by uncertainty theory. The study of uncertain statistics was started by Liu [12] in 2010 in which a questionnaire survey for collecting expert's experimental data was designed and a principle of least squares for estimating uncertainty distributions was suggested.

Risk analysis and reliability analysis have been discussed widely in literature, for example, Bedford and Cooke [1]. In this paper, the risk is defined as the "accidental loss" plus "uncertain measure of such loss". This paper will introduce a definition of risk index and provide some useful formulas for calculating risk index, thus producing an uncertain risk analysis that is a tool to quantify risk via uncertainty theory. Uncertain reliability analysis will be also presented as a tool to deal with system reliability via uncertainty theory.

2 Risk Index

A system usually contains uncertain factors, for example, lifetime, demand, production rate, cost, profit, and resource. Risk index is defined as the uncertain measure that some specified loss occurs.

Definition 1 *Assume a system contains uncertain variables $\xi_1, \xi_2, \ldots, \xi_n$, and there is a loss function L such that some specified loss occurs if and only if $L(\xi_1, \xi_2, \ldots, \xi_n) \le 0$. Then the risk index is*

$$Risk = \mathcal{M}\{L(\xi_1, \xi_2, \ldots, \xi_n) \le 0\}. \tag{1}$$

Example 1: Consider a series system in which there are n elements whose lifetimes are independent uncertain variables $\xi_1, \xi_2, \ldots, \xi_n$ with uncertainty distributions $\Phi_1, \Phi_2, \ldots, \Phi_n$, respectively. Such a system fails if any one element does not work. Thus the system lifetime

$$\xi = \xi_1 \wedge \xi_2 \wedge \cdots \wedge \xi_n \tag{2}$$

is an uncertain variable with uncertainty distribution

$$\Psi(x) = \Phi_1(x) \vee \Phi_2(x) \vee \cdots \vee \Phi_n(x). \tag{3}$$

If the loss is understood as the case that the system fails before time T, i.e.,

$$L(\xi_1, \xi_2, \ldots, \xi_n) = \xi_1 \wedge \xi_2 \wedge \cdots \wedge \xi_n - T, \tag{4}$$

then the risk index is

$$Risk = \mathcal{M}\{L(\xi_1, \xi_2, \ldots, \xi_n) \le 0\} = \mathcal{M}\{\xi_1 \wedge \xi_2 \wedge \cdots \wedge \xi_n \le T\} = \Phi_1(T) \vee \Phi_2(T) \vee \cdots \vee \Phi_n(T). \tag{5}$$

Example 2: Consider a parallel system in which there are n elements whose lifetimes are independent uncertain variables $\xi_1, \xi_2, \ldots, \xi_n$ with uncertainty distributions $\Phi_1, \Phi_2, \ldots, \Phi_n$, respectively. Such a system fails if all elements do not work. Thus the system lifetime

$$\xi = \xi_1 \vee \xi_2 \vee \cdots \vee \xi_n \tag{6}$$

is an uncertain variable with uncertainty distribution

$$\Psi(x) = \Phi_1(x) \wedge \Phi_2(x) \wedge \cdots \wedge \Phi_n(x). \tag{7}$$

If the loss is understood as the case that the system fails before time T, i.e.,

$$L(\xi_1, \xi_2, \ldots, \xi_n) = \xi_1 \vee \xi_2 \vee \cdots \vee \xi_n - T, \tag{8}$$

then the risk index is

$$Risk = \mathcal{M}\{L(\xi_1, \xi_2, \ldots, \xi_n) \le 0\} = \mathcal{M}\{\xi_1 \vee \xi_2 \vee \cdots \vee \xi_n \le T\} = \Phi_1(T) \wedge \Phi_2(T) \wedge \cdots \wedge \Phi_n(T). \tag{9}$$

Theorem 1 *(Risk Index Theorem) Assume that $\xi_1, \xi_2, \ldots, \xi_n$ are independent uncertain variables with uncertainty distributions $\Phi_1, \Phi_2, \ldots, \Phi_n$, respectively, and L is a strictly increasing function. If some specified loss occurs if and only if $L(\xi_1, \xi_2, \ldots, \xi_n) \le 0$, then the risk index is*

$$Risk = \alpha \tag{10}$$

where α is the root of

$$L(\Phi_1^{-1}(\alpha), \Phi_2^{-1}(\alpha), \ldots, \Phi_n^{-1}(\alpha)) = 0. \tag{11}$$

Proof: It follows from the operational law that $L(\xi_1, \xi_2, \ldots, \xi_n)$ is an uncertain variable whose inverse uncertainty distribution is

$$\Psi^{-1}(\alpha) = L(\Phi_1^{-1}(\alpha), \Phi_2^{-1}(\alpha), \ldots, \Phi_n^{-1}(\alpha)).$$

Since $Risk = \mathcal{M}\{L(\xi_1, \xi_2, \ldots, \xi_n) \le 0\} = \Psi(0)$, we get (10).

Theorem 2 *(Risk Index Theorem) Assume that $\xi_1, \xi_2, \ldots, \xi_n$ are independent uncertain variables with uncertainty distributions $\Phi_1, \Phi_2, \ldots, \Phi_n$, respectively, and L is a strictly decreasing function. If some specified loss occurs if and only if $L(\xi_1, \xi_2, \ldots, \xi_n) \le 0$, then the risk index is*

$$Risk = \alpha \tag{12}$$

where α is the root of

$$L(\Phi_1^{-1}(1-\alpha), \Phi_2^{-1}(1-\alpha), \ldots, \Phi_n^{-1}(1-\alpha)) = 0. \tag{13}$$

Proof: It follows from the operational law that $L(\xi_1, \xi_2, \ldots, \xi_n)$ is an uncertain variable whose inverse uncertainty distribution is

$$\Psi^{-1}(\alpha) = L(\Phi_1^{-1}(1-\alpha), \Phi_2^{-1}(1-\alpha), \ldots, \Phi_n^{-1}(1-\alpha)).$$

Since $Risk = \mathcal{M}\{L(\xi_1, \xi_2, \ldots, \xi_n) \le 0\} = \Psi(0)$, we get (12).

Theorem 3 *(Risk Index Theorem) Assume that $\xi_1, \xi_2, \ldots, \xi_n$ are independent uncertain variables with uncertainty distributions $\Phi_1, \Phi_2, \ldots, \Phi_n$, respectively, and the function $L(x_1, x_2, \ldots, x_n)$ is strictly increasing with respect to $x_1, x_2, \ldots, x_m$ and strictly decreasing with respect to $x_{m+1}, x_{m+2}, \ldots, x_n$. If some specified loss occurs if and only if $L(\xi_1, \xi_2, \ldots, \xi_n) \le 0$, then the risk index is*

$$Risk = \alpha \tag{14}$$

where α is the root of

$$L(\Phi_1^{-1}(\alpha), \ldots, \Phi_m^{-1}(\alpha), \Phi_{m+1}^{-1}(1-\alpha), \ldots, \Phi_n^{-1}(1-\alpha)) = 0. \tag{15}$$

Proof: It follows from the operational law that $L(\xi_1, \xi_2, \ldots, \xi_n)$ is an uncertain variable whose inverse uncertainty distribution is

$$\Psi^{-1}(\alpha) = L(\Phi_1^{-1}(\alpha), \ldots, \Phi_m^{-1}(\alpha), \Phi_{m+1}^{-1}(1-\alpha), \ldots, \Phi_n^{-1}(1-\alpha)).$$

Since $Risk = \mathcal{M}\{L(\xi_1, \xi_2, \ldots, \xi_n) \le 0\} = \Psi(0)$, we get (14).

3 Hazard Distribution

Suppose that ξ is the lifetime of some system/element. Here we assume it is an uncertain variable with a prior uncertainty distribution. At some time t, it is observed that the system/element is working. What is the residual lifetime of the system/element? The following definition answers this question.

Definition 2 *Let ξ be a nonnegative uncertain variable representing lifetime of some system/element. If ξ has a prior uncertainty distribution Φ, then the hazard distribution (or failure distribution) at time t is*

$$\Phi(x|t) = \begin{cases} 0, & \text{if } \Phi(x) \le \Phi(t) \\ \dfrac{\Phi(x)}{1-\Phi(t)} \wedge 0.5, & \text{if } \Phi(t) < \Phi(x) \le (1+\Phi(t))/2 \\ \dfrac{\Phi(x)-\Phi(t)}{1-\Phi(t)}, & \text{if } (1+\Phi(t))/2 \le \Phi(x), \end{cases} \tag{16}$$

that is just the conditional uncertainty distribution of ξ given $\xi > t$.

The hazard distribution is essentially the posterior uncertainty distribution just after time t given that it is working at time t.

Theorem 4 *(Conditional Risk Index Theorem) Consider a system that contains n elements whose uncertain lifetimes $\xi_1, \xi_2, \ldots, \xi_n$ are independent and have uncertainty distributions $\Phi_1, \Phi_2, \ldots, \Phi_n$, respectively. Assume L is a strictly increasing function, and some specified loss occurs if and only if $L(\xi_1, \xi_2, \ldots, \xi_n) \le 0$. If it is observed that all elements are working at some time t, then the risk index is*

$$Risk = \alpha \tag{17}$$

where α is the root of

$$L(\Phi_1^{-1}(\alpha|t), \Phi_2^{-1}(\alpha|t), \ldots, \Phi_n^{-1}(\alpha|t)) = 0 \tag{18}$$

where $\Phi_i(x|t)$ are hazard distributions determined by

$$\Phi_i(x|t) = \begin{cases} 0, & \text{if } \Phi_i(x) \le \Phi_i(t) \\ \dfrac{\Phi_i(x)}{1-\Phi_i(t)} \wedge 0.5, & \text{if } \Phi_i(t) < \Phi_i(x) \le (1+\Phi_i(t))/2 \\ \dfrac{\Phi_i(x)-\Phi_i(t)}{1-\Phi_i(t)}, & \text{if } (1+\Phi_i(t))/2 \le \Phi_i(x) \end{cases} \tag{19}$$

for $i = 1, 2, \ldots, n$.

Proof: It follows from Definition 2 that each hazard distribution of element is determined by (19). Thus the conditional risk index is obtained by Theorem 1 immediately.

Theorem 5 *(Conditional Risk Index Theorem) Consider a system that contains n elements whose uncertain lifetimes $\xi_1, \xi_2, \ldots, \xi_n$ are independent and have uncertainty distributions $\Phi_1, \Phi_2, \ldots, \Phi_n$, respectively. Assume L is a strictly decreasing function, and some specified loss occurs if and only if $L(\xi_1, \xi_2, \ldots, \xi_n) \le 0$. If it is observed that all elements are working at some time t, then the risk index is*

$$Risk = \alpha \tag{20}$$

where α is the root of

$$L(\Phi_1^{-1}(1-\alpha|t), \Phi_2^{-1}(1-\alpha|t), \ldots, \Phi_n^{-1}(1-\alpha|t)) = 0 \tag{21}$$

where $\Phi_i(x|t)$ are hazard distributions determined by (19) for $i = 1, 2, \ldots, n$.

Proof: It follows from Definition 2 that each hazard distribution of element is determined by (19). Thus the conditional risk index is obtained by Theorem 2 immediately.

Theorem 6 *(Conditional Risk Index Theorem) Consider a system that contains n elements whose uncertain lifetimes $\xi_1, \xi_2, \ldots, \xi_n$ are independent and have uncertainty distributions $\Phi_1, \Phi_2, \ldots, \Phi_n$, respectively. Assume $L(x_1, x_2, \ldots, x_n)$ is strictly increasing with respect to $x_1, x_2, \ldots, x_m$ and strictly decreasing with respect to $x_{m+1}, x_{m+2}, \ldots, x_n$, and some specified loss occurs if and only if $L(\xi_1, \xi_2, \ldots, \xi_n) \le 0$. If it is observed that all elements are working at some time t, then the risk index is*

$$Risk = \alpha \tag{22}$$

where α is the root of

$$L(\Phi_1^{-1}(\alpha|t), \ldots, \Phi_m^{-1}(\alpha|t), \Phi_{m+1}^{-1}(1-\alpha|t), \ldots, \Phi_n^{-1}(1-\alpha|t)) = 0 \tag{23}$$

where $\Phi_i(x|t)$ are hazard distributions determined by (19) for $i = 1, 2, \ldots, n$.

Proof: It follows from Definition 2 that each hazard distribution of element is determined by (19). Thus the conditional risk index is obtained by Theorem 3 immediately.

4 Boolean System

Many real systems may be simplified to a Boolean system in which each element (including the system itself) has two states: working and failure. This section provides a risk index theorem for such a system.

We use ξ to express an element and use a to express its reliability in uncertain measure. Then the element ξ is essentially an uncertain variable

$$\xi = \begin{cases} 1 \text{ with uncertain measure } a \\ 0 \text{ with uncertain measure } 1-a \end{cases} \tag{24}$$

where $\xi = 1$ means the element is in working state and $\xi = 0$ means ξ is in failure state.

Assume that X is a Boolean system containing elements $\xi_1, \xi_2, \ldots, \xi_n$. Usually there is a function $f : \{0,1\}^n \to \{0,1\}$ such that

$$X = 0 \text{ if and only if } f(\xi_1, \xi_2, \ldots, \xi_n) = 0, \tag{25}$$

$$X = 1 \text{ if and only if } f(\xi_1, \xi_2, \ldots, \xi_n) = 1. \tag{26}$$

Such a Boolean function f is called the *truth function* of X.

Example 3: For a series system, the truth function is a mapping from $\{0,1\}^n$ to $\{0,1\}$, i.e.,

$$f(x_1, x_2, \ldots, x_n) = x_1 \wedge x_2 \wedge \cdots \wedge x_n. \tag{27}$$

Example 4: For a parallel system, the truth function is a mapping from $\{0,1\}^n$ to $\{0,1\}$, i.e.,

$$f(x_1, x_2, \ldots, x_n) = x_1 \vee x_2 \vee \cdots \vee x_n. \tag{28}$$

Example 5: For a k-out-of-n system, the truth function is a mapping from $\{0,1\}^n$ to $\{0,1\}$, i.e.,

$$f(x_1, x_2, \ldots, x_n) = \begin{cases} 1, & \text{if } x_1 + x_2 + \cdots + x_n \geq k \\ 0, & \text{if } x_1 + x_2 + \cdots + x_n < k. \end{cases} \tag{29}$$

For any system with truth function f, if the loss is understood as the system failure, i.e., $X = 0$, then the risk index is

$$Risk = \mathcal{M}\{f(\xi_1, \xi_2, \ldots, \xi_n) = 0\}. \tag{30}$$

Theorem 7 *(Risk Index Theorem for Boolean System) Assume that $\xi_1, \xi_2, \ldots, \xi_n$ are independent elements with reliabilities $a_1, a_2, \ldots, a_n$, respectively. If a system contains $\xi_1, \xi_2, \ldots, \xi_n$ and has truth function f, then the risk index is*

$$Risk = \begin{cases} \sup\limits_{f(x_1,x_2,\ldots,x_n)=0} \min\limits_{1\le i\le n} \nu_i(x_i), & \text{if } \sup\limits_{f(x_1,x_2,\ldots,x_n)=0} \min\limits_{1\le i\le n} \nu_i(x_i) < 0.5 \\ 1 - \sup\limits_{f(x_1,x_2,\ldots,x_n)=1} \min\limits_{1\le i\le n} \nu_i(x_i), & \text{if } \sup\limits_{f(x_1,x_2,\ldots,x_n)=0} \min\limits_{1\le i\le n} \nu_i(x_i) \geq 0.5 \end{cases} \tag{31}$$

where x_i take values either 0 or 1, and ν_i are defined by

$$\nu_i(x_i) = \begin{cases} a_i, & \text{if } x_i = 1 \\ 1 - a_i, & \text{if } x_i = 0 \end{cases} \tag{32}$$

for $i = 1, 2, \ldots, n$, respectively.

Proof: Since $\xi_1, \xi_2, \ldots, \xi_n$ are Boolean uncertain variables and f is a Boolean function, the equation (31) follows from $Risk = \mathcal{M}\{f(\xi_1, \xi_2, \ldots, \xi_n) = 0\}$ immediately.

Example 6: Consider a series system having uncertain elements $\xi_1, \xi_2, \ldots, \xi_n$ with reliabilities $a_1, a_2, \ldots, a_n$, respectively. Note that the truth function is

$$f(x_1, x_2, \ldots, x_n) = x_1 \wedge x_2 \wedge \cdots \wedge x_n. \tag{33}$$

It follows from the risk index theorem that the risk index is

$$Risk = (1 - a_1) \vee (1 - a_2) \vee \cdots \vee (1 - a_n). \tag{34}$$

Example 7: Consider a parallel system having uncertain elements $\xi_1, \xi_2, \ldots, \xi_n$ with reliabilities $a_1, a_2, \ldots, a_n$, respectively. Note that the truth function is

$$f(x_1, x_2, \ldots, x_n) = x_1 \vee x_2 \vee \cdots \vee x_n. \tag{35}$$

It follows from the risk index theorem that the risk index is

$$Risk = (1 - a_1) \wedge (1 - a_2) \wedge \cdots \wedge (1 - a_n). \tag{36}$$

Example 8: Consider a k-out-of-n system having $\xi_1, \xi_2, \ldots, \xi_n$ with reliabilities $a_1, a_2, \ldots, a_n$, respectively. Note that the truth function is

$$f(x_1, x_2, \ldots, x_n) = \begin{cases} 1, & \text{if } x_1 + x_2 + \cdots + x_n \geq k \\ 0, & \text{if } x_1 + x_2 + \cdots + x_n < k. \end{cases} \tag{37}$$

It follows from the risk index theorem that the risk index is

$$Risk = \text{“the } k\text{th smallest value of } 1 - a_1, 1 - a_2, \ldots, 1 - a_n\text{”}. \tag{38}$$

5 Uncertain Reliability Analysis

Uncertain reliability analysis is a tool to deal with system reliability via uncertainty theory. Note that uncertain reliability analysis and uncertain risk analysis have the same root in mathematics. They are separately treated for application convenience in practice rather than theoretical demand.

Definition 3 *Assume a system contains uncertain variables $\xi_1, \xi_2, \ldots, \xi_n$, and there is a function R such that the system is working if and only if $R(\xi_1, \xi_2, \ldots, \xi_n) \geq 0$. Then the reliability index is*

$$Reliability = \mathcal{M}\{R(\xi_1, \xi_2, \ldots, \xi_n) \geq 0\}. \tag{39}$$

Theorem 8 *(Reliability Index Theorem) Assume $\xi_1, \xi_2, \ldots, \xi_n$ are independent uncertain variables with uncertainty distributions $\Phi_1, \Phi_2, \ldots, \Phi_n$, respectively, and R is a strictly increasing function. If some system is working if and only if $R(\xi_1, \xi_2, \ldots, \xi_n) \geq 0$, then the reliability index is*

$$Reliability = \alpha \tag{40}$$

where α is the root of

$$R(\Phi_1^{-1}(1 - \alpha), \Phi_2^{-1}(1 - \alpha), \ldots, \Phi_n^{-1}(1 - \alpha)) = 0. \tag{41}$$

If it is observed that all elements are working at some time t, then α is the root of

$$R(\Phi_1^{-1}(1 - \alpha|t), \Phi_2^{-1}(1 - \alpha|t), \ldots, \Phi_n^{-1}(1 - \alpha|t)) = 0 \tag{42}$$

where $\Phi_i(x|t)$ are hazard distributions for $i = 1, 2, \ldots, n$.

Proof: It follows from the operational law that $R(\xi_1, \xi_2, \ldots, \xi_n)$ is an uncertain variable whose inverse uncertainty distribution is

$$\Psi^{-1}(\alpha) = R(\Phi_1^{-1}(\alpha), \Phi_2^{-1}(\alpha), \ldots, \Phi_n^{-1}(\alpha)).$$

Since $\textit{Reliability} = \mathcal{M}\{R(\xi_1, \xi_2, \ldots, \xi_n) \geq 0\} = 1 - \Psi(0)$, we get (40).

Theorem 9 *(Reliability Index Theorem) Assume $\xi_1, \xi_2, \ldots, \xi_n$ are independent uncertain variables with uncertainty distributions $\Phi_1, \Phi_2, \ldots, \Phi_n$, respectively, and R is a strictly decreasing function. If some system is working if and only if $R(\xi_1, \xi_2, \ldots, \xi_n) \geq 0$, then the reliability index is*

$$\textit{Reliability} = \alpha \tag{43}$$

where α is the root of

$$R(\Phi_1^{-1}(\alpha), \Phi_2^{-1}(\alpha), \ldots, \Phi_n^{-1}(\alpha)) = 0. \tag{44}$$

If it is observed that all elements are working at some time t, then α is the root of

$$R(\Phi_1^{-1}(\alpha|t), \Phi_2^{-1}(\alpha|t), \ldots, \Phi_n^{-1}(\alpha|t)) = 0 \tag{45}$$

where $\Phi_i(x|t)$ are hazard distributions for $i = 1, 2, \ldots, n$.

Proof: It follows from the operational law that $R(\xi_1, \xi_2, \ldots, \xi_n)$ is an uncertain variable whose inverse uncertainty distribution is

$$\Psi^{-1}(\alpha) = R(\Phi_1^{-1}(1-\alpha), \Phi_2^{-1}(1-\alpha), \ldots, \Phi_n^{-1}(1-\alpha)).$$

Since $\textit{Reliability} = \mathcal{M}\{R(\xi_1, \xi_2, \ldots, \xi_n) \geq 0\} = 1 - \Psi(0)$, we get (43).

Theorem 10 *(Reliability Index Theorem) Assume $\xi_1, \xi_2, \ldots, \xi_n$ are independent uncertain variables with uncertainty distributions $\Phi_1, \Phi_2, \ldots, \Phi_n$, respectively, and the function $R(x_1, x_2, \ldots, x_n)$ is strictly increasing with respect to $x_1, x_2, \ldots, x_m$ and strictly decreasing with respect to $x_{m+1}, x_{m+2}, \ldots, x_n$. If some system is working if and only if $R(\xi_1, \xi_2, \ldots, \xi_n) \geq 0$, then the reliability index is*

$$\textit{Reliability} = \alpha \tag{46}$$

where α is the root of

$$R(\Phi_1^{-1}(1-\alpha), \ldots, \Phi_m^{-1}(1-\alpha), \Phi_{m+1}^{-1}(\alpha), \ldots, \Phi_n^{-1}(\alpha)) = 0. \tag{47}$$

If it is observed that all elements are working at some time t, then α is the root of

$$R(\Phi_1^{-1}(1-\alpha|t), \ldots, \Phi_m^{-1}(1-\alpha|t), \Phi_{m+1}^{-1}(\alpha|t), \ldots, \Phi_n^{-1}(\alpha|t)) = 0 \tag{48}$$

where $\Phi_i(x|t)$ are hazard distributions for $i = 1, 2, \ldots, n$.

Proof: It follows from the operational law that $R(\xi_1, \xi_2, \ldots, \xi_n)$ is an uncertain variable whose inverse uncertainty distribution is

$$\Psi^{-1}(\alpha) = R(\Phi_1^{-1}(\alpha), \ldots, \Phi_m^{-1}(\alpha), \Phi_{m+1}^{-1}(1-\alpha), \ldots, \Phi_n^{-1}(1-\alpha)).$$

Since $\textit{Reliability} = \mathcal{M}\{R(\xi_1, \xi_2, \ldots, \xi_n) \geq 0\} = 1 - \Psi(0)$, we get (46).

Consider a Boolean system with n elements $\xi_1, \xi_2, \ldots, \xi_n$ and a truth function f. Since the system is working if and only if $f(\xi_1, \xi_2, \ldots, \xi_n) = 1$, the reliability index is

$$\textit{Reliability} = \mathcal{M}\{f(\xi_1, \xi_2, \ldots, \xi_n) = 1\}. \tag{49}$$

Theorem 11 *(Reliability Index Theorem for Boolean System) Assume $\xi_1, \xi_2, \ldots, \xi_n$ are independent elements with reliabilities $a_1, a_2, \ldots, a_n$, respectively. If a system contains $\xi_1, \xi_2, \ldots, \xi_n$ and has truth function f, then the reliability index is*

$$Reliability = \begin{cases} \sup\limits_{f(x_1,x_2,\ldots,x_n)=1} \min\limits_{1\le i\le n} \nu_i(x_i), & \text{if } \sup\limits_{f(x_1,x_2,\ldots,x_n)=1} \min\limits_{1\le i\le n} \nu_i(x_i) < 0.5 \\ 1 - \sup\limits_{f(x_1,x_2,\ldots,x_n)=0} \min\limits_{1\le i\le n} \nu_i(x_i), & \text{if } \sup\limits_{f(x_1,x_2,\ldots,x_n)=1} \min\limits_{1\le i\le n} \nu_i(x_i) \ge 0.5 \end{cases} \tag{50}$$

where x_i take values either 0 or 1, and ν_i are defined by

$$\nu_i(x_i) = \begin{cases} a_i, & \text{if } x_i = 1 \\ 1 - a_i, & \text{if } x_i = 0 \end{cases} \tag{51}$$

for $i = 1, 2, \cdots, n$, respectively.

Proof: Since $\xi_1, \xi_2, \ldots, \xi_n$ are Boolean uncertain variables and f is a Boolean function, the equation (50) follows from $Reliability = \mathcal{M}\{f(\xi_1, \xi_2, \ldots, \xi_n) = 1\}$ immediately.

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