Journal of Uncertain Systems
Vol.4, No.2, pp.99-115, 2010
Online at: www.jus.org.uk

Non-Archimedean Valued Extension of Logic ŁΠ∀ and a p-Adic Valued Extension of Logic $BL\forall$

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Received 10 March 2009; Revised 4 October 2009

Abstract

In this paper I show that a kind of infinite-order predicate logic can be regarded as non-Archimedean or p-adic valued. I have considered two principal versions of non-Archimedean valued predicate logical calculi: p-adic valued extension of $BL\forall$ and hyper-valued extension of ŁΠ∀. These logical systems are considered for the first time.

Keywords: Fréchet filter, nonstandard extension, infinite-order predicate logic, hyperrational valued predicate logic, hyperreal valued predicate logic, p-adic valued predicate logic

1 Introduction

The paper contains general results concerning non-Archimedean fuzziness, i.e. fuzziness whose values run over an uncountable infinite, non-well-ordered and non-well-founded set. This kind of fuzziness generalizes the notion of non-Archimedean probabilities [11, 13, 14, 20]. Let us remember that Archimedes' axiom affirms: for any positive real or rational number ε, there exists a positive integer n such that $\varepsilon \geq \frac{1}{n}$ or $n \cdot \varepsilon \geq 1$. In this paper we will consider fuzziness defined on the following three sets: (i) the set ${}^*\mathbf{R}$ of hyperreal numbers, (ii) the set ${}^*\mathbf{Q}$ of hyperrational numbers, (iii) the set $\mathbf{Z}_p$ of p-adic integers. It is well known that sets ${}^*\mathbf{R}$, ${}^*\mathbf{Q}$ satisfy properties of field and set $\mathbf{Z}_p$ properties of ring. All those sets contain infinitely large numbers and in addition sets ${}^*\mathbf{R}$, ${}^*\mathbf{Q}$ contain infinitesimals (infinitely small numbers). For more details on infinitesimal and p-adic analysis see: [9, 10, 15, 16, 18].

In this paper I am constructing non-Archimedean valued fuzzy logics $Ł\Pi\forall_\infty$ and $Ł\Pi\frac{1}{2}\forall_\infty$ and a p-adic valued fuzzy logic $BL\forall_\infty$ that are built as ω-order extensions of the logics ŁΠ∀, $Ł\Pi\frac{1}{2}\forall$, and $BL\forall$ respectively. Recall that the logics ŁΠ∀, $Ł\Pi\frac{1}{2}\forall$, $BL\forall$ are considered in [1, 3, 4, 8].

2 Preliminaries

2.1 Hypernumbers

Let Θ be a set and I an infinite set of indices. We consider, in a standard way, the family Θ^I, i.e. the set of all functions: $f\colon I \mapsto \Theta$. Now we define a *filter* $\mathcal{F}$ on I as a family of sets $\mathcal{F} \subset 2^I$ for which: (1) $A \in \mathcal{F}$, $A \subset B \to B \in \mathcal{F}$; (2) $A_1, \ldots, A_n \in \mathcal{F} \to \bigcap\limits_{k=1}^{n} A_k \in \mathcal{F}$ for any $n \geq 1$; (3) $\emptyset \notin \mathcal{F}$. The set of all complements for finite subsets of I is a filter and it is called a *Fréchet filter on* I and it is denoted by $\mathcal{U}_I$.

Further, define a relation $\backsim$ on the set Θ^I by $f \backsim g \equiv \{\alpha \in I\colon f(\alpha) = g(\alpha)\} \in \mathcal{U}_I$. It is easily be proved that the relation $\backsim$ is an equivalence. For each $f \in \Theta^I$ let $[f]$ denote the equivalence class of f under $\backsim$. The *ultrapower* $\Theta^I/\mathcal{U}_I$ is then defined to be the set of all equivalence classes $[f]$ as f ranges over Θ^I : $\Theta^I/\mathcal{U}_I := \{[f]\colon f \in \Theta^I\}$.

The ultrapower $\Theta^I/\mathcal{U}_I$ is said to be a *proper nonstandard extension* of Θ and it is denoted by ${}^*\Theta$. Recall that each element of ${}^*\Theta$ is an equivalence class $[f]$ where $f\colon I \to \Theta$. There exist two groups of members of ${}^*\Theta$: (1) equivalence classes of constant functions, e.g. $f(\alpha) = m \in \Theta$ for all $\alpha \in I$. Such equivalence class is denoted by ${}^*m$ or $[f = m]$, (2) equivalence classes of functions that aren't constant.

The set ${}^\sigma\Theta = \{{}^*m\colon m \in \Theta\}$ is called *standard set*. The members of ${}^\sigma\Theta$ are called *standard*. It is readily seen that ${}^\sigma\Theta$ and Θ are isomorphic: ${}^\sigma\Theta \simeq \Theta$.

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If Θ is a number system, then members of ${}^*\Theta$ will be called *hypernumbers*. We can define operations on them:

$$[f] \odot [g] = [h] \quad \equiv \quad \{\alpha \in I \colon f(\alpha) \odot g(\alpha) = h(\alpha)\} \in \mathcal{U}_I,$$

where $\odot \in \{+, -, \cdot, /\}$.

In this way we can obtain hyperreal numbers of ${}^*\mathbf{R}$ and hyperrational numbers of ${}^*\mathbf{Q}$ satisfying properties of field.

2.2 p-Adic Numbers

Now we are trying to compare p-adic numbers with hypernumbers. For this let us consider a particular case of nonstandard extension when Θ is a finite set such that $|\Theta| = p$, where p is a prime number. Since we stopped to consider the general case there is no need to excuse that we lose generality. Let the set $\mathbf{N}$ of natural numbers be the index set and let $\mathcal{U}_N$ be a Fréchet filter on $\mathbf{N}$. Then there exists a nonstandard extension ${}^*\Theta := \Theta^{\mathbf{N}}/\mathcal{U}_N$.

As usual, every considered function $f \colon \mathbf{N} \to \Theta$ can be meant as an infinite-tuple $\langle f(0), f(1),\ f(2), \ldots\rangle$, where $f(j) \in \Theta$ for any $j = 0, 1, \ldots$

It is obvious that if $n \leq m$ for $n, m \in \Theta$, then we can set ${}^*n \leq {}^*m$ for ${}^*n, {}^*m \in {}^*\Theta$. In other words, the order relation on the members of Θ can be extended to the order relation on the constant functions of ${}^*\Theta$.

Now consider f, such that there is no $k \in \Theta$ for which $[f] = {}^*k$.

Let $f_0, \ldots,\ f_{p-1},\ f_p, \ldots,\ f_{2\cdot p-1}, \ldots,\ f_{p^2-p},\ \ldots,\ f_{p^2-1},\ \ldots$ be functions that satisfy the following condition:

$$[f_m] = [\langle n_0, n_1, n_2, \ldots, n_k, 0, 0, \ldots\rangle],$$

where $n_0, n_1, n_2, \ldots, n_k$ are such that $m = n_0 + n_1 p + n_2 p^2 + \cdots + n_k p^k$.

For instance,

1. $[f_k] = [\langle k, 0, 0, \ldots\rangle]$, for any $k = 0, 1, \ldots p-1$,
2. $[f_k] = [\langle k \mod p, 1, 0, \ldots\rangle]$, for any $k = p, \ldots, 2p-1$,
3. $[f_k] = [\langle k \mod p, p-1, 0, \ldots\rangle]$, for any $k = p^2 - p, \ldots, p^2 - 1$,
4. $\ldots$

We can extend the ordering relation on the members of Θ to the ordering relation $\leq_*$ on the members $[f_0], \ldots,\ [f_{p-1}],\ [f_p], \ldots,\ [f_{2\cdot p-1}], \ldots,\ [f_{p^2-p}],\ \ldots,\ [f_{p^2-1}],\ \ldots$ of ${}^*\Theta$. Define this order structure as follows:

$$\begin{cases} [f_k] \leq_* [f_l] \text{ iff } k \leq l\ , & \text{if } k, l \in \mathbf{N}; \\ [f],\ [f'] \text{ are incompatible under } \leq_*, & \text{if there is no } k \in \mathbf{N} \text{ such that} \\ & [f] = [f_k] \text{ or } [f'] = [f_k]. \end{cases}$$

Notice that $\leq_*$ is partial, because ${}^*\Theta$ contains an uncountable number of its members therefore there exists $[f]$ such that there is no $k \in \mathbf{N}$ for which $[f] = [f_k]$, this $[f]$ is said to be infinitely large integer. We can assign the natural number to each member $[f_k] \in {}^*\Theta$, were k is finite; namely to each $[f_k]$ as $k \to \infty$, we can assign the following expansion

$$f_k(0) + f_k(1) \cdot p + \ldots + f_k(n) \cdot p^n + \ldots = \sum_{n=0}^{\infty} f_k(n) \cdot p^n,$$

where $f_k(n) \in \{0, 1, \ldots, p-1\}$, $\forall n \in \mathbf{N}$. This expansion is called the p-adic integer. This number sometimes has the following notation:

$$\ldots \beta_n \ldots \beta_3 \beta_2 \beta_1 \beta_0,$$

where $\beta_0 = f_k(0)$, $\beta_1 = f_k(1)$, $\beta_2 = f_k(2)$, $\ldots$

Thus, we have shown that the nonstandard extension $\Theta^{\mathbf{N}}/\mathcal{U}_N$, where $\Theta = \{0, \ldots, p-1\}$, is isomorphic with the set $\mathbf{Z}_p$ of p-adic integers. Usual denary operations (/, +, −, ·) can be extrapolated to the case of them, [15], [16]. For example, for 5-adic integers $\ldots 02324$ and $\ldots 003$ we obtain: $(\ldots 02324)/(\ldots 003) =$

$\ldots 0423$ (the operation of division is not defined for all p-adic integers), $\ldots 02324 + \ldots 003 = \ldots 02332$, $\ldots 02324 - \ldots 003 = \ldots 02321$, $\ldots 02324 \cdot \ldots 003 = \ldots 013032$. Finite numbers of $\mathbf{Z}_p$ can be regarded as positive integers, namely each $[f_k]$ considered above such that $k \in \mathbf{N}$ can be identified with k. In this way we can identify $\ldots 02324$ with 339 and $\ldots 003$ with 3.

3 Non-Archimedean Valued and p-adic Valued Logical Matrices

3.1 Hyper-valued ŁΠ-matrix

Let $\mathbf{Q}_{[0,1]} = \mathbf{Q} \cap [0,1]$. We can extend the usual order structure on $\mathbf{Q}_{[0,1]}$ to a partial order structure on ${}^*\mathbf{Q}_{[0,1]} := \mathbf{Q}^{\mathbf{N}}/\mathcal{U}_N$:

1. for any ${}^*x$, ${}^*y \in {}^\sigma\mathbf{Q}_{[0,1]}$ we have ${}^*x \preceq_* {}^*y$ iff $x \leq y$ in $\mathbf{Q}_{[0,1]}$,
2. if ${}^*x \neq {}^*0$, then $[f] \preceq_* {}^*x$, i.e. each positive rational number ${}^*x \in {}^\sigma\mathbf{Q}_{[0,1]}$ is greater than any number $[f] \in {}^*\mathbf{Q}_{[0,1]} \backslash {}^\sigma\mathbf{Q}_{[0,1]}$.

These conditions have the following informal sense: (1) the sets ${}^\sigma\mathbf{Q}_{[0,1]}$ and $\mathbf{Q}_{[0,1]}$ have an isomorphic order structure; (2) the set ${}^*\mathbf{Q}_{[0,1]}$ contains actual infinities that are less than any positive rational number of ${}^\sigma\mathbf{Q}_{[0,1]}$. Define this partial order structure on ${}^*\mathbf{Q}_{[0,1]}$ as follows:

$\mathcal{O}_{{}^*\mathbf{Q}}$ (i) For any $[x]$, $[y] \in {}^*\mathbf{Q}_{[0,1]}$ we have $[x] \preceq_* [y]$ iff $\{\alpha \in \mathbf{N} \colon x(\alpha) \leq y(\alpha)\} \in \mathcal{U}_N$. (ii) For any ${}^*z \in {}^\sigma\mathbf{Q}_{[0,1]}$ and $[y] \in {}^*\mathbf{Q}_{[0,1]} \backslash {}^\sigma\mathbf{Q}_{[0,1]}$ if ${}^*z \neq {}^*0$, then $[y] \prec_* {}^*z$. Notice that we have $[x] \prec_* [y]$ iff $[x] \neq [y]$ and $[x] \preceq_* [y]$.

Introduce two operations sup, inf in the partial order structure $\mathcal{O}_{{}^*\mathbf{Q}}$:

$$\inf([x],[y]) := [\min(x,y)];$$

$$\sup([x],[y]) := [\max(x,y)].$$

Note there exist the maximal number ${}^*1 \in {}^*\mathbf{Q}_{[0,1]}$ and the minimal number ${}^*0 \in {}^*\mathbf{Q}_{[0,1]}$ under the meaning expressed in the condition $\mathcal{O}_{{}^*\mathbf{Q}}$. Indeed, $\{\alpha \in \mathbf{N} \colon x(\alpha) \leq 1\} \in \mathcal{U}_N$ for any $x \in \mathbf{Q}^{\mathbf{N}}$ and $\{\alpha \in \mathbf{N} \colon 0 \leq x(\alpha)\} \in \mathcal{U}_N$ for any $x \in \mathbf{Q}^{\mathbf{N}}$.

Let $+, -, \cdot, /$ be the addition, subtraction, multiplication and division defined on $\mathbf{Q}$; we can extend them on ${}^*\mathbf{Q}$ as follows: $[x] \odot [y] = [z]$ iff $\{\alpha \in \mathbf{N} \colon\ x(\alpha) \odot y(\alpha)) = z(\alpha)\} \in \mathcal{U}_N$, where $\odot \in \{+, -, \cdot, /\}$.

Now introduce the following new operations defined for all $[x], [y] \in {}^*\mathbf{Q}$ in the partial order structure $\mathcal{O}_{{}^*\mathbf{Q}}$:

- $[x] \rightarrow_L [y] = {}^*1 - \sup([x],[y]) + [y]$,
- $[x] \rightarrow_\Pi [y] = \begin{cases} {}^*1 & \text{if } [x] \preceq_* [y], \\ \inf\Big({}^*1, \dfrac{[y]}{[x]}\Big) & \text{otherwise.} \end{cases}$

  Notice that we have $\inf\Big({}^*1, \dfrac{[y]}{[x]}\Big) = [h]$ iff $\{\alpha \in \mathbf{N} \colon\ \min\Big(1, \dfrac{y(\alpha)}{x(\alpha)}\Big) = h(\alpha)\} \in \mathcal{U}_N$, let us also remember that the members $[x]$, $[y]$ can be incompatible under $\mathcal{O}_{{}^*\mathbf{Q}}$,
- $\neg_L[x] = {}^*1 - [x]$, i.e. $[x] \rightarrow_L {}^*0$,
- $\neg_\Pi[x] = \begin{cases} {}^*1 & \text{if } [x] = {}^*0, \\ {}^*0 & \text{otherwise,} \end{cases}$ i.e. $\neg_\Pi[x] = [x] \rightarrow_\Pi {}^*0$,
- $\Delta[x] = \begin{cases} {}^*1 & \text{if } [x] = {}^*1, \\ {}^*0 & \text{otherwise,} \end{cases}$ i.e. $\Delta[x] = \neg_\Pi \neg_L [x]$,
- $[x] \&_L [y] = \sup([x], {}^*1 - [y]) + [y] - {}^*1$, i.e. $[x] \&_L [y] = \neg_L([x] \rightarrow_L \neg_L[y])$,
- $[x] \&_\Pi [y] = [x] \cdot [y]$,

- $[x] \oplus [y] := \neg_L [x] \rightarrow_L [y]$,
- $[x] \ominus [y] := [x] \&_L \neg_L [y]$,
- $[x] \wedge [y] = \inf([x], [y])$, i.e. $[x] \wedge [y] = [x] \&_L ([x] \rightarrow_L [y])$,

  Let us show that $\wedge$ is really derived from $\&_L$ and $\rightarrow_L$. Recall that inf and sup are defined digit by digit. This means that if $[x] = [\langle x_0, x_1, x_2, \dots \rangle]$ and $[y] = [\langle y_0, y_1, y_2, \dots \rangle]$ then

  $$\inf([x], [y]) = [\langle \min(x_0, y_0), \min(x_1, y_1), \min(x_2, y_2), \dots \rangle].$$

  For each $i = 0, 1, 2, \dots$ we have $x_i \leq y_i$ or $x_i > y_i$. At the same time, if $x_i \leq y_i$, then $x_i \&_L (x_i \rightarrow_L y_i) = \max(x_i, \max(x_i, y_i) - y_i) + y_i - \max(x_i, y_i) = x_i$ and if $x_i > y_i$, then $x_i \&_L (x_i \rightarrow_L y_i) = \max(x_i, \max(x_i, y_i) - y_i) + y_i - \max(x_i, y_i) = y_i$.

- $[x] \vee [y] = \sup([x], [y])$, i.e. $[x] \vee [y] = ([x] \rightarrow_L [y]) \rightarrow_L [y]$,

  Indeed, $([x] \rightarrow_L [y]) \rightarrow_L [y] = {}^*1 - \sup({}^*1 - \sup([x], [y]) + [y], [y]) + [y] = \sup([x], [y])$.

- $[x] \leftrightarrow [y] := ([x] \rightarrow_L [y]) \wedge ([y] \rightarrow_L [x])$,
- $[x] \&_G [y] := \inf([x], [y])$,
- $[x] \rightarrow_G [y] = \begin{cases} {}^*1 & \text{if } [x] \preceq_* [y], \\ [y] & \text{otherwise,} \end{cases}$ i.e. $[x] \rightarrow_G [y] = \Delta([x] \rightarrow_L [y]) \vee [y]$.

Proposition 1 *A structure* $\mathfrak{L}_{^*\mathbf{Q}} = \langle {}^*\mathbf{Q}_{[0,1]}, \oplus, \neg_L, \rightarrow_\Pi, \&_\Pi, {}^*0, {}^*1 \rangle$ *is a hyperrational valued ŁΠ-matrix.*

Proof. We should show that

1. $\langle {}^*\mathbf{Q}_{[0,1]}, \oplus, \neg_L, {}^*0 \rangle$ is an MV-algebra, i.e. (1) $\langle {}^*\mathbf{Q}_{[0,1]}, \oplus, {}^*0 \rangle$ is a commutative monoid, (2) $[x] \oplus {}^*1 = {}^*1$, (3) $\neg_L \neg_L [x] = [x]$, (4) $([x] \ominus [y]) \oplus [y] = ([y] \ominus [x]) \oplus [x]$;
2. $\langle {}^*\mathbf{Q}_{[0,1]}, \vee, \wedge, \rightarrow_\Pi, \&_\Pi, {}^*0, {}^*1 \rangle$ is a Π-algebra, i.e. (1) $\langle {}^*\mathbf{Q}_{[0,1]}, \vee, \wedge \rangle$ is a bounded lattice with the order $\preceq_*$, with the top element ^{*1} and the bottom element ^{*0}, (2) $\langle {}^*\mathbf{Q}_{[0,1]}, \&_\Pi, {}^*1 \rangle$ is a commutative semigroup with the unit element ^{*1}, (3) $\rightarrow_\Pi$ and $\&_\Pi$ form an adjoint pair, i.e. $[z] \preceq_* [x] \rightarrow_\Pi [y]$ iff $[x] \&_\Pi [z] \preceq_* [y]$ for all $[x]$, $[y]$, $[z] \in {}^*\mathbf{Q}_{[0,1]}$;
3. $[x] \&_\Pi ([y] \ominus [z]) = ([x] \&_\Pi [y]) \ominus ([x] \&_\Pi [z])$.

However, all three items are readily checked. For instance,

1. $([x] \ominus [y]) \oplus [y] = {}^*1 - \sup[{}^*1 - (\sup([x], [y]) - [y]), [y]] + [y] = \sup([x], [y]) = ([y] \ominus [x]) \oplus [x]$.
2. Show that $[z] \preceq_* [x] \rightarrow_\Pi [y]$ iff $[x] \&_\Pi [z] \preceq_* [y]$ for all $[x]$, $[y]$, $[z] \in {}^*\mathbf{Q}_{[0,1]}$. (i) Suppose that $[x] \preceq_* [y]$. Then $[z] \preceq_* [x] \rightarrow_\Pi [y] = {}^*1$ and $[x] \cdot [z] \preceq_* [y]$. (ii) Otherwise $[x] \rightarrow_\Pi [y] = \inf\Big({}^*1, \dfrac{[y]}{[x]}\Big) \preceq_* \dfrac{[y]}{[x]}$. In this case $[z] \preceq_* \dfrac{[y]}{[x]}$ iff $[x] \cdot [z] \preceq_* [y]$.
3. $[x] \&_\Pi ([y] \ominus [z]) = [x] \cdot (\sup([y], [z]) - [z]) = (\sup([x] \cdot [y], [x] \cdot [z]) - [x] \cdot [z]) = ([x] \&_\Pi [y]) \ominus ([x] \&_\Pi [z])$.□

A *hyperrational valued* ŁΠ$\frac{1}{2}$*-matrix* is a structure $\mathfrak{L}_{^*\mathbf{Q}} = \langle {}^*\mathbf{Q}_{[0,1]}, \oplus, \neg_L, \rightarrow_\Pi, \&_\Pi, {}^*0, {}^*1, {}^*\frac{1}{2} \rangle$, where the reduct $\langle {}^*\mathbf{Q}_{[0,1]}, \oplus, \neg_L, \rightarrow_\Pi, \&_\Pi, {}^*0, {}^*1 \rangle$ is a hyperrational valued ŁΠ-matrix and the identity ${}^*\frac{1}{2} = \neg_L {}^*\frac{1}{2}$ holds.

The truth value ${}^*0 \in {}^*\mathbf{Q}_{[0,1]}$ of a hyperrational valued ŁΠ- (resp. ŁΠ$\frac{1}{2}$)-matrix is falsity, the truth value ${}^*1 \in {}^*\mathbf{Q}_{[0,1]}$ is truth, and other truth values $x \in {}^*\mathbf{Q}_{[0,1]} \backslash \{{}^*0, {}^*1\}$ are called neutral.

If we replace the set $\mathbf{Q}_{[0,1]}$ by $\mathbf{R}_{[0,1]}$ and the set ${}^*\mathbf{Q}_{[0,1]}$ by ${}^*\mathbf{R}_{[0,1]}$ in all above definitions, then we obtain *hyperreal valued* ŁΠ*-matrix* (resp. *hyperreal valued* ŁΠ$\frac{1}{2}$*-matrix*) $\mathfrak{L}_{^*\mathbf{R}}$.

3.2 p-Adic Valued BL-matrix

Extend the standard order structure on $\mathbf{N}$ to a partial order structure on $\mathbf{Z}_p$. We know that each finite number of $\mathbf{Z}_p$ can be identified with a positive integer.

1. For any finite numbers x, $y \in \mathbf{Z}_p$ we have $x \preceq_p y$ iff $x \leq y$ in $\mathbf{N}$.
2. Each finite natural number x is less than any infinite number y, i.e. $x \prec_p y$ for any $x \in \mathbf{N}$ and $y \in \mathbf{Z}_p \backslash \mathbf{N}$, $y \neq 0$. Notice that we have $x \prec_p y$ iff $x \neq y$ and $x \preceq_p y$.

Define this partial order structure on $\mathbf{Z}_p$ as follows:

$\mathcal{O}_{\mathbf{Z}_p}$ Let $x = \ldots x_n \ldots x_1 x_0$ and $y = \ldots y_n \ldots y_1 y_0$ be the canonical expansions of two p-adic integers $x, y \in \mathbf{Z}_p$. (1) We set $x \prec_p y$ if the following three conditions hold: (i) there exists n such that $x_n < y_n$; (ii) $x_k \leq y_k$ for all $k > n$; (iii) x is a finite integer, i.e. there exists l such that $x_m = 0$ for all $m \geq l$. (2) We set $x = y$ if $x_n = y_n$ for each $n = 0, 1, \ldots$ (3) Suppose that both x and y are infinite integers. We set $x \preceq_p y$ if we have $x_n \leq y_n$ for each $n = 0, 1, \ldots$ and we set $x \prec_p y$ if we have $x_n \leq y_n$ for each $n = 0, 1, \ldots$ and there exists n_0 such that $x_{n_0} < y_{n_0}$.

Now introduce two operations sup, inf in the partial order structure on $\mathbf{Z}_p$: $\sup(x, y) = y$ and $\inf(x, y) = x$ iff $x \preceq_p y$. Let $x = \ldots x_n \ldots x_1 x_0$ and $y = \ldots y_n \ldots y_1 y_0$ be the canonical expansions of two p-adic integers $x, y \in \mathbf{Z}_p$ and x, y are incompatible in $\mathcal{O}_{\mathbf{Z}_p}$. We get $\inf(x, y) = z = \ldots z_n \ldots z_1 z_0$, where, for each $n = 0, 1, \ldots$, we set (1) $z_n = y_n$ if $x_n \geq y_n$, (2) $z_n = x_n$ if $x_n \leq y_n$, (3) $z_n = x_n = y_n$ if $x_n = y_n$. We get $\sup(x, y) = z = \ldots z_n \ldots z_1 z_0$, where, for each $n = 0, 1, \ldots$, we set (1) $z_n = y_n$ if $x_n \leq y_n$, (2) $z_n = x_n$ if $x_n \geq y_n$, (3) $z_n = x_n = y_n$ if $x_n = y_n$.

It is important to remark that there exists the maximal number $N_{max} \in \mathbf{Z}_p$ in $\mathcal{O}_{\mathbf{Z}_p}$. It is easy to see: $N_{max} = -1 = (p-1) + (p-1) \cdot p + \ldots + (p-1) \cdot p^k + \ldots$[1]

Further, consider the following new operations defined for all $x, y \in \mathbf{Z}_p$ in the partial order structure $\mathcal{O}_{\mathbf{Z}_p}$:

- $x \rightarrow_L y = N_{max} - \sup(x, y) + y$,
- $x \rightarrow_\Pi y = \begin{cases} N_{max} & \text{if } x \preceq_p y, \\ \text{integral part of } \dfrac{y}{x} & \text{otherwise,} \end{cases}$
- $\neg_L x = N_{max} - x$, i.e. $x \rightarrow_L 0$,
- $\neg_\Pi x = \begin{cases} N_{max} & \text{if } x = 0, \\ 0 & \text{otherwise,} \end{cases}$ i.e. $\neg_\Pi x = x \rightarrow_\Pi 0$,
- $\Delta x = \begin{cases} N_{max} & \text{if } x = N_{max}, \\ 0 & \text{otherwise,} \end{cases}$ i.e. $\Delta x = \neg_\Pi \neg_L x$,
- $x \&_L y = \sup(x, N_{max} - y) + y - N_{max}$, i.e. $x \&_L y = \neg_L(x \rightarrow_L \neg_L y)$,
- $x \&_\Pi y = x \cdot y$,
- $x \oplus y := \neg_L x \rightarrow_L y$,
- $x \ominus y := x \&_L \neg_L y$,
- $x \wedge y = \inf(x, y)$, i.e. $x \wedge y = x \&_L (x \rightarrow_L y)$,
- $x \&_G y := \inf(x, y)$,
- $x \vee y = \sup(x, y)$, i.e. $x \vee y = (x \rightarrow_L y) \rightarrow_L y$,

[1] If $x = \sum_{i=n}^{\infty} x_i \cdot p^i$ then $-x = \sum_{i=n}^{\infty} y_i \cdot p^i$, where $y_n = p - x_n$ and $y_i = (p-1) - x_i$ for $i > n$. For example, $-0 = 0$ and if $p = 5$ we have $\dfrac{1}{3} = \ldots 1313132$ and $-\dfrac{1}{3} = \ldots 131313$.

- $x \leftrightarrow y := (x \rightarrow_L y) \wedge (y \rightarrow_L x)$,
- $x \rightarrow_G y = \begin{cases} N_{max} & \text{if } x \preceq_p y, \\ y & \text{otherwise,} \end{cases}$ i.e. $x \rightarrow_G y = \Delta(x \rightarrow_L y) \vee y$.

Proposition 2 *A structure $\langle \mathbf{Z}_p, \oplus, \neg_L, 0 \rangle$ is a p-adic valued MV-algebra.* □

Proposition 3 *A structure $\langle \mathbf{Z}_p, \oplus, \neg_L, \rightarrow_\Pi, \&_\Pi, 0, N_{max} \rangle$ is not a p-adic valued ŁΠ-matrix.*

Proof. Indeed, it can be easily shown that $\langle \mathbf{Z}_p, \vee, \wedge, \rightarrow_\Pi, \&_\Pi, 0, N_{max} \rangle$ is not a Π-algebra. □

Proposition 4 *A structure $\mathfrak{L}_{\mathbf{Z}_p} = \langle \mathbf{Z}_p, \wedge, \vee, *, \Rightarrow, 0, N_{max} \rangle$, where $* \in \{\&_L, \&_G\}$ and $\Rightarrow \in \{\rightarrow_L, \rightarrow_G\}$ is a p-adic valued BL-matrix. If $* = \&_L$ and $\Rightarrow = \rightarrow_L$, it is called a p-adic valued* **L**-*algebra. If $* = \&_G$ and $\Rightarrow = \rightarrow_G$, it is called a p-adic valued* **G**-*algebra.*

Proof. We can show that (1) $\langle \mathbf{Z}_p, \wedge, \vee, 0, N_{max} \rangle$ is a lattice with the largest element N_{max} and the least element 0, (2) $\langle \mathbf{Z}_p, *, N_{max} \rangle$ is a commutative semigroup with the unit element N_{max}, i.e. $*$ is commutative, associative, and $N_{max} * x = x$ for all $x \in \mathbf{Z}_p$, (3) the following conditions hold

$$z \preceq_p (x \Rightarrow y) \text{ iff } x * z \preceq_p y \text{ for all } x, y, z \in \mathbf{Z}_p;$$

$$x \wedge y = x * (x \Rightarrow y);$$

$$x \vee y = ((x \Rightarrow y) \Rightarrow y) \wedge ((y \Rightarrow x) \Rightarrow x);$$

$$(x \Rightarrow y) \vee (y \Rightarrow x) = N_{max}.$$

The proof is completed. □

The truth value $0 \in \mathbf{Z}_p$ of a p-adic valued BL-matrix is called falsity, the truth value N_{max} is called truth, and other truth values $x \in \mathbf{Z}_p \backslash \{0, N_{max}\}$ are called neutral.

We can dualize the order $\preceq_p$ in the following natural way: $x \succeq_p^N y$ iff $x \preceq_p y$ and $x \neq 0$. As we see, 1 was the least positive p-adic integer due to $\preceq_p$ and became the maximal number due to $\preceq_p^N$ (respectively, -1 was the largest p-adic integer and became the least positive integer). Let us set 0 as the minimal.

Reintroduce two operations sup, inf in the new partial order structure on $\mathbf{Z}_p$: $\sup(x, y) = y$ and $\inf(x, y) = x$ iff $x \preceq_p^N y$. If two p-adic integers x, y are incompatible, their maximum and minimum are defined digit by digit too. Consider the following new operations defined for all $x, y \in \mathbf{Z}_p$ in the partial order structure ordered by $\preceq_p^N$:

- $x \rightarrow_L y = 1 - \sup(x, y) + y$,
- $x \rightarrow_\Pi y = \begin{cases} 1 & \text{if } x \preceq_p^N y, \\ \text{integral part of } \dfrac{y}{x} & \text{otherwise,} \end{cases}$
- $\neg_L x = 1 - x$, i.e. $x \rightarrow_L 0$,
- $\neg_\Pi x = \begin{cases} 1 & \text{if } x = 0, \\ 0 & \text{otherwise,} \end{cases}$ i.e. $\neg_\Pi x = x \rightarrow_\Pi 0$,
- $\Delta x = \begin{cases} 1 & \text{if } x = 1, \\ 0 & \text{otherwise,} \end{cases}$ i.e. $\Delta x = \neg_\Pi \neg_L x$,
- $x \&_L y = \sup(x, 1 - y) + y - 1$, i.e. $x \&_L y = \neg_L(x \rightarrow_L \neg_L y)$,
- $x \&_\Pi y = x \cdot y$,
- $x \oplus y := \neg_L x \rightarrow_L y$,
- $x \ominus y := x \&_L \neg_L y$,
- $x \wedge y = \inf(x, y)$, i.e. $x \wedge y = x \&_L (x \rightarrow_L y)$,

- $x \vee y = \sup(x, y)$, i.e. $x \vee y = (x \rightarrow_L y) \rightarrow_L y$,
- $x \leftrightarrow y := (x \rightarrow_L y) \wedge (y \rightarrow_L x)$,
- $x \rightarrow_G y = \begin{cases} 1 & \text{if } x \preceq_p^N y, \\ y & \text{otherwise,} \end{cases}$ i.e. $x \rightarrow_G y = \Delta(x \rightarrow_L y) \vee y$.

Proposition 5 *A structure $\mathfrak{L}'_{\mathbf{Z}_p} = \langle \mathbf{Z}_p, \wedge, \vee, *, \Rightarrow, 0, 1\rangle$, where $* \in \{\&_L, \&_G\}$ and $\Rightarrow \in \{\rightarrow_L, \rightarrow_G\}$ is a p-adic valued BL-matrix.* □

The truth value $0 \in \mathbf{Z}_p$ of a p-adic valued BL-matrix is called falsity, the truth value $1 \in \mathbf{Z}_p$ is called truth, and other truth values $x \in \mathbf{Z}_p \backslash \{0, 1\}$ are called neutral.

4 Non-Archimedean and p-adic Valued Logics

4.1 Non-Archimedean and p-adic Valued Logical Language

Definition 1 *An infinite-order predicate logical language $\mathcal{L}^{\infty}_{^*V}$ (with the set of truth values $^*V \in \{^*\mathbf{Q}_{[0,1]}, {}^*\mathbf{R}_{[0,1]}, \mathbf{Z}_p\}$) consists of the following symbols:*

- *Variables:*
  - *denumerable set of free variables $a_0, a_1, a_2, \ldots$;*
  - *denumerable set of bound variables $x_0, x_1, x_2, \ldots$;*
  - *infinite set of i-order monadic predicate variables $\{Q_i^{j_i}$: where $i \in \omega$ and j_i belongs to the set K_i, for any $i \in \omega\}$, where $\{K_i\}_{i\in\omega}$ is a family of set of indices;*
  - *infinite set of infinite-order (ω-order) monadic predicate variables $\{Q_\omega^{j_i}$: where j_i belongs to the set K_i, for any $i \in \omega\}$, where $\{K_i\}_{i\in\omega}$ is a family of set of indices.*
- *Constants:*
  - *denumerable set of constant symbols $c_0, c_1, c_2, \ldots$;*
  - *denumerable set of function symbols of arity n: $f_0^n, f_1^n, f_2^n, \ldots$;*
  - *first-order monadic predicate signs: $R_0, R_1, R_2, \ldots$.*
- *Logical symbols:*
  - *propositional connectives $\rightarrow_L, \rightarrow_G, \rightarrow_\Pi, \&_L, \&_G, \&_\Pi, \neg_L, \neg_\Pi$;*
  - *quantifiers $\forall, \exists$.*
- *Auxiliary symbols (,).*

Terms, well-formed atomic formulas, and well-formed first-order formulas of $\mathcal{L}^{\infty}_{^*V}$ are defined in the standard way. Let us set P_j, P_ω, N as metavariables for j-order predicate variables, ω-order predicate variables, predicate signs respectively. The sequence of quantifiers $\mathrm{Q}_i P_i \mathrm{Q}_{i-1} P_{i-1} \ldots \mathrm{Q}_1 P_1$ of the formula

$$\mathrm{Q}_i P_i \mathrm{Q}_{i-1} P_{i-1} \ldots \mathrm{Q}_1 P_1 \;\; \Psi(P_i(P_{i-1}(\ldots(P_1(t))\ldots))),$$

where $\mathrm{Q}_i, \mathrm{Q}_{i-1}, \ldots, \mathrm{Q}_1 \in \{\forall, \exists\}$ and t is a term, will be denoted by $\mathrm{Q}^i P_i$ and Q^i will be said to be $i+1$-order vertical quantifier. The sequence of quantifiers $\mathrm{Q}_\infty P_\omega \ldots \mathrm{Q}_{i-1} P_{i-1} \ldots \mathrm{Q}_1 P_1$ of the formula

$$\mathrm{Q}_\infty P_\omega \ldots \mathrm{Q}_i P_i \mathrm{Q}_{i-1} P_{i-1} \ldots \mathrm{Q}_1 P_1 \;\; \Psi(P_\omega(\ldots(P_i(P_{i-1}(\ldots(P_1(t))\ldots)))\ldots)),$$

where $\mathrm{Q}_\infty, \ldots, \mathrm{Q}_i, \mathrm{Q}_{i-1}, \ldots, \mathrm{Q}_1 \in \{\forall, \exists\}$ and t is a term, will be denoted by $\mathrm{Q}^\infty P_\omega$ and Q^∞ will be said to be ω-order vertical quantifier.

Let us define *substitution* for predicate variables of higher order: if P_i (resp. P_ω) is an i-order (resp. ω-order) predicate variable and a first-order formula Ψ is a formula possibly containing the predicate sign N, then $\Psi[N/P_i]$ (resp. $\Psi[N/P_\omega]$) is the result of replacing all instances of N (they always are free) by P_i (resp. P_ω) in Ψ.

Well-formed higher-order formulas of $\mathcal{L}^{\infty}_{^*V}$ are inductively defined as follows:

1. If Ψ is a first-order formula not containing the predicate variables P_i, P_{i-1}, …, P_1, and N is a first-order predicate, Q^i is an $i+1$-order vertical quantifier, then
$$\mathrm{Q}_i P_i \mathrm{Q}_{i-1} P_{i-1} \dots \mathrm{Q}_1 P_1 \;\; \Psi[N/P_i(P_{i-1}(\dots(P_1(\cdot))\dots))],$$
where $\Psi[N/P_i(P_{i-1}(\dots(P_1(\cdot))\dots))]$ is obtained from Ψ by substitution, i.e. by replacing $N(\cdot)$ by
$$P_i(P_{i-1}(\dots(P_1(\cdot))\dots))$$
at every occurrence of N in Ψ, is an $i+1$-order formula.

2. If Ψ is a first-order not containing the predicate variables P_ω, …, P_i, P_{i-1}, …, P_1, and N is a first-order predicate, Q^∞ is an ω-order vertical quantifier, then
$$\mathrm{Q}_\infty P_\omega \dots \mathrm{Q}_i P_i \mathrm{Q}_{i-1} P_{i-1} \dots \mathrm{Q}_1 P_1$$
$$\Psi[N/P_\omega(\dots(P_i(P_{i-1}(\dots(P_1(\cdot))\dots)))\dots)],$$
where $\Psi[N/P_\omega(\dots(P_i(P_{i-1}(\dots(P_1(\cdot))\dots)))\dots)]$ is obtained from Ψ by replacing $N(\cdot)$ by
$$P_\omega(\dots(P_i(P_{i-1}(\dots(P_1(\cdot))\dots)))\dots)$$
at every occurrence of N in Ψ, is an ω-order formula.

A higher-order formula is called *vertical bounded* if it doesn't contain first-order predicates and all predicate variables are quantified.

4.2 Semantics for Non-Archimedean and p-adic Valued Predicate Logical Language

Let $\mathfrak{L}_{*V}$ be a non-Archimedean valued ŁΠ-matrix (or p-adic valued BL-matrix). An $\mathfrak{L}_{*V}$-structure $\mathcal{M} = \langle D, v_\mathcal{M}\rangle$ for first-order formulas of $\mathcal{L}^\infty_{*V}$ consists of the following:

- a non-empty set D;
- a mapping $v_\mathcal{M}$ such that
  - each constant symbol c is mapped to an element d of D,
  - each n-place function symbol f^n to a function $F^n : D^n \mapsto D$,
  - each monadic predicate symbol R to a *fuzzy* subset $\mathcal{P}$ of D (i.e. $\mathcal{P} \in V^D$, where $V \in \{\mathbf{Q}, \mathbf{R}, \{0,1,\dots, p-1\}\}$),
  - each free variable symbol a is mapped to an element d' of D,
  - for every n-place function symbol f^n and terms t_1, …, t_n, $v_\mathcal{M}(f^n(t_1, \dots, t_n)) = F^n(v_\mathcal{M}(t_1), \dots, v_\mathcal{M}(t_n))$,
  - for every monadic predicate symbol R and term t, $v_\mathcal{M}\big(R(t)\big) = \mathcal{P}(v_\mathcal{M}(t))$.

Now extend an $\mathfrak{L}_{*V}$-structure $\mathcal{M} = \langle D, v_\mathcal{M}\rangle$ to a *higher-order* $\mathfrak{L}_{*V}$*-structure* $\mathcal{M}_\infty = \langle(\mathcal{F}_\infty(D))_{n\in\omega}, v^\infty_\mathcal{M}\rangle$ as follows: (1) each monadic first-order predicate variable $Q_1^{j_1}$ is mapped to $\mathcal{P}_1$, a fuzzy subset of D (i.e. $\mathcal{P}_1 \in V^D$, where $V \in \{\mathbf{Q}, \mathbf{R}, \{0, 1, \dots, p-1\}\}$); (2) each monadic i-order predicate variable $Q_i^{j_i}$ is mapped to $\mathcal{P}_i$, a member of $\mathcal{F}_i(D) := \underbrace{V^{\cdot^{\cdot^{\cdot^{V^{V^D}}}}}}_{i+1}$; (3) each monadic ω-order predicate variable $Q_\omega^{j_\omega}$ is mapped to $\mathcal{P}_\omega$, a member of $\mathcal{F}_\omega(D) := \underbrace{V^{\cdot^{\cdot^{\cdot^{V^{V^D}}}}}}_{\omega}$; (4) for each monadic i-order predicate variable $Q_i^{j_i}$ and term t, $v^\infty_\mathcal{M}(Q_i^{j_i}(t)) = \mathcal{P}_i(v^\infty_\mathcal{M}(t))$; (5) for each monadic ω-order predicate variable $Q_\omega^{j_\omega}$ and term t, $v^\infty_\mathcal{M}(Q_\omega^{j_\omega}(t)) = \mathcal{P}_\omega(v^\infty_\mathcal{M}(t))$.

Let $\mathfrak{L}_{*V}$ be an ŁΠ-matrix (BL-matrix) and $\mathcal{M}_\infty$ be an $\mathfrak{L}_{*V}$-structure for $\mathcal{L}^\infty_{*V}$.

Definition 2 *An i-order truth assignment on a higher-order $\mathfrak{L}_{*V}$-structure $\mathcal{M}_\infty$ is a function $\Big[\cdot\Big]^i$ whose domain is the set of all i-order formulas of $\mathcal{L}^\infty_{*V}$ and whose range is the set V^i of truth values such that:*

1. *For any first-order formula Φ, $\Big[\Phi\Big]^1$ is a truth assignment of the logic $Ł\Pi\forall$ (respectively, $BL\forall$). Extend $\Big[\cdot\Big]^1$ as follows:*

$$\Big[P_i(t)\Big]^1 = v^\infty_{\mathcal{M}}(P_i(t)) = \mathcal{P}_i(v^\infty_{\mathcal{M}}(t)).$$

2. *For any formula Φ without vertical quantifiers,*

$$\Big[\Phi\Big]^i = \langle\underbrace{\Big[\Phi\Big]^1, \ldots, \Big[\Phi\Big]^1}_{i}\rangle.$$

3. *For any i-order formula Φ, $\Big[\neg_L\Phi\Big]^i = \neg_L\Big[\Phi\Big]^i = \langle \top - k_1, \top - k_2, \ldots, \top - k_i\rangle$, where $\Big[\Phi\Big]^i = \langle k_1, k_2, \ldots, k_i\rangle$ and $\top$ is the largest element of V.*

4. *For any i-order formulas Φ and Ψ,*

$$\Big[\Phi \odot \Psi\Big]^i = \Big[\Phi\Big]^i \odot \Big[\Psi\Big]^i = \langle (x_1 \odot y_1), (x_2 \odot y_2), \ldots, (x_i \odot y_i)\rangle,$$

where $\odot \in \{\rightarrow_L, \rightarrow_G, \rightarrow_\Pi, \&_L, \&_G, \&_\Pi\}$, $\Big[\Phi\Big]^i = \langle x_1, x_2, \ldots, x_i\rangle$ and $\Big[\Psi\Big]^i = \langle y_1, y_2, \ldots, y_i\rangle$.

5. *For any i-order vertical bounded formula $\mathrm{Q}^{i-1}P_{i-1}\Psi(P_{i-1})$,*

$$\Big[\mathrm{Q}^{i-1}P_{i-1}\Psi(P_{i-1})\Big]^i = \langle k_0, \ldots, k_{i-1}\rangle,$$

where $k_0 = \Big[\Psi(P_j)\Big]^1$ and for $j \in \{1, \ldots, i-1\}$ we have

$$k_j = \sup\Big(\bigcup_{\mathcal{P}_j \in \mathcal{F}_j(D)} \Big[\Psi(P_j)\Big]^1\Big),$$

if $\mathrm{Q}_j = \exists$ and

$$k_j = \inf\Big(\bigcup_{\mathcal{P}_j \in \mathcal{F}_j(D)} \Big[\Psi(P_j)\Big]^1\Big),$$

if $\mathrm{Q}_j = \forall$, where $\mathcal{P}_j(v^\infty_{\mathcal{M}}(\cdot)) = v^\infty_{\mathcal{M}}(P_j(\cdot))$.

6. *For any i-order vertical bounded formula $\forall x\, \mathrm{Q}^{i-1}P_{i-1}\,\Phi(P_{i-1}(x))$,*

$$\begin{aligned}\Big[\forall x\, \mathrm{Q}^{i-1}P_{i-1}\,\Phi(P_{i-1}(x))\Big]^i &= \Big[\mathrm{Q}^{i-1}P_{i-1}\,\forall x\,\Phi(P_{i-1}(x))\Big]^i \\ &= \inf_{a\in D}\Big[\Phi(P_{i-1}(x))\Big]^1 \wedge \Big[\mathrm{Q}^{i-1}P_{i-1}\,\Phi(P_{i-1}(x))\Big]^i,\end{aligned}$$

where $a = v^\infty_{\mathcal{M}}(x)$.

Notice that for the p-adic multiple-validity, conditions 2, 3, 4 are other:

2'. For any k-order formula Φ_k $(k < i)$, $\Big[\Phi_k\Big]^i = \langle y_1, y_2, \ldots, y_k, \underbrace{0, \ldots, 0}_{k-i}\rangle$ iff we have $\Big[\Phi_k\Big]^k =$

$\langle y_1, y_2, \ldots, y_k\rangle$.

3'. For any k-order formula Φ_k $(k < i)$, $\Big[\neg_L \Phi_k\Big]^i = \neg_L \Big[\Phi_k\Big]^i = (p^i - 1) - \Big[\Phi_k\Big]^i$.

4'. For any higher-order formulas Φ_k and Ψ_m such that $\max(k, m) \leq i$,

$$\Big[\Phi_k \odot \Psi_m\Big]^i = \Big[\Phi_k\Big]^i \odot \Big[\Psi_m\Big]^i = \langle (x_1 \odot y_1), (x_2 \odot y_2), \ldots, (x_i \odot y_i)\rangle,$$

where $\odot \in \{\rightarrow_L, \rightarrow_G, \rightarrow_\Pi, \&_L, \&_G, \&_\Pi\}$, $\Big[\Phi_k\Big]^i = \langle x_1, x_2, \ldots, x_i\rangle$, and $\Big[\Psi_m\Big]^i = \langle y_1, y_2, \ldots, y_i\rangle$.

Definition 3 *An infinite-order (ω-order) truth assignment on a higher-order $\mathfrak{L}_{^*V}$-structure $\mathcal{M}_\infty$ is a function $\Big[\cdot\Big]^\infty$ whose domain is the set of all infinite-order formulas of $\mathcal{L}^\infty_{^*V}$ and whose range is the set $^*V$ of truth values such that:*

1. *Extend $\Big[\cdot\Big]^1$ as follows:*

$$\Big[P_\omega(t)\Big]^1 = v^\infty_{\mathcal{M}}(P_\omega(t)) = \mathcal{P}_\omega(v^\infty_{\mathcal{M}}(t)).$$

2. *For any formula Φ without vertical quantifiers,*

$$\Big[\Phi\Big]^\infty = {}^*y_1 = \langle y_1, y_1, \ldots\rangle \text{ iff } \Big[\Phi\Big]^1 = y_1.$$

3. *For any formula Φ, $\Big[\neg_L \Phi\Big]^\infty = \neg_L \Big[\Phi\Big]^\infty = {}^*\top - \Big[\Phi\Big]^\infty$, where $^*\top$ is the largest member of $^*V$, the symbol $\neg_L$ on the right-hand side is the corresponding operation in $\mathfrak{L}_{^*V}$.*

4. *For any formulas Φ and Ψ,*

$$\Big[\Phi \odot \Psi\Big]^\infty = \Big[\Phi\Big]^\infty \odot \Big[\Psi\Big]^\infty,$$

where $\odot \in \{\rightarrow_L, \rightarrow_G, \rightarrow_\Pi, \&_L, \&_G, \&_\Pi\}$, the symbol $\odot$ on the right-hand side is an appropriate operation in $\mathfrak{L}_{^*V}$.

5. *For any ω-order vertical bounded formula $\mathrm{Q}^\infty P_\omega \Psi(P_\omega)$,*

$$\Big[\mathrm{Q}^\infty P_\omega \Psi(P_\omega)\Big]^\infty = [f] = [\langle k_0, \ldots, k_{i-1}, \ldots\rangle],$$

where $k_0 = \Big[\Psi(P_\omega)\Big]^1$ and for $j \in \{1, 2, \ldots\}$ we have

$$k_j = \sup\Big(\bigcup_{\mathcal{P}_j \in \mathcal{F}_j(D)} \Big[\Psi(P_j)\Big]^1\Big)$$

if $\mathrm{Q}_j = \exists$ and

$$k_j = \inf\Big(\bigcup_{\mathcal{P}_j \in \mathcal{F}_j(D)} \Big[\Psi(P_j)\Big]^1\Big)$$

if $\mathrm{Q}_j = \forall$, where $\mathcal{P}_j(v^\infty_{\mathcal{M}}(\cdot)) = v^\infty_{\mathcal{M}}(P_j(\cdot))$.

6. *For any ω-order vertical bounded formula $\forall x\, \mathrm{Q}^\infty P_\omega\, \Phi(P_\omega(x))$,*

$$\Big[\forall x\, \mathrm{Q}^\infty P_\omega\, \Phi(P_\omega(x))\Big]^\infty = \Big[\mathrm{Q}^\infty P_\omega\, \forall x\, \Phi(P_\omega(x))\Big]^\infty =$$
$$= \inf_{a \in D} \Big[\Phi(P_\omega(x))\Big]^1 \wedge \Big[\mathrm{Q}^\infty P_\omega\, \Phi(P_\omega(x))\Big]^\infty,$$

where $a = v^\infty_{\mathcal{M}}(x)$.

Notice that for the p-adic multiple-validity, condition 2 is another:

2'. For any i-order formula Φ_i, $\Big[\Phi_i\Big]^\infty = \ldots 00 \ldots 00 y_{i-1} \ldots y_1 y_0$ iff $\Big[\Phi_i\Big]^i = \langle y_0, y_1, \ldots, y_{i-1}\rangle$.

We say that an i-order formula Φ_i (resp. ω-order formula Φ_∞) is *logically valid*/is an $\mathfrak{L}_{^*V}$*-tautology* if $\Big[\Phi_i\Big]^i = \langle \top, \ldots, \top\rangle$ (resp. $\Big[\Phi_\infty\Big]^\infty = {}^*\top$) for each higher-order $\mathfrak{L}_{^*V}$-structure $\mathcal{M}_\infty$. An i-order formula Φ_i (resp. ω-order formula Φ_∞) is *logically satisfiable* if $\Big[\Phi_i\Big]^i \neq \langle \bot, \ldots, \bot\rangle$ (resp. $\Big[\Phi_\infty\Big]^\infty \neq {}^*\bot$) for some $\mathfrak{L}_{^*V}$-structures $\mathcal{M}_\infty$, where $\bot$ is the smallest member of V.

We say that an $\mathfrak{L}_{^*V}$-structure $\mathcal{M}_\infty$ is an i-order $\mathfrak{L}_{^*V}$*-model* (resp. an ω-order $\mathfrak{L}_{^*V}$*-model*) of an $\mathcal{L}^\infty_{^*V}$-theory T iff $\Big[\Phi\Big]^i = \langle \top, \ldots, \top\rangle$ (resp. $\Big[\Phi\Big]^\infty = {}^*\top$) for each $\Phi \in T$.

For any higher-order formula Ψ_i (resp. Ψ_∞), a *logical matrix* of Ψ_i (resp. Ψ_∞) is said to be a result of replacing each i-order (resp. ω-order) predicate variable of Ψ_i (resp. of Ψ_∞) by an appropriate first-order predicate constant with rejecting all vertical quantifiers, i.e. we eliminate vertical quantifiers and in addition, different i-order (resp. ω-order) predicate variables of Ψ_i (resp. of Ψ_∞) bounded by different quantifiers are mapped by a substitution into different first-order predicate constants (namely this substitution is a bijection).

Theorem 1 *Suppose that Ψ_i (resp. Ψ_∞) is an i-order (resp. ω-order) formula and Ψ is its logical matrix, i.e. Ψ is a first-order formula in that every i-order (resp. ω-order) predicate variable of Ψ_i (resp. of Ψ_∞) is replaced by a first-order predicate constant, then Ψ_i (resp. Ψ_∞) is logically valid/satisfiable iff Ψ is logically valid/satisfiable.*

Proof. It follows from the semantic rules of $\mathcal{L}^\infty_{^*V}$. □

4.3 Non-Archimedean Valued Logics $\text{Ł}\Pi\forall_\infty$ and $\text{Ł}\Pi\frac{1}{2}\forall_\infty$

Let us construct a non-Archimedean (infinite-order) extension of ŁΠ∀ denoted by $\text{Ł}\Pi\forall_\infty$. The logic $\text{Ł}\Pi\forall_\infty$ is given by the following axioms:

- the axioms of Łukasiewicz propositional logic;
- the axioms of product propositional logic;
- $\Delta(\Phi \to_L \Psi) \to_L (\Phi \to_\Pi \Psi)$, where Φ, Ψ are either first-order or higher-order formulas;
- $\Delta(\Phi \to_\Pi \Psi) \to_L (\Phi \to_L \Psi)$, where Φ, Ψ are either first-order or higher-order formulas;
- $\Phi \&_\Pi (\Gamma \ominus \Psi) \leftrightarrow_L (\Phi \&_\Pi \Gamma) \ominus (\Phi \&_\Pi \Psi)$, where Φ, Γ, Ψ are either first-order or higher-order formulas;
- $\forall x\, \Phi(x, y_1, \ldots, y_k) \to \Phi(t, y_1, \ldots, y_k)$, where t is substitutable for x in Φ and Φ is either a first-order or higher-order formula;
- $\forall x\, (\Gamma \to_L \Phi) \to (\Gamma \to_L \forall x\, \Phi)$, where x is not free in Γ and Φ, Γ are either first-order or higher-order formulas;
- $x = x$;
- $(x = y) \to \Delta(\Phi(x, y_1, \ldots, y_k) \leftrightarrow \Phi(y, y_1, \ldots, y_k))$, where Φ is either a first-order or higher-order formula.

These axioms are said to be *horizontal*. Introduce also some new axioms that show basic properties of non-Archimedean ordered structures. These express a connection between formulas of various level. It is well known that there exist infinitesimals that are less than any positive number of $[0, 1]$. This property can be expressed by means of the following logical axiom:

$$(\neg_L(\Psi_1 \leftrightarrow \Psi_\infty) \wedge \neg_L(\Phi_1 \leftrightarrow {}^*\bot)) \to_L (\Psi_\infty \to_L \Phi_1), \tag{1}$$

where Ψ_1, Φ_1 are first-order formulas and Ψ_∞ is an ω-order formula that has the logical matrix Ψ_1.

Axiom (1) is said to be *vertical*.

The deduction rules of $Ł\Pi\forall_\infty$ are modus ponens, Δ-necessitation (from Φ infer $\Delta\Phi$), and generalization.

The logic $Ł\Pi\frac{1}{2}\forall_\infty$ results from $Ł\Pi\forall_\infty$ by adding the axiom ${}^*\dfrac{1}{2} \leftrightarrow \neg_L {}^*\dfrac{1}{2}$, where ${}^*\dfrac{1}{2}$ is a new truth constant.

The notions of proof, derivability $\vdash$, theorem, and theory over $Ł\Pi\forall_\infty$ and $Ł\Pi\frac{1}{2}\forall_\infty$ are defined as usual.

Theorem 2 *Take $\langle r, \ldots, r\rangle \in V^i$ (resp. ${}^*r \in {}^*V$), where $V = \mathbf{Q}_{[0,1]}$ or $V = \mathbf{R}_{[0,1]}$. Then there is an i-order (resp. ω-order) formula Φ of $Ł\Pi\frac{1}{2}{}_\infty$ such that $\Big[\Phi\Big]^i = \langle r, \ldots, r\rangle$ (resp. $\Big[\Phi\Big]^\infty = {}^*r$) for any valuation.*

Proof. The first-order case is proved in [3] and the higher-order case follows from the semantic rules of $\mathcal{L}^\infty_{*V}$.□

Theorem 3 (Completeness) *Let Φ be an i-order (resp. ω-order) formula of $\mathcal{L}^\infty_{*V}$, T an i-order (resp. ω-order) $\mathcal{L}^\infty_{*V}$-theory. Then the following conditions are equivalent:*

- *$T \vdash \Phi$;*
- *$\Big[\Phi\Big]^i = \langle\underbrace{\top, \ldots, \top}_{i}\rangle$ (resp. $\Big[\Phi\Big]^\infty = {}^*\top$) for each $\mathfrak{L}_{*V}$-model $\mathcal{M}_\infty$ of T;*

Proof. The first-order case is proved in [4] and the higher-order case follows from Theorem 1. □

4.4 p-Adic valued Logic $BL\forall_\infty$

Let us construct an infinite-order extension of $BL\forall$ denoted by $BL\forall_\infty$. Remember that the logic $BL\forall$ has just two propositional operations: &, $\rightarrow$, which are understood as t-norm and its residuum respectively.

In $BL\forall$ we can define the following new operations:

- $\Phi \wedge \Psi := \Phi \& (\Phi \rightarrow \Psi)$,
- $\Phi \vee \Psi := ((\Phi \rightarrow \Psi) \rightarrow \Psi) \wedge ((\Psi \rightarrow \Phi) \rightarrow \Phi)$,
- $\neg\Phi := \Phi \rightarrow 0$,
- $\Phi \leftrightarrow \Psi := (\Phi \rightarrow \Psi) \& (\Psi \rightarrow \Phi)$,
- $\Phi \oplus \Psi := \neg\Phi \rightarrow \Psi$,
- $\Phi \ominus \Psi := \Phi \& \neg\Psi$.

The logic $BL\forall_\infty$ is given by the following axioms, where all subformulas are either first-order or i-order (ω-order):

$$(\Phi \rightarrow \Psi) \rightarrow ((\Psi \rightarrow \Gamma) \rightarrow (\Phi \rightarrow \Gamma)), \tag{2}$$

$$(\Phi \& \Psi) \rightarrow \Phi, \tag{3}$$

$$(\Phi \& \Psi) \rightarrow (\Psi \& \Phi), \tag{4}$$

$$(\Phi \& (\Phi \rightarrow \Psi)) \rightarrow (\Psi \& (\Psi \rightarrow \Phi)), \tag{5}$$

$$(\Phi \rightarrow (\Psi \rightarrow \Gamma)) \rightarrow ((\Phi \& \Psi) \rightarrow \Gamma), \tag{6}$$

$$((\Phi \& \Psi) \rightarrow \Gamma) \rightarrow (\Phi \rightarrow (\Psi \rightarrow \Gamma)), \tag{7}$$

$$((\Phi \rightarrow \Psi) \rightarrow \Gamma) \rightarrow (((\Psi \rightarrow \Phi) \rightarrow \Gamma) \rightarrow \Gamma), \tag{8}$$

$$0 \to \Psi, \tag{9}$$

$$\forall x\, \Psi(x) \to \Psi(t), \tag{10}$$

where t is substitutable for x in $\Psi(x)$.

$$\Psi(t) \to \exists x\, \Psi(x), \tag{11}$$

where t is substitutable for x in $\Psi(x)$.

$$\forall x(\Gamma \to \Psi) \to (\Gamma \to \forall x\, \Psi), \tag{12}$$

where x is not free Γ.

$$\forall x(\Psi \to \Gamma) \to (\exists x\, \Psi \to \Gamma), \tag{13}$$

where x is not free Γ.

$$\forall x(\Gamma \vee \Psi) \to (\Gamma \vee \forall x\, \Psi), \tag{14}$$

where x is not free Γ.

These axioms are said to be *horizontal*.

There is a well known theorem according to that every equivalence class a for which $|a|_p \leq 1$ (this means that a is a p-adic integer) has exactly one representative Cauchy sequence for which:

1. $0 \leq a_i < p^i$ for $i = 1, 2, 3, \ldots$;
2. $a_i \equiv a_{i+1} \mod p^i$ for $i = 1, 2, 3, \ldots$

This property can be expressed by means of the following logical axioms:

$$\begin{gathered}((\overline{p^{i+1}-1} \ominus \overline{p^i-1}) \to_L \Psi_{i+1}) \to_L \\ (\Psi_{i+1} \leftrightarrow (\underbrace{\overline{p-1} \oplus \cdots \oplus \overline{p-1}}_{p^i} \oplus \Psi_i),\end{gathered} \tag{15}$$

$$\begin{gathered}(\Psi_{i+1} \leftrightarrow (\overbrace{\overline{k} \oplus \cdots \oplus \overline{k}}^{p^i}) \oplus \Psi_i) \to_L \\ ((((\ldots(\overline{p^{i+1}-1} \ominus \underbrace{\overline{p^i-1}) \ominus \ldots) \ominus \overline{p^i-1})}_{0<p-k\leq p} \ominus \neg_L \overline{k}) \to_L \Psi_{i+1}),\end{gathered} \tag{16}$$

$$\begin{gathered}(\Psi_{i+1} \leftrightarrow (\overbrace{\overline{k} \oplus \cdots \oplus \overline{k}}^{p^i}) \oplus \Psi_i) \to_L \\ (\Psi_{i+1} \to_L (((\ldots(\overline{p^{i+1}-1} \ominus \underbrace{\overline{p^i-1}) \ominus \ldots) \ominus \overline{p^i-1})}_{0\leq(p-1)-k\leq p-1} \ominus \neg_L \overline{k})),\end{gathered} \tag{17}$$

$$(\Psi_{i+1} \to_L \overline{p^i-1}) \to_L (\Psi_{i+1} \leftrightarrow \Psi_i), \tag{18}$$

$$\begin{gathered}(\Psi_{i+1} \leftrightarrow \Psi_i) \vee (\Psi_{i+1} \leftrightarrow (\Psi_i \oplus p^i \cdot \overline{1})) \vee \ldots \\ \vee(\Psi_{i+1} \leftrightarrow (\Psi_i \oplus p^i \cdot \overline{p-1})),\end{gathered} \tag{19}$$

where $\overline{p-1}$ is a tautology at the first-order level and $\overline{p^i-1}$ (respectively $\overline{p^{i+1}-1}$) a tautology at i-th order level (respectively at $(i+1)$-th order level); $i+1$-order formula Ψ_{i+1} and i-order formula Ψ_i have the same logical matrix; $\neg_L \overline{k}$ is a first-order formula that has the truth value $((p-1)-k) \in \{0, \ldots, p-1\}$ for any its interpretations and $\overline{k}$ is a first-order formula that has the truth value $k \in \{0, \ldots, p-1\}$ for any its interpretations; $\overline{1}$ is a first-order formula that has the truth value 1 for any its interpretations, etc. The denoting $p^i \cdot \overline{k}$ means $\underbrace{\overline{k} \oplus \cdots \oplus \overline{k}}_{p^i}$.

Axioms (15) – (19) are said to be *vertical*.

Theorem 4 (Completeness) *Let Φ be an i-order (resp. ω-order) formula of $\mathcal{L}^{\infty}_{*V}$, T an i-order (resp. ω-order) $\mathcal{L}^{\infty}_{*V}$-theory. Then the following conditions are equivalent:*

- $T \vdash \Phi$;
- $\left[\Phi\right]^{i} = \langle \underbrace{\top, \ldots, \top}_{i} \rangle$ *(resp.* $\left[\Phi\right]^{\infty} = {}^{*}\top$*) for each* $\mathfrak{L}_{\mathbf{Z}_p}$*-model* $\mathcal{M}_{\infty}$ *of* T.

Proof. The first-order case is proved in [8] and the higher-order case follows from Theorem 1. □

5 Examples and Discussion

Non-Archimedean valued fuzzy logics (including the p-adic case) considered above may find many applications to rule-based systems and demonstrate their importance as a powerful design methodology. So, on the one hand, novel logics are complete (see theorems 3 and 4) and, on the other hand, they have been extended to *infinite hierarchy of fuzzy sets*, therefore they have more design degrees of freedom than do conventional fuzzy logics with values distributed in the standard unit interval $[0, 1]$.

The point is that geometrically we can imagine a system of non-Archimedean numbers in two ways:

- In case of hyperreal or hyperrational numbers they may be regarded as a homogeneous infinite tree with $[0, 1]$-branches splitting at each vertex.
- In case of p-adic integers as a homogeneous infinite tree with p-branches splitting at each vertex.

The first one is said to be *non-Archimedean tree*, the second one *p-adic tree*. For instance, a 2-adic tree is pictured in Fig. 1 and a non-Archimedean tree with branches whose number runs over the set $[0, 1]$ in Fig. 2. This geometrical presentation allows us to consider infinite hierarchies of conditional fuzzy (resp. finitely many-valued) data described in the standard form as that distributed in the set $[0, 1]$ (resp. $\{0, \ldots, p-1\}$). In order to exemplify this feature, let us take some fuzzy (resp. finitely many-valued) data and build up using them *vectors of fuzzy information* generating a hierarchical structure between digits of these vectors. If $x = \langle x_1, x_2, \ldots, x_n, \ldots \rangle$, where $x_j \in [0, 1]$ in the hypervalued case (resp. $x_j \in \{0, 1, \ldots, p-1\}$ in the p-adic case), is a fuzzy information vector, then digits x_j have different weights. The digit x_0 is the most important, x_1 dominates over x_2, x_2 over x_3 in turn and so on. The distance between uncertain states $x = \langle x_1, x_2, \ldots, x_n, \ldots \rangle$ and $y = \langle y_1, y_2, \ldots, y_n, \ldots \rangle$ is determined just by the length of their common root: close uncertain states have a long common root (i.e. then there exists a large integer j such that $x_i = y_i$ for any $i = 0, \ldots, j$).

This distance between uncertain states could be regarded within ultrametric space $\langle X, \rho \rangle$, where the distance ρ satisfies the strong triangle inequality: $\rho(x, y) \leq \max(\rho(x, z), \rho(z, y))$. For $r \in \mathbf{R}_+, a \in X$, we set

$$U_r(a) := \{x \in X \colon \rho(x, a) \leq r\}, \;\; U_r^-(a) := \{x \in X \colon \rho(x, a) < r\}.$$

Both sets are called *balls* of radius r with center a. It can be readily shown that balls in ultrametric space have the following properties [15], [16]:

- For any balls U and V in X, either they are ordered by inclusion (i.e. $U \subset V$ or $V \subset U$) or they are disjoint.
- Each point of a ball is a center.

Example 1 *Let us try to present results of a pattern recognition in the form of 2-adic valued logic. Suppose that our sensor gathers the observations of people to be classified and the first task is to detect the sex of a person. Then $x_0 = 1$ if a person who is observed is female and $x_0 = 0$ if male. Further, we have $x_1 = 1$ if a person is young and $x = 0$ otherwise. Then $x_2 = 1$ if a person is tall and $x_2 = 0$ otherwise. After that we set $x_3 = 1$ if (s)he is blond and $x_3 = 0$ otherwise, etc. As a result, we will obtain a 2-adic tree if we continue. Results of such pattern recognition can be regarded as the following balls in 2-adic ultrametric space:*

- *the set of women* $U_{1/2} = \{x = \langle x_0, x_1, \ldots, x_n, \ldots \rangle \colon x_0 = 1\}$,

- *the set of young women* $U_{1/4} = \{x = \langle x_0, x_1, \ldots, x_n, \ldots\rangle \colon x_0 = 1, x_1 = 1\}$,
- *the set of tall young women* $U_{1/8} = \{x = \langle x_0, x_1, \ldots, x_n, \ldots\rangle \colon x_0 = 1, x_1 = 1, x_2 = 1\}$,
- *the set of blond tall young women* $U_{1/16} = \{x = \langle x_0, x_1, \ldots, x_n, \ldots\rangle \colon x_0 = 1, x_1 = 1, x_2 = 1, x_3 = 1\}$,
- $\ldots$

For processing these results we can use operations of 2-adic valued logic $BL\forall_\infty$.

Thus, while the key notion of conventional fuzzy logics is that truth values are indicated by a value on the range $[0, 1]$, with 0.0 representing absolute falseness and 1.0 representing absolute truth, in non-Archimedean and p-adic fuzzy logics we come cross non-Archimedean or p-adic trees (Fig. 1 and 2) and truth values are indicated by a value on the range ${}^*[0, 1]$ of hypernumbers or on the range $\mathbf{Z}_p$ of p-adic integers. We build up infinite tuples of the form $\langle\mu_1, \mu_2, \ldots, \mu_n, \ldots\rangle$, where μ_j is a fuzzy measure such that μ_j depends on μ_i for any natural number $i < j$. For instance, we can interpret this dependence as types of higher-order logic. Let us take a fuzzy set $\mathcal{P}_i \in \mathcal{F}_i(D)$, where

$$\mathcal{F}_i(D) := \underbrace{[0, 1]^{\cdot^{\cdot^{\cdot [0,1]^{[0,1]^D}}}}}_{i+1},$$

then a membership function $\mu_{\mathcal{P}_i}(x)$ is defined as the degree of membership of x in $\mathcal{P}_i$.

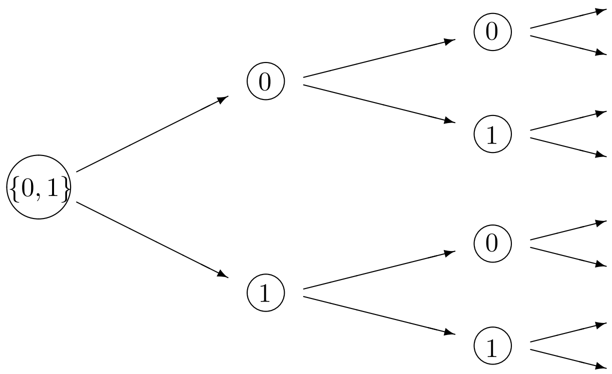

Figure 1: The 2-adic tree.

The further advantages of non-Archimedean and p-adic valued fuzzy logics consist in a possibility to be considered as behavior fuzzy logics. Let us recall that behaviors can be viewed as a labelled transition system. The set of finite sequences over a set A will be denoted by A^*, and the empty sequence by ϵ. The transition system is understood as a tuple $\Upsilon = \langle S, S_0, L, \longrightarrow\rangle$, where S is a potentially infinite collection of states, S_0 is the set of initial states, L is a potentially infinite collection of labels, $\longrightarrow \subseteq S \times L \times S$ is a transition relation that models how a state $s \in S$ can evolve into another state $s' \in S$ due to an interaction $\alpha \in L$. Usually, $\langle s, \alpha, s'\rangle \in \longrightarrow$ is denoted by $s \xrightarrow{\alpha} s'$. A state s' is reachable from a state s if $s \xrightarrow{\alpha} s'$.

The *finite trace of transition system* is a pair $\langle s_0, \sigma\rangle$, where $s_0 \in S_0$ and $\sigma = \alpha_1\alpha_2 \ldots \alpha_n$ is a finite sequence of labels such that there are states $s_0, s_1, \ldots, s_n$ satisfying $s_i \xrightarrow{\alpha_{i+1}} s_{i+1}$ for all i such that $0 \leq i < n$. The *infinite trace of transition system* is a pair $\langle s_0, \sigma\rangle$, where $s_0 \in S_0$ and $\sigma = \alpha_1\alpha_2 \ldots$ is an infinite sequence of labels such that there are states $s_0, s_1, \ldots$ satisfying $s_i \xrightarrow{\alpha_{i+1}} s_{i+1}$ for all $i \geq 0$. The set of all finite (resp. infinite) traces is denoted by $trace^*(\Upsilon)$ (resp. by $trace^\omega(\Upsilon)$). Notice that non-Archimedean and p-adic trees may be naturally interpreted as sets of traces of transition system, where the set of states runs either over $[0, 1]$ or $\{0, \ldots, p-1\}$. In other words, non-Archimedean and p-adic trees may be regarded as behavior trees.

Now let us construct a *fuzzification of transition system* Υ by defining the following functions

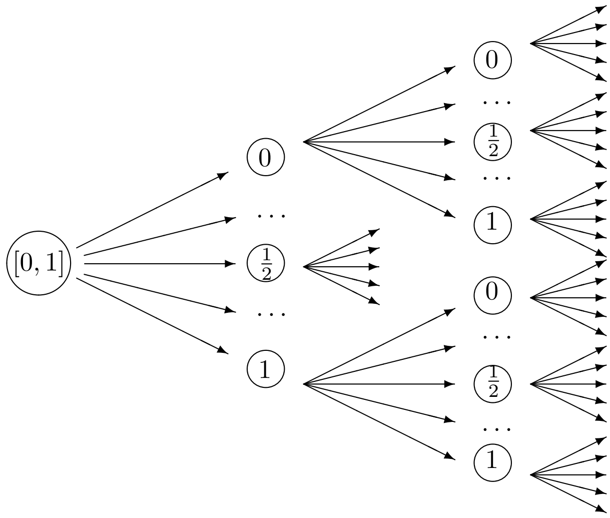

Figure 2: The non-Archimedean tree.

- in case S is infinite, $fuz^*: trace^*(\Upsilon) \mapsto [0,1]^*$, where $[0,1]^*$ denotes the set of finite tuples of members of $[0,1]$.
- in case S is infinite, $fuz^\omega: trace^\omega(\Upsilon) \mapsto {}^*[0,1]$.
- in case S is finite and $|S| = |\{0,\ldots,p-1\}|$, i.e. they are of the same cardinality, then

$$fuz^*: trace^*(\Upsilon) \mapsto \{0,\ldots,p-1\}^*,$$

where $\{0,\ldots,p-1\}^*$ denotes the set of finite tuples of members of $\{0,\ldots,p-1\}$.
- in case S is finite and $|S| = |\{0,\ldots,p-1\}|$, i.e. they are of the same cardinality, then

$$fuz^\omega: trace^\omega(\Upsilon) \mapsto \mathbf{Z}_p.$$

Suppose that all these functions are injections such that

- $fuz^*(\epsilon) = \langle 0,\ldots,0\rangle$, where ϵ is the empty sequence of $trace^*(\Upsilon)$,
- $fuz^\omega(\epsilon) = {}^*0$, where ϵ is the empty sequence of $trace^\omega(\Upsilon)$,
- an i-th projection of $fuz^*(\langle s_0, \sigma\rangle)$ (resp. $fuz^\omega(\langle s_0, \sigma\rangle)$) is a fuzzy measure of s_i for any $i = 0,1,2,\ldots$

Thus, we can analyze an evolution of transition system Υ by setting its fuzzification and further by using the non-Archimedean valued logic ŁΠ∀ and the p-adic valued logic $BL\forall$ for processing results.

Example 2 *Let us assume that the following twelve attributes should be identified for a prediction of weather: date D, hour H, cloud amount CA, cloud ceiling height CC, visibility V, wind direction WD, wind speed WS, precipitation type PT, precipitation intensity PI, dew point temperature DP, dry bulb temperature DB, and pressure trend P. All of these attributes are continuous except for precipitation PT, which is nominal (e.g. rain, snow, etc.). Each experiment consists of obtaining data for each attribute. Let us denote by S^{12} the set of all possible data (states) that could be obtained in those experiments. By S^{12}_0 we will denote results of the first experiment. Our prediction model could be presented as a transition system $\Upsilon = \langle S^{12}, S^{12}_0, L^{12}, \longrightarrow\rangle$, where L^{12} is a set of labels for each attribute. In order to set up such a model we should solve the task how it is possible to define labels on the base of experimental data. This task could be solved by a fuzzification, i.e. we could, firstly, identify the attributes to be used to indicate similarity between cases (experiments) and, secondly, describe fuzzy degrees of similarity in relations between states of an appropriate attribute and obtained in different experiments. This similarity will be defined in a forecasting scenario step by step for* *each new experiment. The sequence of forecasting scenarios will be presented as a sequence of fuzzy measures of the resultant effects on forecast accuracy in respect to fixed parameters of a previous experiment. Thus, in a prediction model Υ we could deal with labels defined as fuzzy relations between states s_i and s_{i+1} of an appropriate attribute, where s_i and s_{i+1} are states obtained in the i-th experiment and $i+1$-st experiment respectively. The sequences of such fuzzy labels could be regarded as traces of Υ within the non-Archimedean valued logic $Ł\Pi\forall_\infty$.*

6 Conclusions

So, we have constructed infinite-order logical calculi and shown that these calculi have non-Archimedean and p-adic multiple-validity. In particular, if a first-order logic L is infinite-valued, then we will obtain hyper-valued logic. If L is m-valued and m is a finite natural number, then we will obtain m-adic valued logic.

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