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Designing Location-Allocation Model in a Service Network considering Chance-Constrained Programming: A Queuing Based Analysis

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Abstract

This paper presents a new mathematical model for location-allocation (LA) problem considering uncertain parameters. In real-world cases, demand, distance, traveling time or any parameters in classical models may change over the period of time. So, considering uncertainty yields more flexibility for the results and its applications. The aim of this study is to optimize the location of petrol stations subject to an uncertain constraint denoting a service level. It is assumed that traveling time between customers and stations and also, the required time to receive service will be uncertain, too. In this model, the best location of petrol stations and the best assignment of customers will be selected. In this way, since service time and inter-arrival time are stochastic, so waiting time for customers is also stochastic and it must be less than a critical time. Queuing theory will be applied in order to compute waiting time in a service network. Finally, some numerical examples and sensitivity analysis are illustrated to show the effectiveness of this proposed model.

Keyword: facilities design, uncertainty modeling, stochastic programming, queuing theory, location-allocation

1 Introduction

Location-allocation (LA) problem aims to locate a set of new facilities such that the transportation cost between facilities and customers is optimized. LA problem has been studied widely because of its practical application, such as facilities design problem and design of service networks. Since LA problem was proposed and applied to a weighted network [7], it is introduced as Network LA problem, much work has been done on the problem, most of them is developed for the deterministic conditions.

Facility location decisions are costly and difficult to reverse, and they have impact for a long time horizon. During this time, when design decisions are made in effect, any of the parameters of the problem (costs, demands, distances, times) may fluctuate widely. Knowing this, researchers have presented models for facilities design under uncertainty. LA problem was studied in detail by Gen and Cheng [5] and Larson's hypercube model [9] was the first to applied queuing theory in facility location problems. In the main model, Larson examines problems relating to vehicle location and response district design for emergency service centers. Berman et al. [2] presented the problem of locating a vehicle in a congested network by explicitly considering the arrival process of customer calls for service. The authors show that when the server is busy and customers are queued, the mean time spent in the queue may be much greater than the mean travel time, and thus they consider queuing delay in modeling the objective function.

In uncertain LA problem, Wen and Iwamura [14] considered fuzzy location allocation (FLA) problem where a new model named α-cost model under the Hurwicz criterion with fuzzy demands was proposed. They also, considered the FLA problem [15] under random fuzzy environment using (α,β)-cost minimization model under the Hurwicz criterion. Taaffe et al. [12] presented a type of capacity acquisition and assignment problems with stochastic customers' demand found in operations planning framework.

Syam and Côté [11] introduced and solved a model for the location and allocation of specialized health care services such as traumatic brain injury (TBI) treatment. Ernst et al. [4] studied uncapacitated hub center problems with either single or multiple allocation.

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Recently, Daskin et al. [3] studied inventory-location models that considered expected cost of inventory with cost of location and allocation simultaneously. Aikens [1] has studied previous location-allocation models. Some researches in uncertainty conducted in the area of multi product and multi echelon in supply chain system by Tsiakis et al. [13].

In certain situations, all parameters are deterministic and known, while uncertain situations involve arbitrariness. In uncertain situations, parameters are unsure, and in addition, in some cases no information about probabilities is known but in the other ones probability of distribution function is specified. Problems in the first level are identified as robust optimization problems and often aim to optimize the worst-case performance of the system. The problems in the second situations are denoted as stochastic optimization problems; an ordinary goal is to optimize the expected value of objective functions. The goal of both stochastic and robust optimization is to determine a solution that will present well under all possible realizations of the uncertain parameter. In this way, we can describe random parameters either by continuous distribution functions or discrete scenarios. Scenario based planning is an approach in which uncertainty is described by determining a number of possible future conditions by decision making process. The scenario based approach has generally advantage to allow parameters to be statistically dependent. In robust optimization problems, continuous parameters are generally assumed to lie in some pre-determined intervals, because it is often impossible to consider a worst-case scenario when parameter values are unlimited. We will describe this case of uncertainty as “interval uncertainty” and describe parameters modeled this manner as “interval-uncertain” parameters [10].

In any stochastic programming (SP) problem using uncertain parameters, one must decide which decision variables are first stage and which are second stage; that is, which variables must be set now and which may be set after the uncertainty has been resolved [6]. In other word, which variables should be set at the beginning of the planning period and which of them must be set after uncertainty is realized. In stochastic locations problem which we are interested, location decisions must be made now (design variable), before it is known which scenario will come to pass, while allocation decisions are determined in future after uncertainty has been realized (control variable). Therefore, we must minimize total expected costs of system.

The structure of this paper is organized as follows. Stochastic location-allocation model is presented in Section 2. We present computational results in Section 3. In Section 4, we summarize our conclusions and discuss avenues for future research.

2 Problem Definition

In this section, we describe in more detail the designing service networks under uncertainty which we are interested. There is a set of customers zones (*N*), at which requests for service are generated, and a set of potential sites for petrol stations (*M*), where facilities may be opened. We assume that the population for each customer zone *i* is generated according to a Poisson process, independent of the population at other customers in (*N*). At each open petrol station a random customers arrive to receive service in random situations. For this purpose each customer may wait in queue line for the next available server. Thus, for each customer total time spends in petrol station consists of two parts: waiting time and service time. So, the aim of this research is to optimize location-allocation problem where the probability that a customer waits more than a service level time is at most α. We then say that (1 - α) is the fill rate of the system. In the modeling process, it’s assumed that the inter-arrival time between two sequenced customers in zone *i* is described by exponential distribution with the rate λ_i. Also, service time for customers follows exponential distribution with the rate μ in each petrol station.

2.1 Application of Queuing Theory to Location-Allocation Problem

In this study, we formulate *L-LA* problem as a queue system. Also, assume a birth-death process with constant arrival (birth) and service completion (death) rates. Specifically, let λ_i and μ be the arrival and service rate of customers, respectively, per unit time. If arrival rate is greater than the service rate, the queue will grow infinitely. The ratio of λ to μ is named utilization factor or the probability that a machine is busy and is defined as $\rho = \lambda / \mu$. Therefore, for a system in steady state, this ratio must be less than one. In this research, we assume M/M/1 queue system for each petrol station where customers in zone *i* arrived to station with rate λ_i where parts served by servers. In this condition, since it may be possible that different customer zones attended to receive service to each, so for each station (server) ρ is computed using the following property.

Property 1 The minimum of independent exponential random variables is also exponential. Let $F_1, F_2, ..., F_n$ be independent random variables with parameters $\lambda_1, \lambda_2, ..., \lambda_n$. Let $F_{\min} = \min\{F_1, F_2, ..., F_n\}$. Then for any $t \geq 0$,

$$P(F_{\min} > t) = P(F_1 > t) \times P(F_2 > t) \times ... \times P(F_n > t) = e^{-\lambda_1 t} e^{-\lambda_2 t} ... e^{-\lambda_n t} = e^{-[\lambda_1 + \lambda_2 + ... + \lambda_n]t}.$$

An interesting implication of this property to inter-arrival times is discussed in [8]. Suppose n types of customers, with the ith type of customer having an exponential inter-arrival time distribution with parameter λ_i, arrive at a queue system. Let us assume that an arrival has just taken place. Then from a no-memory property of exponential distribution, it follows that the time remaining until the next arrival is also exponential. Using mentioned property, we can see that the inter-arrival time for entire queue system or efficient arrival rate (which is the minimum among all inter-arrival times) has an exponential distribution with parameter

$$\lambda_{eff} = \sum_{i=1}^{N} \lambda_i .$$

Hence, utilization factor or the probability that each server (j) is busy is as follow (efficient arrival rate divided by service rate)

$$\rho_j = \frac{\lambda_{eff}}{\mu} = \frac{\sum_{i=1}^{N} \lambda_i}{\mu} . \quad (1)$$

2.2 Notations

We use the following notation.

Sets

N: Index set of customer zones.

M: Index set of potential sites for install petrol stations.

Parameters

λ_i : Mean arrival rate for customer zone i.

μ : Number of customers served per unit time by server (Mean Service Rate).

t : Critical time for customers to be in system.

α : The probability that a customer waits in system more than critical time.

c_{ij} : Cost of transporting a customer in zone i to potential station at site j.

P : Number of facilities to be located in the network.

F_j : Fixed cost for opening and operating petrol station at site j.

Decisions variables

$$x_{ij} = \begin{cases} 1, & \text{if customer zone } i \text{ assigned to site } j \\ 0, & \text{otherwise.} \end{cases}$$

$$U_j = \begin{cases} 1, & \text{if a petrol station located at site } j \\ 0, & \text{othrwise.} \end{cases}$$

2.3 Chance Constraint Programming

Since, both arrival time and service time are uncertain so the time in which each customer spends in petrol station will be uncertain, too. Thus, in order to prevent long waiting time for each customer, a chance constraint must be considered in the formulation. It's known that distribution function denoting total time for each customer in a system is as follows

$$P(W_s \geq t) = e^{-\mu(1-\rho)t} . \quad (2)$$

Proof: Assume that there are N customers in a system once a new customer is arrived. Thus, based on the conditional probability theory

$$P(W_s \geq t) = \sum_{n=0}^{\infty} P(W_s \geq t \mid N = n) \times P(N = n) . \quad (3)$$

On the other side, total time in which a new customer has to wait is equal to

$$W_q = F_1 + F_2 + ... + F_n \quad (4)$$

where F_i denotes service time for customer i. So

$$W_s = W_q + F_{n+1} \quad (5)$$

where F_{n+1} denotes service time for new arrived customer. It is obvious that sum of the n+1 random variables with exponential distribution with rate μ will be an Erlang random variable with parameters n+1 and μ . So

$$P(W_s \geq t \mid N = n) = P(\sum_{i=1}^{n+1} F_i > t) = \int_t^{\infty} \mu \times e^{-\mu \cdot y} \frac{(\mu y)^n}{n!} d_y . \tag{6}$$

It is known in the literature that the probability of being n customers in a M/M/1 model system is

$$p_n = \rho^n (1-\rho) \quad \text{where} \quad \rho = \frac{\lambda}{\mu} . \tag{7}$$

Based on the equations (6) and (7), equation (3) will be computed as

$$P(W_s \geq t) = \sum_{n=0}^{\infty} \rho^n (1-\rho) \int_t^{\infty} \mu \times e^{-\mu \cdot y} \frac{(\mu y)^n}{n!} d_y \tag{8}$$

$$= \mu(1-\rho) \int_t^{\infty} e^{-\mu \cdot y} \sum_{n=0}^{\infty} \frac{\rho^n (\mu y)^n}{n!} d_y . \tag{9}$$

Also, based on the exponential series, we have

$$\sum_{n=0}^{\infty} \frac{\rho^n (\mu y)^n}{n!} = e^{\rho \cdot \mu \cdot y} = e^{\lambda \cdot y} . \tag{10}$$

If we replace equation (10) to the equation (9), the equation (2) will be proved. It can be found that W_s has an exponential distribution function with parameter $\mu - \lambda$.

In order to satisfy service level this probability must be at most α. So, the chance constraint will be determined as $P(W_s \geq t) \leq \alpha$. In order to linearize this nonlinear constraint the following procedure is performed

$$P(W_s \geq t) \leq \alpha \tag{11}$$

$$\Rightarrow e^{-\mu(1-\rho)t} \leq \alpha \tag{12}$$

$$\Rightarrow -\mu(1-\rho)t \leq Ln(\alpha) \tag{13}$$

$$\Rightarrow -\mu(1 - \frac{\sum_i \lambda_i x_{ij}}{\mu}) \leq \frac{1}{t} Ln(\alpha) \tag{14}$$

$$\Rightarrow -\mu + \sum_i \lambda_i x_{ij} \leq \frac{1}{t} Ln(\alpha) \tag{15}$$

$$\Rightarrow \sum_i \lambda_i x_{ij} \leq \mu + \frac{1}{t} Ln(\alpha) \quad \text{for each pertol station } (j) . \tag{16}$$

The achieved constraint indicates that a customer will be in system more than critical time t with probability at most α.

2.4 Model Formulation

$$\text{Min} \quad Z = \sum_{i \in N} \sum_{j \in M} [\lambda_i \times c_{ij}] \times x_{ij} + \sum_{j \in M} F_j \times U_j \tag{17}$$

$$\text{s.t.} \quad \sum_{i \in N} x_{ij} = 1 \qquad \text{for all } j \tag{18}$$

$$x_{ij} \leq U_j \qquad \text{for all } i \text{ and } j \tag{19}$$

$$\sum_{j \in M} U_j = p \tag{20}$$

$$\sum_i \lambda_i x_{ij} \leq \mu + \frac{1}{t} Ln(\alpha) \quad \text{for each pertol station } (j) \tag{21}$$

$$x_{ij}, U_j \in \{0,1\} . \tag{22}$$

The objective function (17) of the proposed model minimizes the total expected cost covered so far and fixed cost in order to opening new stations. Constraint (18) ensures that each customer zone have to be allocated to the exactly one petrol station. Constraint (19) ensures that a customer can allocate to emergency center j when this emergency center is installed. Constraint (20) indicates that the total number of petrol stations to be located should be equal to *p*. Constraint (21) guarantees that the probability a customer waits more that critical time *t* in system is at most α. Constraint (22) determines the type of variables.

3 Computational Experiments

To measure the effectiveness of the proposed approach, we generate some random examples and solve them by branch-and-bound algorithm using Lingo 8 software package. All algorithms considered in this paper are run on a Pentium IV PC with 3 GHz CPU and 512 MB RAM.

Suppose it's decided to design new service network locating new petrol stations, in which there are 40 customer zones and 10 sites to install stations (servers). The decision maker needs to assign zones to new petrol stations to serve these customers. Tables 1 and 2 illustrate data sets for parameters λ_i and C_{ij}, respectively. Also, in Table 3, associated results are reported when proposed example is solved for 10 times. In these run all parameters applied in the model are fix expect service level rate. The aim of this example is to show a vital role of service level (or the probability that a customer waits more than critical time) of servers in order to determine the best location decisions. This major factor will be base for our computational analysis. Therefore, in this example which is run for 10 times, service level rate is varied and number of opened facilities is measured. Table 3 illustrates characteristics, the other problem information and the final results.

Table 1: Data set describing parameter λ_i

Cu.1	Cu.2	Cu.3	Cu.4	Cu.5	Cu.6	Cu.7	Cu.8	Cu.9	Cu.10	Cu.11	Cu.12	Cu.13	Cu.14	Cu.15	Cu.16	Cu.17	Cu.18	Cu.19	Cu.20
3	4	5	7	9	7	8	3	8	2	2	6	5	8	5	5	8	2	10	1
Cu.21	Cu.22	Cu.23	Cu.24	Cu.25	Cu.26	Cu.27	Cu.28	Cu.29	Cu.30	Cu.31	Cu.32	Cu.33	Cu.34	Cu.35	Cu.36	Cu.37	Cu.38	Cu.39	Cu.40
5	8	4	10	7	1	7	7	1	5	7	7	9	4	2	7	7	8	10	4

Table 2: Data set describing parameter C_{ij} between customer zone i and petrol station site *j*

	S.1	S.2	S.3	S.4	S.5	S.6	S.7	S.8	S.9	S.10
Cu.1	4	5	10	1	8	8	6	4	9	2
Cu.2	3	1	8	7	6	4	2	3	3	4
Cu.3	3	3	4	10	7	5	4	6	3	10
Cu.4	2	7	10	5	5	2	4	1	5	4
Cu.5	2	7	6	9	3	5	10	3	10	4
Cu.6	1	9	10	1	5	9	5	2	6	6
Cu.7	10	8	10	6	3	7	1	10	3	7
Cu.8	2	7	7	9	1	4	9	5	1	4
Cu.9	2	1	10	7	6	8	2	3	5	1
Cu.10	8	5	4	10	3	8	4	1	3	3
Cu.11	10	7	6	5	7	5	6	3	3	7
Cu.12	8	8	6	8	7	10	3	8	7	6
Cu.13	5	1	6	9	7	6	2	3	1	4
Cu.14	8	3	5	5	7	3	9	5	4	3
Cu.15	2	8	9	9	10	4	6	10	6	7
Cu.16	2	5	4	4	9	2	10	10	3	3
Cu.17	9	9	4	10	7	3	3	10	10	7
Cu.18	3	9	7	5	9	2	4	6	7	4
Cu.19	5	1	2	9	5	1	2	6	4	10
Cu.20	2	2	7	4	7	1	3	1	1	2

	S.1	S.2	S.3	S.4	S.5	S.6	S.7	S.8	S.9	S.10
Cu.21	9	5	3	4	4	9	2	8	2	3
Cu.22	9	5	7	2	10	7	6	2	2	10
Cu.23	4	5	1	3	5	4	3	9	7	1
Cu.24	8	1	6	7	10	9	10	3	2	3
Cu.25	4	1	3	2	4	8	3	5	3	9
Cu.26	1	9	4	5	7	7	10	3	6	7
Cu.27	1	2	2	8	9	6	4	7	7	9
Cu.28	5	3	6	10	4	9	6	7	6	6
Cu.29	1	1	1	4	10	5	2	2	5	10
Cu.30	1	9	9	1	10	1	4	7	7	3
Cu.31	5	9	6	1	10	10	4	10	10	8
Cu.32	8	2	1	2	5	9	4	7	9	9
Cu.33	6	6	4	8	4	7	7	3	4	8
Cu.34	3	5	2	9	5	5	5	10	9	6
Cu.35	5	7	6	10	5	5	9	4	6	7
Cu.36	9	3	1	2	6	5	6	8	7	9
Cu.37	8	10	3	5	4	2	8	1	2	1
Cu.38	4	2	4	6	10	3	6	10	9	2
Cu.39	4	1	3	9	1	9	3	9	7	5
Cu.40	4	3	2	3	10	7	2	6	1	6

Table 3: Associated results relating to location-allocation and queuing theory

Problem info.			Achieved Results			
Prob. No.	No. of customers	No. of Sites	*a*	Objec. Function	No. of opened Stations	Average utilization factor
1	40	10	0.02	1362	7	0.54
2	40	10	0.05	1216	6	0.63
3	40	10	0.07	1207	6	0.63
4	40	10	0.10	1113	5	0.74
5	40	10	0.15	1101	5	0.76
6	40	10	0.20	1095	5	0.76
7	40	10	0.25	1084	5	0.77
8	40	10	0.40	1082	5	0.78
9	40	10	0.60	1008	4	0.95
10	40	10	0.80	1007	4	0.96

In Figure 1, relation between the probability that each customer is not satisfied (a) and total cost the network is illustrated. From Figure 1, the more customers' satisfaction rate (1-a), the more cost for system is needed. As it can be expected if decision makers wants to improve customers' satisfaction rate (in other words, he wants to decrease a rate), he have to pay more cost for installing the network. This result can be predictable as in each system in order to have more customers' satisfaction, the more cost will be required. So, this result confirms correctness of the model performance.

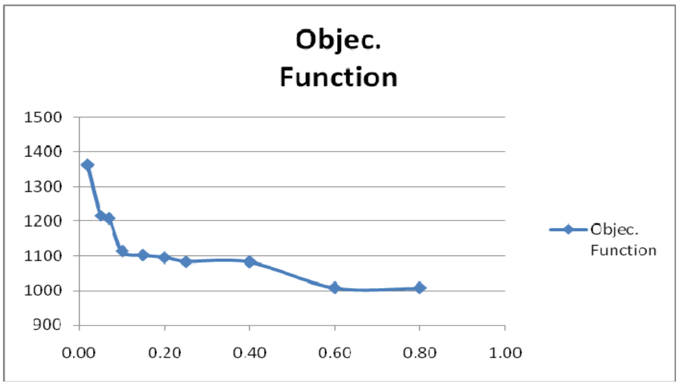

Figure 1: Relationship curve between objective function and the probability that each customer waits more than critical time

In Figure 2, aslo the manner between number of opened sations and the probability that customers are not satisfied is explained. As it's shown, in order to have more customers' satisfication, the more stations must be installed by opening more stations custormers and increasing number of servers, the custoemrs will be satisfied more. This result can be obtained from Figure 2.

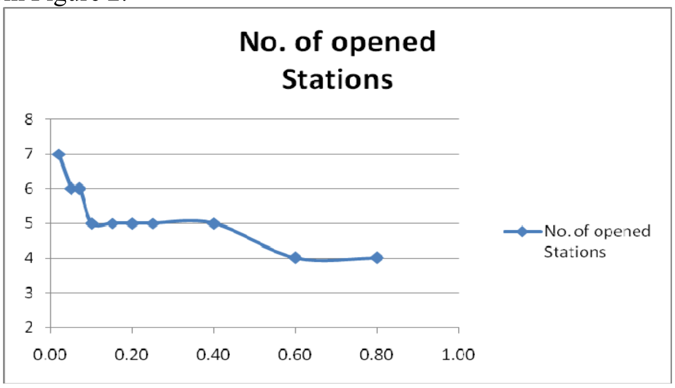

Figure 2: Relationship curve between number of opened stations and the probability that each customer waits more than critical time

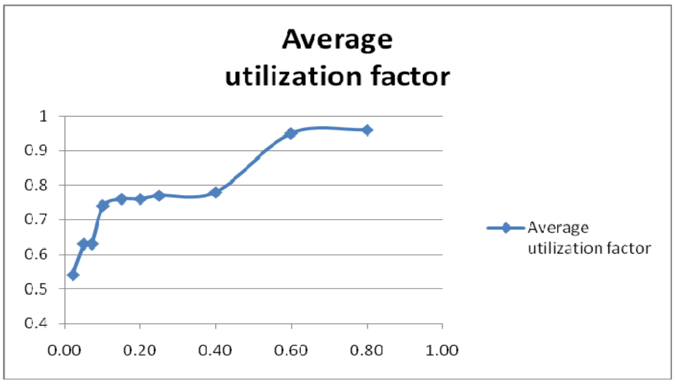

Figure 3: Relationship curve between average utilization factor and the probability that each customer waits more than critical time

In the last comparision, changes of average utilization factor are drwan rather than the probability that a custorme is not satisfied (a). From Figure 3, if the probability that each customer dissatisfied is increases, then the customers must wait more in stations and therfore stations are more populated, so the srevers will be busier than normal condtions and then utilization factor for each server increases. Figure 3 points up this result.

4 Conclusion and Future Directions

In this paper, we defined a notation of stochastic location-allocation problem considering stochastic inter-arrival and service times which have been described by exponential distribution. A conceptual framework and a mathematical model were proposed as a queue system and therefore by optimizing queue system measurements, location of petrol sataions and allocation of customer zones can be determined optimality. Our contributions research field consists of: considering stochastic parameters which yield to more flexibility and practical aspects in real world cases and formulating the stochastic problem as a queue system.

For future research, we suggest three directions:

(i) Development of the model under more and the other stochastic parameters such as costs, routings and server availability.

(ii) Optimizing the other queue measurements such as average queue length for customers in sations (servers) to have allocations with high quality.

(iii) Aggregating proposed model with the other assumptions like layout problem considerations.

These remain critical issues for future study.

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