

# Uncertain Set Theory and Uncertain Inference Rule with Application to Uncertain Control

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## Abstract

Uncertain set theory is a generalization of uncertainty theory to the domain of uncertain sets. This paper will introduce some concepts of uncertain set, membership degree, membership function, uncertainty distribution, independence, expected value, critical values, Hausdorff distance, and conditional uncertainty. This paper will also propose an uncertain inference rule that derives consequences from uncertain knowledge or evidence via the tool of conditional uncertain set. Finally, an uncertain system will be presented and applied to uncertain inference control.

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**Keywords:** uncertainty theory, uncertain measure, uncertain set, uncertain inference, uncertain control

## 1 Introduction

In many cases, some information and knowledge are represented by human language like “about 100km”, “approximately 39 °C”, “roughly 80kg”, “low speed”, “middle age”, and “big size”. How do we understand them? Perhaps some people think that they are subjective probability or they are fuzzy concepts. However, a lot of surveys showed that those imprecise quantities behave neither like randomness nor like fuzziness. In other words, those imprecise quantities cannot be quantified by probability measure (Kolmogorov [4]), capacity (Choquet [2]), fuzzy measure (Sugeno [15]), possibility measure (Zadeh [18]), and credibility measure (Liu and Liu [6]). In order to model them, Liu [7] defined a new measure by an axiomatic method and named it *uncertain measure*. In order to develop a theory of uncertain measure, Liu [7] founded an *uncertainty theory* that is a branch of mathematics based on normality, monotonicity, self-duality, and countable subadditivity axioms.

As an application of uncertainty theory, Liu [11] proposed a spectrum of uncertain programming that is a type of mathematical programming involving uncertain variables. In addition, Liu [9] designed a canonical process that is a Lipschitz continuous uncertain process with stationary and independent increments, and developed an uncertain calculus to deal with differentiation and integration of functions of uncertain processes. Based on uncertain calculus, Liu [8] proposed a tool of uncertain differential equations. After that, an existence and uniqueness theorem of solution of uncertain differential equation was proved by Chen and Liu [1]. Uncertainty theory was also applied to uncertain logic by Li and Liu [5] in which the truth value is defined as the uncertain measure that the proposition is true. Furthermore, uncertain entailment was proposed by Liu [10] that is a methodology for calculating the truth value of an uncertain formula when the truth values of other uncertain formulas are given. In addition, Gao [3] provided some mathematical properties of uncertain measure when continuity is assumed, and You [16] proved some convergence theorems of uncertain sequences. For exploring the recent developments of uncertainty theory, the readers may consult Liu [12].

Different from random set (Robbins [14]), fuzzy set (Zadeh [17]) and rough set (Pawlak [13]), this paper will introduce a concept of uncertain set, including membership degree, membership function, uncertainty distribution, independence, expected value, critical values, Hausdorff distance, and conditional uncertain set. This paper will also introduce an uncertain inference rule via the tool of conditional uncertain set, and apply it to uncertain system and uncertain control.

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## 2 Uncertain Set

Roughly speaking, an uncertain set is a set-valued function on an uncertainty space. A formal definition is given as follows.

**Definition 1** An uncertain set is a measurable function  $\xi$  from an uncertainty space  $(\Gamma, \mathcal{L}, \mathcal{M})$  to a collection of sets of real numbers, i.e., for any Borel set  $B$  of real numbers, the set

$$\{\xi \subset B\} = \{\gamma \in \Gamma \mid \xi(\gamma) \subset B\} \quad (1)$$

is an event.

**Example 1:** Take an uncertainty space  $(\Gamma, \mathcal{L}, \mathcal{M})$  to be  $\{\gamma_1, \gamma_2, \gamma_3\}$  with power set  $\mathcal{L}$ . Then the set-valued function

$$\xi(\gamma) = \begin{cases} [1, 3], & \text{if } \gamma = \gamma_1 \\ [2, 4], & \text{if } \gamma = \gamma_2 \\ [3, 5], & \text{if } \gamma = \gamma_3 \end{cases} \quad (2)$$

is an uncertain set on  $(\Gamma, \mathcal{L}, \mathcal{M})$ .

**Example 2:** Take an uncertainty space  $(\Gamma, \mathcal{L}, \mathcal{M})$  to be  $\mathfrak{R}$  with Borel algebra  $\mathcal{L}$ . Then the set-valued function

$$\xi(\gamma) = [\gamma, \gamma + 1], \quad \forall \gamma \in \Gamma \quad (3)$$

is an uncertain set on  $(\Gamma, \mathcal{L}, \mathcal{M})$ .

**Theorem 1** Let  $\xi$  be an uncertain set and let  $B$  be a Borel set of real numbers. Then  $\{\xi \not\subset B\}$  is an event.

**Proof:** Since  $\xi$  is an uncertain set and  $B$  is a Borel set, the set  $\{\xi \subset B\}$  is an event. Thus  $\{\xi \not\subset B\}$  is an event by using the relation  $\{\xi \not\subset B\} = \{\xi \subset B\}^c$ .

**Theorem 2** Let  $\xi$  be an uncertain set and let  $B$  be a Borel set. Then  $\{\xi \cap B = \emptyset\}$  is an event.

**Proof:** Since  $\xi$  is an uncertain set and  $B$  is a Borel set, the set  $\{\xi \subset B^c\}$  is an event. Thus  $\{\xi \cap B = \emptyset\}$  is an event by using the relation  $\{\xi \cap B = \emptyset\} = \{\xi \subset B^c\}$ .

**Theorem 3** Let  $\xi$  be an uncertain set and let  $B$  be a Borel set. Then  $\{\xi \cap B \neq \emptyset\}$  is an event.

**Proof:** Since  $\xi$  is an uncertain set and  $B$  is a Borel set, the set  $\{\xi \cap B = \emptyset\}$  is an event. Thus  $\{\xi \cap B \neq \emptyset\}$  is an event by using the relation  $\{\xi \cap B \neq \emptyset\} = \{\xi \cap B = \emptyset\}^c$ .

**Theorem 4** Let  $\xi$  be an uncertain set and let  $a$  be a real number. Then  $\{a \in \xi\}$  is an event.

**Proof:** Since  $\xi$  is an uncertain set and  $a$  is a real number, the set  $\{\xi \not\subset \{a\}^c\}$  is an event. Thus  $\{a \in \xi\}$  is an event by using the relation  $\{a \in \xi\} = \{\xi \not\subset \{a\}^c\}$ .

**Theorem 5** Let  $\xi$  be an uncertain set and let  $a$  be a real number. Then  $\{a \notin \xi\}$  is an event.

**Proof:** Since  $\xi$  is an uncertain set and  $a$  is a real number, the set  $\{a \in \xi\}$  is an event. Thus  $\{a \notin \xi\}$  is an event by using the relation  $\{a \notin \xi\} = \{a \in \xi\}^c$ .

**Definition 2** Let  $\xi$  and  $\eta$  be two uncertain sets on the uncertainty space  $(\Gamma, \mathcal{L}, \mathcal{M})$ . Then the complement  $\xi^c$  of uncertain set  $\xi$  is

$$\xi^c(\gamma) = \xi(\gamma)^c, \quad \forall \gamma \in \Gamma. \quad (4)$$

The union  $\xi \cup \eta$  of uncertain sets  $\xi$  and  $\eta$  is

$$(\xi \cup \eta)(\gamma) = \xi(\gamma) \cup \eta(\gamma), \quad \forall \gamma \in \Gamma. \quad (5)$$

The intersection  $\xi \cap \eta$  of uncertain sets  $\xi$  and  $\eta$  is

$$(\xi \cap \eta)(\gamma) = \xi(\gamma) \cap \eta(\gamma), \quad \forall \gamma \in \Gamma. \quad (6)$$

**Theorem 6** (Law of Excluded Middle) Let  $\xi$  be an uncertain set. Then  $\xi \cup \xi^c \equiv \mathfrak{R}$ .

**Proof:** For each  $\gamma \in \Gamma$ , it follows from the definition of  $\xi$  and  $\xi^c$  that the union is

$$(\xi \cup \xi^c)(\gamma) = \xi(\gamma) \cup \xi^c(\gamma) = \xi(\gamma) \cup \xi(\gamma)^c = \mathfrak{R}.$$

Thus we have  $\xi \cup \xi^c \equiv \mathfrak{R}$ .

**Theorem 7** (Law of Contradiction) Let  $\xi$  be an uncertain set. Then  $\xi \cap \xi^c \equiv \emptyset$ .

**Proof:** For each  $\gamma \in \Gamma$ , it follows from the definition of  $\xi$  and  $\xi^c$  that the intersection is

$$(\xi \cap \xi^c)(\gamma) = \xi(\gamma) \cap \xi^c(\gamma) = \xi(\gamma) \cap \xi(\gamma)^c = \emptyset.$$

Thus we have  $\xi \cap \xi^c \equiv \emptyset$ .

**Theorem 8** (Double-Negation Law) Let  $\xi$  be an uncertain set. Then  $(\xi^c)^c = \xi$ .

**Proof:** For each  $\gamma \in \Gamma$ , it follows from the definition of complement that

$$(\xi^c)^c(\gamma) = (\xi^c(\gamma))^c = (\xi(\gamma)^c)^c = \xi(\gamma).$$

Thus we have  $(\xi^c)^c = \xi$ .

**Theorem 9** (De Morgan's Law) Let  $\xi$  and  $\eta$  be uncertain sets. Then  $(\xi \cup \eta)^c = \xi^c \cap \eta^c$  and  $(\xi \cap \eta)^c = \xi^c \cup \eta^c$ .

**Proof:** For each  $\gamma \in \Gamma$ , it follows from the definition of complement that

$$(\xi \cup \eta)^c(\gamma) = ((\xi(\gamma) \cup \eta(\gamma))^c) = \xi(\gamma)^c \cap \eta(\gamma)^c = (\xi^c \cap \eta^c)(\gamma).$$

Thus we have  $(\xi \cup \eta)^c = \xi^c \cap \eta^c$ . In addition, since

$$(\xi \cap \eta)^c(\gamma) = ((\xi(\gamma) \cap \eta(\gamma))^c) = \xi(\gamma)^c \cup \eta(\gamma)^c = (\xi^c \cup \eta^c)(\gamma),$$

we get  $(\xi \cap \eta)^c = \xi^c \cup \eta^c$ .

**Definition 3** Let  $\xi_1, \xi_2, \dots, \xi_n$  be uncertain sets on the uncertainty space  $(\Gamma, \mathcal{L}, \mathcal{M})$ , and  $f$  a measurable function. Then  $\xi = f(\xi_1, \xi_2, \dots, \xi_n)$  is an uncertain set defined by

$$\xi(\gamma) = f(\xi_1(\gamma), \xi_2(\gamma), \dots, \xi_n(\gamma)), \quad \forall \gamma \in \Gamma. \quad (7)$$

**Definition 4** Let  $\xi$  and  $\eta$  be two uncertain sets. We say  $\xi$  is included in  $\eta$  (i.e.,  $\xi \subset \eta$ ) if  $\xi(\gamma) \subset \eta(\gamma)$  for almost all  $\gamma \in \Gamma$  in the sense of classical set theory.

**Definition 5** Let  $\xi$  and  $\eta$  be two uncertain sets. We say  $\xi$  is equal to  $\eta$  (i.e.,  $\xi = \eta$ ) if  $\xi(\gamma) = \eta(\gamma)$  for almost all  $\gamma \in \Gamma$  in the sense of classical set theory.

**Definition 6** An uncertain set  $\xi$  is said to be nonempty if  $\xi(\gamma) \neq \emptyset$  for almost all  $\gamma \in \Gamma$  in the sense of classical set theory.

### 3 Membership Degree

Let  $\xi$  and  $\eta$  be two nonempty uncertain sets. What is the degree that  $\eta$  is included in  $\xi$ ? In other words, what is the degree that  $\eta$  is a subset of  $\xi$ ? Unfortunately, this problem is not as simple as you think. In order to discuss this issue, we introduce some symbols. At first, the set

$$\{\eta \subset \xi\} = \{\gamma \in \Gamma \mid \eta(\gamma) \subset \xi(\gamma)\} \quad (8)$$

is an event that  $\eta$  is strongly included in  $\xi$ ; and the set

$$\{\eta \not\subset \xi^c\} = \{\gamma \in \Gamma \mid \eta(\gamma) \not\subset \xi(\gamma)^c\} = \{\gamma \in \Gamma \mid \eta(\gamma) \cap \xi(\gamma) \neq \emptyset\} \quad (9)$$

is an event that  $\eta$  is weakly included in  $\xi$ . It is easy to verify that  $\{\eta \subset \xi\} \subset \{\eta \not\subset \xi^c\}$ . That is, "strong inclusion" is a subset of "weak inclusion".

**Definition 7** Let  $\xi$  and  $\eta$  be two nonempty uncertain sets. Then the strong membership degree of  $\eta$  to  $\xi$  is defined as the uncertain measure that  $\eta$  is strongly included in  $\xi$ , i.e.,  $\mathcal{M}\{\eta \subset \xi\}$ .

**Definition 8** Let  $\xi$  and  $\eta$  be two nonempty uncertain sets. Then the weak membership degree of  $\eta$  to  $\xi$  is defined as the uncertain measure that  $\eta$  is weakly included in  $\xi$ , i.e.,  $\mathcal{M}\{\eta \not\subset \xi^c\}$ .

What is the appropriate event that  $\eta$  is included in  $\xi$ ? Intuitively, it is too conservative if we take the strong inclusion  $\{\eta \subset \xi\}$ , and it is too adventurous if we take the weak inclusion  $\{\eta \not\subset \xi^c\}$ . Thus we have to introduce a new symbol  $\triangleright$  to represent this inclusion relationship called *imaginary inclusion*. That is,  $\eta \triangleright \xi$  represents the event that  $\eta$  is imaginarily included in  $\xi$ .

How do we determine  $\mathcal{M}\{\eta \triangleright \xi\}$ ? It is too conservative if we take the strong membership degree  $\mathcal{M}\{\eta \subset \xi\}$ , and it is too adventurous if we take the weak membership degree  $\mathcal{M}\{\eta \not\subset \xi^c\}$ . In fact, it is reasonable to take the middle value between  $\mathcal{M}\{\eta \subset \xi\}$  and  $\mathcal{M}\{\eta \not\subset \xi^c\}$ .

**Definition 9** Let  $\xi$  and  $\eta$  be two nonempty uncertain sets. Then the membership degree of  $\eta$  to  $\xi$  is defined as the average of strong and weak membership degrees, i.e.,

$$\mathcal{M}\{\eta \triangleright \xi\} = \frac{1}{2} (\mathcal{M}\{\eta \subset \xi\} + \mathcal{M}\{\eta \not\subset \xi^c\}). \quad (10)$$

The membership degree is understood as the uncertain measure that  $\eta$  is imaginarily included in  $\xi$ .

For any uncertain sets  $\xi$  and  $\eta$ , the membership degree  $\mathcal{M}\{\eta \triangleright \xi\}$  reflects the truth degree that  $\eta$  is a subset of  $\xi$ . If  $\mathcal{M}\{\eta \triangleright \xi\} = 1$ , then  $\eta$  is completely included in  $\xi$ . If  $\mathcal{M}\{\eta \triangleright \xi\} = 0$ , then  $\eta$  and  $\xi$  have no intersection at all. It is always true that

$$\mathcal{M}\{\eta \subset \xi\} \leq \mathcal{M}\{\eta \triangleright \xi\} \leq \mathcal{M}\{\eta \not\subset \xi^c\}. \quad (11)$$

**Theorem 10** Let  $\xi$  be a nonempty uncertain set, and let  $A$  be a Borel set of real numbers. Then

$$\mathcal{M}\{\xi \triangleright A\} + \mathcal{M}\{\xi \triangleright A^c\} = 1. \quad (12)$$

**Proof:** Since  $A$  is a special uncertain set, we have

$$\mathcal{M}\{\xi \triangleright A\} = \frac{1}{2} (\mathcal{M}\{\xi \subset A\} + \mathcal{M}\{\xi \not\subset A^c\}), \mathcal{M}\{\xi \triangleright A^c\} = \frac{1}{2} (\mathcal{M}\{\xi \subset A^c\} + \mathcal{M}\{\xi \not\subset A\}).$$

By using the self-duality of uncertain measure, we get

$$\begin{aligned} \mathcal{M}\{\xi \triangleright A\} + \mathcal{M}\{\xi \triangleright A^c\} &= \frac{1}{2} (\mathcal{M}\{\xi \subset A\} + \mathcal{M}\{\xi \not\subset A^c\}) + \frac{1}{2} (\mathcal{M}\{\xi \subset A^c\} + \mathcal{M}\{\xi \not\subset A\}) \\ &= \frac{1}{2} (\mathcal{M}\{\xi \subset A\} + \mathcal{M}\{\xi \not\subset A\}) + \frac{1}{2} (\mathcal{M}\{\xi \subset A^c\} + \mathcal{M}\{\xi \not\subset A^c\}) = \frac{1}{2} + \frac{1}{2} = 1. \end{aligned}$$

The theorem is verified.

## 4 Membership Function

This section will introduce a concept of membership function for a special type of uncertain set that takes values in a nested class of sets. Keep in mind that only some special uncertain sets have their own membership functions.

**Definition 10** A real-valued function  $\mu$  is called a membership function if  $0 \leq \mu(x) \leq 1$  for any  $x \in \mathfrak{R}$ .

**Definition 11** Let  $\mu$  be a membership function. Then for any number  $\alpha \in [0, 1]$ , the set

$$\mu_\alpha = \{x \in \mathfrak{R} \mid \mu(x) \geq \alpha\} \quad (13)$$

is called the  $\alpha$ -cut of  $\mu$  (see Figure 1).

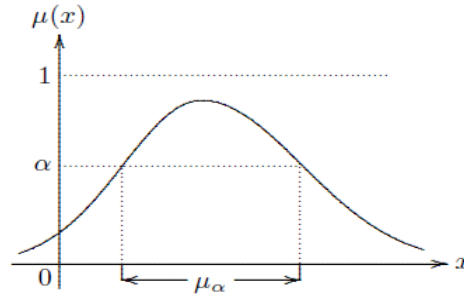


Figure 1: The  $\alpha$ -cut  $\mu_\alpha$  of membership function  $\mu(x)$

**Theorem 11** *The  $\alpha$ -cut  $\mu_\alpha$  is a monotone decreasing set with respect to  $\alpha$ . That is, for any real numbers  $\alpha$  and  $\beta$  in  $[0, 1]$  with  $\alpha > \beta$ , we have  $\mu_\alpha \subset \mu_\beta$ .*

**Proof:** For any  $x \in \mu_\alpha$ , we have  $\mu(x) \geq \alpha$ . Since  $\alpha > \beta$ , we have  $\mu(x) > \beta$  and  $x \in \mu_\beta$ . Hence  $\mu_\alpha \subset \mu_\beta$ .

**Definition 12** *Let  $\mu$  be a membership function. Then for any number  $\alpha \in [0, 1]$ , the set*

$$W_\alpha = \{ \mu_\beta \mid \beta \leq \alpha \} \tag{14}$$

*is called the  $\alpha$ -class of  $\mu$ . Especially, the 1-class is called the total class of  $\mu$ .*

Note that each element in  $W_\alpha$  is a  $\beta$ -cut of  $\mu$  where  $\beta$  is a number less than or equal to  $\alpha$ . In addition,  $\mu_{\beta_1}$  and  $\mu_{\beta_2}$  are regarded as distinct elements in  $W_\alpha$  whenever  $\beta_1 \neq \beta_2$ . Each  $\alpha$ -class (including total class) forms a family of nested sets. In the sense that the universe is assumed to be the total class, the complement  $W_\alpha^c$  is the class of  $\beta$ -cuts with  $\beta > \alpha$ , i.e.,

$$W_\alpha^c = \{ \mu_\beta \mid \beta > \alpha \} . \tag{15}$$

Thus  $W_\alpha \cup W_\alpha^c$  is just the total class.

Now it is ready to assign a membership function to an uncertain set. Roughly speaking, an uncertain set  $\xi$  is said to have a membership function  $\mu$  if  $\xi$  takes values in the total class of  $\mu$  and contains each  $\alpha$ -cut with uncertain measure  $\alpha$ . Precisely, we have the following definition.

**Definition 13** *An uncertain set  $\xi$  is said to have a membership function  $\mu$  if the range of  $\xi$  is just the total class of  $\mu$ , and*

$$\mathcal{M}\{\xi \in W_\alpha\} = \alpha, \quad \forall \alpha \in [0, 1] \tag{16}$$

*where  $W_\alpha$  is the  $\alpha$ -class of  $\mu$ .*

Since  $W_\alpha^c$  is the complement of  $W_\alpha$ , it follows from the self-duality of uncertain measure that

$$\mathcal{M}\{\xi \in W_\alpha^c\} = 1 - \alpha, \quad \forall \alpha \in [0, 1]. \tag{17}$$

In addition, it is easy to verify that  $\{\xi \notin W_\alpha\} = \{\xi \in W_\alpha^c\}$ . Hence

$$\mathcal{M}\{\xi \notin W_\alpha\} = 1 - \alpha, \quad \forall \alpha \in [0, 1]. \tag{18}$$

**Theorem 12** *(Representation Theorem) Let  $\xi$  be an uncertain set with membership function  $\mu$ . Then  $\xi$  may be represented by*

$$\xi = \bigcup_{0 \leq \alpha \leq 1} \alpha \cdot \mu_\alpha \tag{19}$$

*where  $\mu_\alpha$  is the  $\alpha$ -cut of membership function  $\mu$ .*

**Proof:** The representation theorem is essentially nothing but an alternative explanation of membership function. The equation (19) tells us that the range of  $\xi$  is just the total class of  $\mu$ , and  $\mathcal{M}\{\xi \in W_\alpha\} = \alpha$  for any  $\alpha \in [0, 1]$ .

**Remark 1:** What uncertain set does the representation theorem stand for? Take an uncertainty space  $(\Gamma, \mathcal{L}, \mathcal{M})$  to be  $[0, 1]$  with  $\mathcal{M}\{[0, \gamma]\} = \gamma$  for each  $\gamma \in [0, 1]$ . Then the set-valued function

$$\xi(\gamma) = \mu_\gamma \quad (20)$$

on the uncertainty space  $(\Gamma, \mathcal{L}, \mathcal{M})$  is just the uncertain set.

**Remark 2:** It is not true that any uncertain set has its own membership function. For example, the uncertain set

$$\xi = \begin{cases} [1, 2] & \text{with uncertain measure 0.5} \\ [2, 3] & \text{with uncertain measure 0.5} \end{cases}$$

has no membership function.

**Example 3:** The set  $\mathfrak{R}$  of real numbers is a special uncertain set  $\xi(\gamma) \equiv \mathfrak{R}$ . Such an uncertain set  $\xi$  has a membership function

$$\mu(x) \equiv 1, \quad \forall x \in \mathfrak{R}. \quad (21)$$

For this case, the membership function  $\mu$  is identical with the characteristic function of  $\mathfrak{R}$ .

**Example 4:** The empty set  $\emptyset$  is a special uncertain set  $\xi(\gamma) \equiv \emptyset$ . Such an uncertain set  $\xi$  has a membership function

$$\mu(x) \equiv 0, \quad \forall x \in \mathfrak{R}. \quad (22)$$

For this case, the membership function  $\mu$  is identical with the characteristic function of  $\emptyset$ .

**Example 5:** Let  $a$  be a number in  $\mathfrak{R}$  and let  $\alpha$  be a number in  $(0, 1)$ . Then the membership function

$$\mu(x) = \begin{cases} \alpha, & \text{if } x = a \\ 0, & \text{if } x \neq a \end{cases} \quad (23)$$

represents the uncertain set

$$\xi = \begin{cases} \{a\} & \text{with uncertain measure } \alpha \\ \emptyset & \text{with uncertain measure } 1 - \alpha \end{cases} \quad (24)$$

that takes values either the singleton  $\{a\}$  or the empty set  $\emptyset$ . This means that uncertainty exists even when there is a unique element in the universal set.

**Example 6:** By a *rectangular uncertain set* we mean the uncertain set fully determined by the pair  $(a, b)$  of crisp numbers with  $a < b$ , whose membership function is

$$\mu(x) = 1, \quad a \leq x \leq b.$$

**Example 7:** By a *triangular uncertain set* we mean the uncertain set fully determined by the triplet  $(a, b, c)$  of crisp numbers with  $a < b < c$ , whose membership function is

$$\mu(x) = \begin{cases} \frac{x-a}{b-a}, & \text{if } a \leq x \leq b \\ \frac{x-c}{b-c}, & \text{if } b \leq x \leq c. \end{cases}$$

**Example 8:** By a *trapezoidal uncertain set* we mean the uncertain set fully determined by the quadruplet  $(a, b, c, d)$  of crisp numbers with  $a < b < c < d$ , whose membership function is

$$\mu(x) = \begin{cases} \frac{x-a}{b-a}, & \text{if } a \leq x \leq b \\ 1, & \text{if } b \leq x \leq c \\ \frac{x-d}{c-d}, & \text{if } c \leq x \leq d. \end{cases}$$

**Theorem 13** Let  $\xi$  be a nonempty uncertain set with membership function  $\mu$ . Then for any number  $x \in \mathfrak{R}$ , we have

$$\mathcal{M}\{x \in \xi\} = \mu(x), \quad \mathcal{M}\{x \notin \xi\} = 1 - \mu(x), \quad \mathcal{M}\{x \notin \xi^c\} = \mu(x), \quad \mathcal{M}\{x \in \xi^c\} = 1 - \mu(x). \quad (25)$$

**Proof:** Since  $\mu$  is the membership function of  $\xi$ , we have  $\{x \in \xi\} = \{\xi \in W_\alpha\}$  where  $\alpha = \mu(x)$ . Thus

$$\mathcal{M}\{x \in \xi\} = \mathcal{M}\{\xi \in W_\alpha\} = \alpha = \mu(x).$$

In addition, it follows from the self-duality of uncertain measure that

$$\mathcal{M}\{x \notin \xi\} = 1 - \mathcal{M}\{x \in \xi\} = 1 - \mu(x).$$

Finally, it is easy to verify that  $\{x \in \xi^c\} = \{x \notin \xi\}$ . Hence  $\mathcal{M}\{x \in \xi^c\} = 1 - \mu(x)$  and  $\mathcal{M}\{x \notin \xi^c\} = \mu(x)$ .

**Theorem 14** Let  $\xi$  be a nonempty uncertain set with membership function  $\mu$ , and let  $x$  be a constant. Then

$$\mathcal{M}\{x \triangleright \xi\} = \mu(x). \quad (26)$$

**Proof:** Note that  $\mathcal{M}\{x \in \xi\} = \mu(x)$  and  $\mathcal{M}\{x \notin \xi^c\} = \mu(x)$ . It follows that

$$\mathcal{M}\{x \triangleright \xi\} = \frac{1}{2} (\mathcal{M}\{x \in \xi\} + \mathcal{M}\{x \notin \xi^c\}) = \mu(x).$$

**Theorem 15** Let  $\xi$  be a nonempty uncertain set with membership function  $\mu$ . Then for any number  $\alpha$ , we have

$$\mathcal{M}\{\mu_\alpha \subset \xi\} = \alpha, \quad \mathcal{M}\{\mu_\alpha \not\subset \xi\} = 1 - \alpha. \quad (27)$$

**Proof:** Since  $\{\mu_\alpha \subset \xi\}$  is just the  $\alpha$ -class of  $\mu$ , we immediately have  $\mathcal{M}\{\mu_\alpha \subset \xi\} = \alpha$ . In addition, by the self-duality of uncertain measure, we obtain  $\mathcal{M}\{\mu_\alpha \not\subset \xi\} = 1 - \mathcal{M}\{\mu_\alpha \subset \xi\} = 1 - \alpha$ .

**Theorem 16** Let  $\xi$  be a nonempty uncertain set with membership function  $\mu$ , and let  $A$  be a set of real numbers. Then

$$\begin{aligned} \mathcal{M}\{A \subset \xi\} &= \inf_{x \in A} \mu(x), \quad \mathcal{M}\{A \not\subset \xi\} = 1 - \inf_{x \in A} \mu(x), \\ \mathcal{M}\{A \not\subset \xi^c\} &= \sup_{x \in A} \mu(x), \quad \mathcal{M}\{A \subset \xi^c\} = 1 - \sup_{x \in A} \mu(x). \end{aligned} \quad (28)$$

**Proof:** Since  $\mu$  is the membership function of  $\xi$ , we immediately have

$$\{A \subset \xi\} = \{\xi \in W_\alpha\}, \quad \text{with } \alpha = \inf_{x \in A} \mu(x).$$

Thus we get

$$\mathcal{M}\{A \subset \xi\} = \mathcal{M}\{\xi \in W_\alpha\} = \alpha = \inf_{x \in A} \mu(x).$$

Since  $\{A \not\subset \xi\} = \{A \subset \xi\}^c$ , it follows from the self-duality of uncertain measure that

$$\mathcal{M}\{A \not\subset \xi\} = 1 - \mathcal{M}\{A \subset \xi\} = 1 - \inf_{x \in A} \mu(x).$$

In addition, we have

$$\{A \not\subset \xi^c\} = \{\xi \in W_\alpha\}, \quad \text{with } \alpha = \sup_{x \in A} \mu(x).$$

Thus

$$\mathcal{M}\{A \not\subset \xi^c\} = \mathcal{M}\{\xi \in W_\alpha\} = \alpha = \sup_{x \in A} \mu(x).$$

Since  $\{A \subset \xi^c\} = \{A \not\subset \xi^c\}^c$ , it follows from the self-duality of uncertain measure that

$$\mathcal{M}\{A \subset \xi^c\} = 1 - \mathcal{M}\{A \not\subset \xi^c\} = 1 - \sup_{x \in A} \mu(x).$$

The theorem is verified.

**Theorem 17** Let  $\xi$  be an uncertain set with membership function  $\mu$ , and let  $A$  be a set of real numbers. Then

$$\mathcal{M}\{A \triangleright \xi\} = \frac{1}{2} \left( \inf_{x \in A} \mu(x) + \sup_{x \in A} \mu(x) \right). \quad (29)$$

**Proof:** Since  $\mu$  is the membership function of  $\xi$ , we immediately have

$$\mathcal{M}\{A \subset \xi\} = \inf_{x \in A} \mu(x), \quad \mathcal{M}\{A \not\subset \xi^c\} = \sup_{x \in A} \mu(x).$$

The theorem follows from the definition of membership degree directly.

## 5 Uncertainty Distribution

This section introduces the concept of uncertainty distribution for nonempty uncertain sets, and gives a sufficient and necessary condition for uncertainty distribution.

**Definition 14** Let  $\xi$  be a nonempty uncertain set. Then the function  $\Phi(x) = \mathcal{M}\{\xi \triangleright (-\infty, x]\}$  is called the uncertainty distribution of  $\xi$ .

**Theorem 18** (Measure Inversion Theorem) Let  $\xi$  be a nonempty uncertain set with continuous uncertainty distribution  $\Phi$ . Then

$$\mathcal{M}\{\xi \triangleright (-\infty, x]\} = \Phi(x), \quad \mathcal{M}\{\xi \triangleright [x, +\infty)\} = 1 - \Phi(x) \quad (30)$$

for any  $x \in \mathfrak{R}$ .

**Proof:** The first equation follows from the definition of uncertainty distribution, and the second equation follows from the self-duality of uncertain measure.

**Theorem 19** (Sufficient and Necessary Condition for Uncertainty Distribution) A function  $\Phi : \mathfrak{R} \rightarrow [0, 1]$  is an uncertainty distribution of an uncertain set if and only if it is an increasing function except  $\Phi(x) \equiv 0$  and  $\Phi(x) \equiv 1$ .

**Proof:** Suppose  $\Phi$  is an uncertainty distribution. Since an uncertain variable is a special uncertain set, it follows that  $\Phi$  is an increasing function except  $\Phi(x) \equiv 0$  and  $\Phi(x) \equiv 1$ . Conversely, suppose  $\Phi$  is an increasing function but  $\Phi(x) \not\equiv 0$  and  $\Phi(x) \not\equiv 1$ . Then there is an uncertain variable (a degenerate uncertain set) whose uncertainty distribution is just  $\Phi$ .

**Theorem 20** Let  $\xi$  be a nonempty uncertain set with continuous membership function  $\mu$ . If  $x_0$  is a point with  $\mu(x_0) = 1$ , then the uncertainty distribution of  $\xi$  is

$$\Phi(x) = \begin{cases} \sup_{y \leq x} \mu(y)/2, & \text{if } x \leq x_0 \\ 1 - \sup_{y \geq x} \mu(y)/2, & \text{if } x \geq x_0. \end{cases} \quad (31)$$

Especially, if  $\mu$  is unimodal, then

$$\Phi(x) = \begin{cases} \mu(x)/2, & \text{if } x \leq x_0 \\ 1 - \mu(x)/2, & \text{if } x \geq x_0. \end{cases} \quad (32)$$

**Proof:** When  $x \leq x_0$ , it follows from the continuity of membership function that

$$\mathcal{M}\{\xi \subset (-\infty, x]\} = 0, \quad \mathcal{M}\{\xi \not\subset (x, +\infty)\} = \sup_{y \leq x} \mu(y).$$

Thus we have

$$\Phi(x) = \mathcal{M}\{\xi \triangleright (-\infty, x]\} = \frac{1}{2} \left( 0 + \sup_{y \leq x} \mu(y) \right) = \sup_{y \leq x} \mu(y)/2.$$

When  $x \geq x_0$ , we get

$$\mathcal{M}\{\xi \subset (-\infty, x]\} = 1 - \sup_{y \geq x} \mu(y), \quad \mathcal{M}\{\xi \not\subset (x, +\infty)\} = 1.$$

Thus we have

$$\Phi(x) = \mathcal{M}\{\xi \triangleright (-\infty, x]\} = \frac{1}{2} \left( 1 - \sup_{y \geq x} \mu(y) + 1 \right) = 1 - \sup_{y \geq x} \mu(y)/2.$$

The theorem is proved.



## 6 Independence

**Definition 15** The uncertain sets  $\xi_1, \xi_2, \dots, \xi_m$  are said to be independent if

$$\mathcal{M}\left\{\bigcap_{i=1}^m (\xi_i \subset B_i)\right\} = \min_{1 \leq i \leq m} \mathcal{M}\{\xi_i \subset B_i\}, \quad \mathcal{M}\left\{\bigcup_{i=1}^m (\xi_i \subset B_i)\right\} = \max_{1 \leq i \leq m} \mathcal{M}\{\xi_i \subset B_i\} \quad (33)$$

for any Borel sets  $B_1, B_2, \dots, B_m$  of real numbers.

It is easy to verify that the uncertain sets  $\xi_1, \xi_2, \dots, \xi_m$  are independent if and only if

$$\mathcal{M}\left\{\bigcap_{i=1}^m (\xi_i \not\subset B_i)\right\} = \min_{1 \leq i \leq m} \mathcal{M}\{\xi_i \not\subset B_i\}, \quad \mathcal{M}\left\{\bigcup_{i=1}^m (\xi_i \not\subset B_i)\right\} = \max_{1 \leq i \leq m} \mathcal{M}\{\xi_i \not\subset B_i\} \quad (34)$$

for any Borel sets  $B_1, B_2, \dots, B_m$  of real numbers.

## 7 Operational Law

This section will discuss the operational law on independent uncertain sets via uncertainty distributions.

**Theorem 21 (Operational Law)** Let  $\xi_1, \xi_2, \dots, \xi_n$  be independent uncertain sets, and  $f$  a measurable function. Then  $\xi = f(\xi_1, \xi_2, \dots, \xi_n)$  is an uncertain set such that

$$\mathcal{M}\{\xi \subset B\} = \begin{cases} \sup_{f(B_1, B_2, \dots, B_n) \subset B} \min_{1 \leq k \leq n} \mathcal{M}_k\{\xi_k \subset B_k\}, & \\ \text{if } \sup_{f(B_1, B_2, \dots, B_n) \subset B} \min_{1 \leq k \leq n} \mathcal{M}_k\{\xi_k \subset B_k\} > 0.5 & \\ 1 - \sup_{f(B_1, B_2, \dots, B_n) \subset B^c} \min_{1 \leq k \leq n} \mathcal{M}_k\{\xi_k \subset B_k\}, & \\ \text{if } \sup_{f(B_1, B_2, \dots, B_n) \subset B^c} \min_{1 \leq k \leq n} \mathcal{M}_k\{\xi_k \subset B_k\} > 0.5 & \\ 0.5, & \text{otherwise} \end{cases}$$

for Borel sets  $B, B_1, B_2, \dots, B_n$  of real numbers.

**Proof:** Write  $\Lambda = \{\xi \subset B\}$  and  $\Lambda_k = \{\xi_k \subset B_k\}$  for  $k = 1, 2, \dots, n$ . It is easy to verify that

$$\Lambda_1 \times \Lambda_2 \times \dots \times \Lambda_n \subset \Lambda \text{ if and only if } f(B_1, B_2, \dots, B_n) \subset B,$$

$$\Lambda_1 \times \Lambda_2 \times \dots \times \Lambda_n \subset \Lambda^c \text{ if and only if } f(B_1, B_2, \dots, B_n) \subset B^c.$$

Thus the operational law follows from the product measure axiom immediately.

### Increasing Function of Uncertain Sets

**Theorem 22** Let  $\xi_1, \xi_2, \dots, \xi_n$  be independent uncertain sets with uncertainty distributions  $\Phi_1, \Phi_2, \dots, \Phi_n$ , respectively. If  $f$  is a strictly increasing function, then  $\xi = f(\xi_1, \xi_2, \dots, \xi_n)$  is an uncertain set whose inverse uncertainty distribution is

$$\Phi^{-1}(\alpha) = f(\Phi_1^{-1}(\alpha), \Phi_2^{-1}(\alpha), \dots, \Phi_n^{-1}(\alpha)) \quad (35)$$

for any  $\alpha$  with  $0 < \alpha < 1$ .

**Proof:** For simplicity, we only prove the case  $n = 2$ . Since  $\xi_1$  and  $\xi_2$  are independent uncertain sets and  $f$  is a strictly increasing function, we have

$$\begin{aligned} \mathcal{M}\{\xi \triangleright (-\infty, \Phi^{-1}(\alpha))\} &= \mathcal{M}\{f(\xi_1, \xi_2) \triangleright (-\infty, f(\Phi_1^{-1}(\alpha), \Phi_2^{-1}(\alpha)))\} \\ &\geq \mathcal{M}\{(\xi_1 \triangleright (-\infty, \Phi_1^{-1}(\alpha))) \cap (\xi_2 \triangleright (-\infty, \Phi_2^{-1}(\alpha)))\} \\ &= \mathcal{M}\{\xi_1 \triangleright (-\infty, \Phi_1^{-1}(\alpha))\} \wedge \mathcal{M}\{\xi_2 \triangleright (-\infty, \Phi_2^{-1}(\alpha))\} \\ &= \alpha \wedge \alpha = \alpha. \end{aligned}$$

On the other hand, there exists some index  $i$  such that

$$\{f(\xi_1, \xi_2) \triangleright (-\infty, f(\Phi_1^{-1}(\alpha), \Phi_2^{-1}(\alpha)))\} \subset \{\xi_i \triangleright (-\infty, \Phi_i^{-1}(\alpha))\}.$$

Thus

$$\mathcal{M}\{\xi \triangleright (-\infty, \Phi^{-1}(\alpha))\} \leq \mathcal{M}\{\xi_i \triangleright (-\infty, \Phi_i^{-1}(\alpha))\} = \alpha.$$

It follows that  $\mathcal{M}\{\xi \triangleright (-\infty, \Phi^{-1}(\alpha))\} = \alpha$ . In other words,  $\Phi$  is just the uncertainty distribution of  $\xi$ . The theorem is proved.

**Example 9:** Let  $\xi_1$  and  $\xi_2$  be independent uncertain sets with uncertainty distributions  $\Phi_1(x)$  and  $\Phi_2(x)$ , respectively, and let  $a_1$  and  $a_2$  be nonnegative numbers. Then the inverse uncertainty distribution of the weighted sum  $a_1\xi_1 + a_2\xi_2$  is

$$\Phi^{-1}(\alpha) = a_1\Phi_1^{-1}(\alpha) + a_2\Phi_2^{-1}(\alpha). \quad (36)$$

**Example 10:** Let  $\xi_1$  and  $\xi_2$  be independent and nonnegative uncertain sets with uncertainty distributions  $\Phi_1$  and  $\Phi_2$ , respectively. Then the inverse uncertainty distribution of the product  $\xi_1 \times \xi_2$  is

$$\Phi^{-1}(\alpha) = \Phi_1^{-1}(\alpha) \times \Phi_2^{-1}(\alpha). \quad (37)$$

## Decreasing Function of Uncertain Sets

**Theorem 23** *Let  $\xi_1, \xi_2, \dots, \xi_n$  be independent uncertain sets with uncertainty distributions  $\Phi_1, \Phi_2, \dots, \Phi_n$ , respectively. If  $f$  is a strictly decreasing function, then  $\xi = f(\xi_1, \xi_2, \dots, \xi_n)$  is an uncertain set whose inverse uncertainty distribution is*

$$\Phi^{-1}(\alpha) = f(\Phi_1^{-1}(1-\alpha), \Phi_2^{-1}(1-\alpha), \dots, \Phi_n^{-1}(1-\alpha)) \quad (38)$$

for any  $\alpha$  with  $0 < \alpha < 1$ .

**Proof:** For simplicity, we only prove the case  $n = 2$ . Since  $\xi_1$  and  $\xi_2$  are independent uncertain sets and  $f$  is a strictly decreasing function, we have

$$\begin{aligned} \mathcal{M}\{\xi \triangleright (-\infty, \Phi^{-1}(\alpha))\} &= \mathcal{M}\{f(\xi_1, \xi_2) \triangleright (-\infty, f(\Phi_1^{-1}(1-\alpha), \Phi_2^{-1}(1-\alpha)))\} \\ &\geq \mathcal{M}\{(\xi_1 \triangleright [\Phi_1^{-1}(1-\alpha), +\infty)) \cap (\xi_2 \triangleright [\Phi_2^{-1}(1-\alpha), +\infty))\} \\ &= \mathcal{M}\{\xi_1 \triangleright [\Phi_1^{-1}(1-\alpha), +\infty)\} \wedge \mathcal{M}\{\xi_2 \triangleright [\Phi_2^{-1}(1-\alpha), +\infty)\} \\ &= \alpha \wedge \alpha = \alpha. \end{aligned}$$

On the other hand, there exists some index  $i$  such that

$$\{f(\xi_1, \xi_2) \triangleright (-\infty, f(\Phi_1^{-1}(1-\alpha), \Phi_2^{-1}(1-\alpha)))\} \subset \{\xi_i \triangleright [\Phi_i^{-1}(1-\alpha), +\infty)\}.$$

Thus

$$\mathcal{M}\{\xi \triangleright (-\infty, \Phi^{-1}(\alpha))\} \leq \mathcal{M}\{\xi_i \triangleright [\Phi_i^{-1}(1-\alpha), +\infty)\} = \alpha.$$

It follows that  $\mathcal{M}\{\xi \triangleright (-\infty, \Phi^{-1}(\alpha))\} = \alpha$ . In other words,  $\Phi$  is just the uncertainty distribution of  $\xi$ . The theorem is proved.

## Alternating Monotone Function of Uncertain Sets

**Theorem 24** *Let  $\xi_1, \xi_2, \dots, \xi_n$  be independent uncertain sets with uncertainty distributions  $\Phi_1, \Phi_2, \dots, \Phi_n$ , respectively. If  $f(x_1, x_2, \dots, x_n)$  is strictly increasing with respect to  $x_1, x_2, \dots, x_m$  and strictly decreasing with respect to  $x_{m+1}, x_{m+2}, \dots, x_n$ , then  $\xi = f(\xi_1, \xi_2, \dots, \xi_n)$  is an uncertain set whose inverse uncertainty distribution is*

$$\Phi^{-1}(\alpha) = f(\Phi_1^{-1}(\alpha), \dots, \Phi_m^{-1}(\alpha), \Phi_{m+1}^{-1}(1-\alpha), \dots, \Phi_n^{-1}(1-\alpha)) \quad (39)$$

for any  $\alpha$  with  $0 < \alpha < 1$ .

**Proof:** For simplicity, we only prove the case  $n = 2$ . Since  $\xi_1$  and  $\xi_2$  are independent uncertain sets and the function  $f(x_1, x_2)$  is strictly increasing with respect to  $x_1$  and strictly decreasing with respect to  $x_2$ , we have

$$\begin{aligned} \mathcal{M}\{\xi \triangleright (-\infty, \Phi^{-1}(\alpha))\} &= \mathcal{M}\{f(\xi_1, \xi_2) \triangleright (-\infty, f(\Phi_1^{-1}(\alpha), \Phi_2^{-1}(1 - \alpha)))\} \\ &\geq \mathcal{M}\{(\xi_1 \triangleright (-\infty, \Phi_1^{-1}(\alpha))) \cap (\xi_2 \triangleright [\Phi_2^{-1}(1 - \alpha), +\infty))\} \\ &= \mathcal{M}\{\xi_1 \triangleright (-\infty, \Phi_1^{-1}(\alpha))\} \wedge \mathcal{M}\{\xi_2 \triangleright [\Phi_2^{-1}(1 - \alpha), +\infty)\} \\ &= \alpha \wedge \alpha = \alpha. \end{aligned}$$

On the other hand, the event  $\{\xi \triangleright (-\infty, \Phi^{-1}(\alpha))\}$  is a subset of either  $\{\xi_1 \triangleright (-\infty, \Phi_1^{-1}(\alpha))\}$  or  $\{\xi_2 \triangleright [\Phi_2^{-1}(1 - \alpha), +\infty)\}$ . Thus

$$\mathcal{M}\{\xi \triangleright (-\infty, \Phi^{-1}(\alpha))\} \leq \alpha.$$

It follows that  $\mathcal{M}\{\xi \triangleright (-\infty, \Phi^{-1}(\alpha))\} = \alpha$ . In other words,  $\Phi$  is just the uncertainty distribution of  $\xi$ . The theorem is proved.

**Example 11:** Let  $\xi_1$  and  $\xi_2$  be independent uncertain sets with uncertainty distributions  $\Phi_1$  and  $\Phi_2$ , respectively. Then the inverse uncertainty distribution of the difference  $\xi_1 - \xi_2$  is

$$\Phi^{-1}(\alpha) = \Phi_1^{-1}(\alpha) - \Phi_2^{-1}(1 - \alpha), \quad 0 < \alpha < 1. \tag{40}$$

## 8 Expected Value

**Definition 16** Let  $\xi$  be a nonempty uncertain set. Then the expected value of  $\xi$  is defined by

$$E[\xi] = \int_0^{+\infty} \mathcal{M}\{\xi \triangleright [r, +\infty)\} dr - \int_{-\infty}^0 \mathcal{M}\{\xi \triangleright (-\infty, r]\} dr \tag{41}$$

provided that at least one of the two integrals is finite.

**Example 12:** Consider an uncertain set  $\xi$  that has no membership function but may be represented by

$$\xi = \begin{cases} [1, 2] & \text{with uncertain measure 0.5} \\ [2, 3] & \text{with uncertain measure 0.5.} \end{cases}$$

Intuitively, the expected value of  $\xi$  should be 2. At first, we have

$$\mathcal{M}\{\xi \triangleright [r, +\infty)\} = \begin{cases} 1, & \text{if } 0 \leq r \leq 1 \\ 0.75, & \text{if } 1 < r \leq 2 \\ 0.25, & \text{if } 2 < r \leq 3 \\ 0, & \text{if } r > 3, \end{cases}$$

$$\mathcal{M}\{\xi \triangleright (-\infty, r]\} \equiv 0, \quad \forall r \leq 0.$$

Thus

$$E[\xi] = \int_0^1 1 dr + \int_1^2 0.75 dr + \int_2^3 0.25 dr = 2.$$

**Theorem 25** Let  $\xi$  be a nonempty uncertain set with uncertainty distribution  $\Phi$ . If  $\xi$  has a finite expected value, then

$$E[\xi] = \int_0^{+\infty} (1 - \Phi(x)) dx - \int_{-\infty}^0 \Phi(x) dx. \tag{42}$$

**Proof:** The theorem follows immediately from  $\Phi(x) = \mathcal{M}\{\xi \triangleright (-\infty, x]\}$  and  $1 - \Phi(x) = \mathcal{M}\{\xi \triangleright (x, +\infty)\}$  for any  $x \in \mathfrak{R}$ .

**Theorem 26** Let  $\xi$  be a nonempty uncertain set with uncertainty distribution  $\Phi$ . If  $\xi$  has a finite expected value, then

$$E[\xi] = \int_0^1 \Phi^{-1}(\alpha) d\alpha. \quad (43)$$

**Proof:** It follows from the definitions of expected value operator and uncertainty distribution that

$$\begin{aligned} E[\xi] &= \int_0^{+\infty} \mathcal{M}\{\xi \triangleright [r, +\infty)\} dr - \int_{-\infty}^0 \mathcal{M}\{\xi \triangleright (-\infty, r]\} dr \\ &= \int_{\Phi(0)}^1 \Phi^{-1}(\alpha) d\alpha + \int_0^{\Phi(0)} \Phi^{-1}(\alpha) d\alpha = \int_0^1 \Phi^{-1}(\alpha) d\alpha. \end{aligned}$$

The theorem is proved.

**Theorem 27** Let  $\xi$  be a nonempty uncertain set with membership function  $\mu$ . If  $\xi$  has a finite expected value and  $\mu$  is a unimodal function about  $x_0$  (i.e., increasing on  $(-\infty, x_0)$  and decreasing on  $(x_0, +\infty)$ ), then the expected value of  $\xi$  is

$$E[\xi] = x_0 + \frac{1}{2} \int_{x_0}^{+\infty} \mu(x) dx - \frac{1}{2} \int_{-\infty}^{x_0} \mu(x) dx. \quad (44)$$

**Proof:** Since  $\mu$  is increasing on  $(-\infty, x_0)$  and decreasing on  $(x_0, +\infty)$ , it follows from the definition of membership degree that

$$\mathcal{M}\{\xi \triangleright (-\infty, x]\} = \begin{cases} \mu(x)/2, & \text{if } x \leq x_0 \\ 1 - \mu(x)/2, & \text{if } x \geq x_0 \end{cases} \quad (45)$$

and

$$\mathcal{M}\{\xi \triangleright [x, +\infty)\} = \begin{cases} 1 - \mu(x)/2, & \text{if } x \leq x_0 \\ \mu(x)/2, & \text{if } x \geq x_0 \end{cases} \quad (46)$$

for almost all  $x \in \mathfrak{R}$ . It follows from the definition of expected value operator that the theorem holds.

The rectangular uncertain set  $\xi = (a, b)$  has an expected value  $E[\xi] = (a + b)/2$ . The triangular uncertain set  $\xi = (a, b, c)$  has an expected value  $E[\xi] = (a + 2b + c)/4$ . The trapezoidal uncertain set  $\xi = (a, b, c, d)$  has an expected value  $E[\xi] = (a + b + c + d)/4$ .

**Theorem 28** Let  $\xi_1$  and  $\xi_2$  be independent nonempty uncertain sets with finite expected values. Then for any real numbers  $a_1$  and  $a_2$ , we have

$$E[a_1\xi_1 + a_2\xi_2] = a_1E[\xi_1] + a_2E[\xi_2]. \quad (47)$$

**Proof:** Suppose that  $\xi_1$  and  $\xi_2$  have uncertainty distributions  $\Phi_1$  and  $\Phi_2$ , respectively. It follows that  $a_1\xi_1 + a_2\xi_2$  has an inverse uncertainty distribution,

$$\Phi^{-1}(\alpha) = a_1\Phi_1^{-1}(\alpha) + a_2\Phi_2^{-1}(\alpha)$$

and

$$\int_0^1 \Phi^{-1}(\alpha) d\alpha = a_1 \int_0^1 \Phi_1^{-1}(\alpha) d\alpha + a_2 \int_0^1 \Phi_2^{-1}(\alpha) d\alpha.$$

Then Theorem 26 tells us that

$$E[a_1\xi_1 + a_2\xi_2] = \int_0^1 \Phi^{-1}(\alpha) d\alpha = a_1E[\xi_1] + a_2E[\xi_2].$$

The theorem is proved.

**Theorem 29** Assume  $\xi_1, \xi_2, \dots, \xi_n$  are independent uncertain sets with uncertainty distributions  $\Phi_1, \Phi_2, \dots, \Phi_n$ , respectively. If  $f$  is a strictly monotone function, then the uncertain set  $\xi = f(\xi_1, \xi_2, \dots, \xi_n)$  has an expected value

$$E[\xi] = \int_0^1 f(\Phi_1^{-1}(\alpha), \Phi_2^{-1}(\alpha), \dots, \Phi_n^{-1}(\alpha)) d\alpha \quad (48)$$

provided that the expected value  $E[\xi]$  exists.

**Proof:** Suppose that  $f$  is a strictly increasing function. It follows that the inverse uncertainty distribution of  $\xi$  is

$$\Phi^{-1}(\alpha) = f(\Phi_1^{-1}(\alpha), \Phi_2^{-1}(\alpha), \dots, \Phi_n^{-1}(\alpha)).$$

Thus we obtain (48). When  $f$  is a strictly decreasing function, it follows that the inverse uncertainty distribution of  $\xi$  is

$$\Phi^{-1}(\alpha) = f(\Phi_1^{-1}(1 - \alpha), \Phi_2^{-1}(1 - \alpha), \dots, \Phi_n^{-1}(1 - \alpha)).$$

By using the change of variable of integral, we obtain (48). The theorem is proved.

**Example 13:** Let  $\xi$  and  $\eta$  be independent and nonnegative uncertain sets with uncertainty distributions  $\Phi$  and  $\Psi$ , respectively. Then

$$E[\xi\eta] = \int_0^1 \Phi^{-1}(\alpha)\Psi^{-1}(\alpha)d\alpha. \tag{49}$$

## 9 Critical Values

In order to rank uncertain sets, we may use two critical values: optimistic value and pessimistic value.

**Definition 17** Let  $\xi$  be an uncertain set, and  $\alpha \in (0, 1]$ . Then

$$\xi_{\text{sup}}(\alpha) = \sup \{r \mid \mathcal{M} \{\xi \triangleright [r, +\infty)\} \geq \alpha\} \tag{50}$$

is called the  $\alpha$ -optimistic value to  $\xi$ , and

$$\xi_{\text{inf}}(\alpha) = \inf \{r \mid \mathcal{M} \{\xi \triangleright (-\infty, r]\} \geq \alpha\} \tag{51}$$

is called the  $\alpha$ -pessimistic value to  $\xi$ .

**Theorem 30** Let  $\xi$  be an uncertain set with uncertainty distribution  $\Phi$ . Then its  $\alpha$ -optimistic value and  $\alpha$ -pessimistic value are

$$\xi_{\text{sup}}(\alpha) = \Phi^{-1}(1 - \alpha), \quad \xi_{\text{inf}}(\alpha) = \Phi^{-1}(\alpha) \tag{52}$$

for any  $\alpha$  with  $0 < \alpha < 1$ .

**Proof:** Since  $\Phi$  is a strictly monotone function when  $0 < \Phi(x) < 1$ , we have

$$\xi_{\text{sup}}(\alpha) = \sup \{r \mid \mathcal{M} \{\xi \triangleright [r, +\infty)\} \geq \alpha\} = \sup \{r \mid 1 - \Phi(r) \geq \alpha\} = \Phi^{-1}(1 - \alpha),$$

$$\xi_{\text{inf}}(\alpha) = \inf \{r \mid \mathcal{M} \{\xi \triangleright (-\infty, r]\} \geq \alpha\} = \inf \{r \mid \Phi(r) \geq \alpha\} = \Phi^{-1}(\alpha).$$

The theorem is proved.

**Theorem 31** Let  $\xi_1, \xi_2, \dots, \xi_n$  be independent uncertain sets with uncertainty distributions. If  $f$  is a continuous and strictly increasing function, then  $\xi = f(\xi_1, \xi_2, \dots, \xi_n)$  is an uncertain set, and

$$\xi_{\text{sup}}(\alpha) = f(\xi_{1 \text{ sup}}(\alpha), \xi_{2 \text{ sup}}(\alpha), \dots, \xi_{n \text{ sup}}(\alpha)), \tag{53}$$

$$\xi_{\text{inf}}(\alpha) = f(\xi_{1 \text{ inf}}(\alpha), \xi_{2 \text{ inf}}(\alpha), \dots, \xi_{n \text{ inf}}(\alpha)). \tag{54}$$

**Proof:** Since  $f$  is a strictly increasing function, it follows that the inverse uncertainty distribution of  $\xi$  is

$$\Phi^{-1}(\alpha) = f(\Phi_1^{-1}(\alpha), \Phi_2^{-1}(\alpha), \dots, \Phi_n^{-1}(\alpha))$$

where  $\Phi_1, \Phi_2, \dots, \Phi_n$  are uncertainty distributions of  $\xi_1, \xi_2, \dots, \xi_n$ , respectively. Thus we get (53) and (54). The theorem is proved.

**Example 14:** Let  $\xi$  and  $\eta$  be independent uncertain sets with uncertainty distributions. Then

$$(\xi + \eta)_{\text{sup}}(\alpha) = \xi_{\text{sup}}(\alpha) + \eta_{\text{sup}}(\alpha), \quad (\xi + \eta)_{\text{inf}}(\alpha) = \xi_{\text{inf}}(\alpha) + \eta_{\text{inf}}(\alpha). \tag{55}$$

**Example 15:** Let  $\xi$  and  $\eta$  be independent and positive uncertain sets with uncertainty distributions. Then

$$(\xi\eta)_{\text{sup}}(\alpha) = \xi_{\text{sup}}(\alpha)\eta_{\text{sup}}(\alpha), \quad (\xi\eta)_{\text{inf}}(\alpha) = \xi_{\text{inf}}(\alpha)\eta_{\text{inf}}(\alpha). \tag{56}$$

**Theorem 32** Let  $\xi_1, \xi_2, \dots, \xi_n$  be independent uncertain sets with uncertainty distributions. If  $f$  is a continuous and strictly decreasing function, then

$$\xi_{\sup}(\alpha) = f(\xi_{1 \inf}(\alpha), \xi_{2 \inf}(\alpha), \dots, \xi_{n \inf}(\alpha)), \quad (57)$$

$$\xi_{\inf}(\alpha) = f(\xi_{1 \sup}(\alpha), \xi_{2 \sup}(\alpha), \dots, \xi_{n \sup}(\alpha)). \quad (58)$$

**Proof:** Since  $f$  is a strictly decreasing function, it follows that the inverse uncertainty distribution of  $\xi$  is

$$\Phi^{-1}(\alpha) = f(\Phi_1^{-1}(1 - \alpha), \Phi_2^{-1}(1 - \alpha), \dots, \Phi_n^{-1}(1 - \alpha)).$$

Thus we get (57) and (58). The theorem is proved.

## 10 Hausdorff Distance

Let  $\xi$  and  $\eta$  be two uncertain sets on the uncertainty space  $(\Gamma, \mathcal{L}, \mathcal{M})$ . For each  $\gamma \in \Gamma$ , it is clear that  $\xi(\gamma)$  and  $\eta(\gamma)$  are two sets of real numbers. Thus the Hausdorff distance between them is

$$\rho(\gamma) = \left( \sup_{a \in \xi(\gamma)} \inf_{b \in \eta(\gamma)} |a - b| \right) \vee \left( \sup_{b \in \eta(\gamma)} \inf_{a \in \xi(\gamma)} |a - b| \right). \quad (59)$$

Note that  $\rho$  is a function from  $(\Gamma, \mathcal{L}, \mathcal{M})$  to the set of nonnegative numbers, and is just a nonnegative uncertain variable.

**Definition 18** Let  $\xi$  and  $\eta$  be two uncertain sets. Then the Hausdorff distance between  $\xi$  and  $\eta$  is

$$d(\xi, \eta) = \int_0^{+\infty} \mathcal{M}\{\rho \geq r\} dr \quad (60)$$

where  $\rho$  is a nonnegative uncertain variable determined by (59).

If the uncertain sets degenerate to uncertain variables, then the Hausdorff distance between uncertain sets degenerates to the distance between uncertain variables. Let  $\xi, \eta, \tau$  be uncertain sets, and let  $d(\cdot, \cdot)$  be the Hausdorff distance. Then we have (a) (Nonnegativity)  $d(\xi, \eta) \geq 0$ ; (b) (Identification)  $d(\xi, \eta) = 0$  if and only if  $\xi = \eta$ ; (c) (Symmetry)  $d(\xi, \eta) = d(\eta, \xi)$ .

## 11 Conditional Uncertainty

Let  $\xi$  be an uncertain set on  $(\Gamma, \mathcal{L}, \mathcal{M})$ . What is the conditional uncertain set of  $\xi$  after it has been learned that some event  $B$  has occurred? This section will answer this question.

**Definition 19** Let  $\xi$  be an uncertain set with membership function  $\mu$ , and let  $B$  be an event with  $\mathcal{M}\{B\} > 0$ . Then the conditional membership function of  $\xi$  given  $B$  is defined by  $\mu(x|B) = \mathcal{M}\{\xi \in W_{\mu(x)}|B\}$  where  $W_{\mu(x)}$  is the  $\mu(x)$ -class of  $\mu$ .

**Definition 20** Let  $\xi$  be an uncertain set and let  $B$  be an event with  $\mathcal{M}\{B\} > 0$ . Then the conditional uncertainty distribution  $\Phi: \mathbb{R} \rightarrow [0, 1]$  of  $\xi$  given  $B$  is defined by  $\Phi(x|B) = \mathcal{M}\{\xi \triangleright (-\infty, x]|B\}$ .

**Definition 21** Let  $\xi$  be an uncertain set and let  $B$  be an event with  $\mathcal{M}\{B\} > 0$ . Then the conditional expected value of  $\xi$  given  $B$  is defined by

$$E[\xi|B] = \int_0^{+\infty} \mathcal{M}\{\xi \triangleright [r, +\infty) | B\} dr - \int_{-\infty}^0 \mathcal{M}\{\xi \triangleright (-\infty, r] | B\} dr$$

provided that at least one of the two integrals is finite.

## 12 Inference Rule

Uncertain inference is a process of deriving consequences from uncertain knowledge or evidence via the tool of conditional uncertain set. Let  $\mathbb{X}$  and  $\mathbb{Y}$  be two concepts. It is assumed that we only have a rule “if  $\mathbb{X}$  is  $\xi$  then  $\mathbb{Y}$  is  $\eta$ ” where  $\xi$  and  $\eta$  are two uncertain sets. We first have the following inference rule.

**Inference Rule** Let  $\mathbb{X}$  and  $\mathbb{Y}$  be two concepts. Assume a rule “if  $\mathbb{X}$  is an uncertain set  $\xi$  then  $\mathbb{Y}$  is an uncertain set  $\eta$ ”. From  $\mathbb{X}$  is an uncertain set  $\xi^*$  we infer that  $\mathbb{Y}$  is an uncertain set

$$\eta^* = \eta|_{\xi^* \triangleright \xi} \quad (61)$$

which is the conditional uncertain set of  $\eta$  given  $\xi^* \triangleright \xi$ . The inference rule is represented by

$$\begin{array}{l} \text{Rule: If } \mathbb{X} \text{ is } \xi \text{ then } \mathbb{Y} \text{ is } \eta \\ \text{From: } \mathbb{X} \text{ is } \xi^* \\ \hline \text{Infer: } \mathbb{Y} \text{ is } \eta^* = \eta|_{\xi^* \triangleright \xi} \end{array} \quad (62)$$

**Theorem 33** Let  $\xi$  and  $\eta$  be independent uncertain sets with membership functions  $\mu$  and  $\nu$ , respectively. If  $\xi^*$  is a constant  $a$ , then the inference rule yields that  $\eta^*$  has a membership function

$$\nu^*(y) = \begin{cases} \frac{\nu(y)}{\mu(a)}, & \text{if } \nu(y) < \mu(a)/2 \\ \frac{\nu(y) + \mu(a) - 1}{\mu(a)}, & \text{if } \nu(y) > 1 - \mu(a)/2 \\ 0.5, & \text{otherwise.} \end{cases} \quad (63)$$

**Proof:** It follows from the inference rule that  $\eta^*$  has a membership function

$$\nu^*(y) = \mathcal{M}\{y \in \eta | a \triangleright \xi\}.$$

By using the definition of conditional uncertainty, we have

$$\mathcal{M}\{y \in \eta | a \triangleright \xi\} = \begin{cases} \frac{\mathcal{M}\{y \in \eta\}}{\mathcal{M}\{a \triangleright \xi\}}, & \text{if } \frac{\mathcal{M}\{y \in \eta\}}{\mathcal{M}\{a \triangleright \xi\}} < 0.5 \\ 1 - \frac{\mathcal{M}\{y \notin \eta\}}{\mathcal{M}\{a \triangleright \xi\}}, & \text{if } \frac{\mathcal{M}\{y \notin \eta\}}{\mathcal{M}\{a \triangleright \xi\}} < 0.5 \\ 0.5, & \text{otherwise.} \end{cases}$$

The equation (63) follows from  $\mathcal{M}\{y \in \eta\} = \nu(y)$ ,  $\mathcal{M}\{y \notin \eta\} = 1 - \nu(y)$  and  $\mathcal{M}\{a \triangleright \xi\} = \mu(a)$  immediately. The theorem is proved.

## 13 Uncertain System and Inference Control

An uncertain system is a function from its inputs to outputs based on the inference rule. As an application of uncertain system, an inference controller is a controller based on the inference rule. An inference control system (see Figure 2) consists of an inference controller and a process. Note that  $t$  represents time,  $\alpha_1(t), \alpha_2(t), \dots, \alpha_m(t)$  are not only the inputs of inference controller but also the outputs of process, and  $\beta_1(t), \beta_2(t), \dots, \beta_n(t)$  are not only the outputs of inference controller but also the inputs of process.

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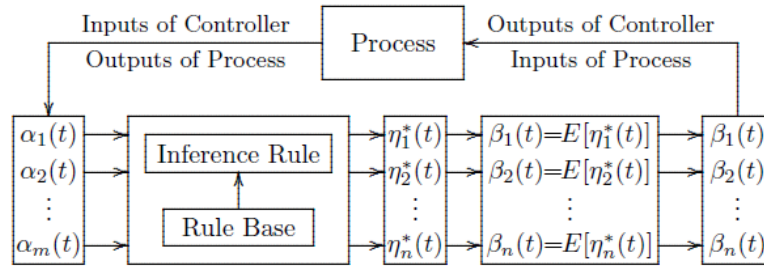


Figure 2: An inference control system

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