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Different Types of Convergence for Random Variables with Respect to Separately Coherent Upper Conditional Probabilities Defined by Hausdorff Outer Measures

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Abstract

Let (Ω, d) be a metric space and let **B** be a partition of Ω. For every B in **B** let $\boldsymbol{L}(B)$ be the class of all bounded random variables on B. Upper conditional previsions $\overline{P}(X|B)$ are defined on $\boldsymbol{L}(B) \times \boldsymbol{B}$ with respect to a class of Hausdorff outer measures when the conditioning event B has positive and finite Hausdorff outer measure in its dimension; otherwise they are defined by a 0-1 valued finitely additive (but not countably additive) probability. For every conditioning event B these upper conditional previsions $\overline{P}(X|B)$ are proven to be the upper envelopes of all linear previsions, defined on the class of all bounded random variables on B and dominated by $\overline{P}(X|B)$. Upper conditional probabilities are obtained as a particular case when $\boldsymbol{L}(B)$ is the class of all 0-1 valued random variables on B. The unconditional upper probability is defined when the conditioning event is Ω. Relations among different types of convergence of sequences of random variables are investigated with respect to these upper conditional probabilities. If B has finite and positive Hausdorff outer measure in its dimension the given upper conditional probabilities are continuous from above on the Borel σ-field. In this case we obtain that the pointwise convergence implies the μ-stochastic convergence. Moreover, since the outer measure is subadditive then stochastic convergence with respect to the given upper conditional probabilities implies convergence in μ-distribution. It is proven that the given upper conditional previsions satisfy the Monotone Convergence Theorem and on the class of all Borel-measurable random variables convergence in distribution is equivalent to the pointwise convergence of the expectation functionals on all bounded continuous functions.

Keywords: upper conditional previsions, upper envelope, Hausdorff outer measures, convergences for sequences of random variables, monotone convergence theorem, convergence in distribution

1 Introduction

One of the main topics in probability theory and stochastic processes theory is the convergence of sequences of random variables, which plays a key role in asymptotic inference. Different kinds of convergence and their relations are considered in literature [1] when all the random variables are defined on the same probability space: convergence with probability 1 or strong convergence and convergence in probability or weak convergence. It is well known that convergence with probability 1 implies convergence in probability but the converse in not true (see for example Billingsley [1] p.274 and p.340). These convergences are used respectively for the strong law and the weak law of large numbers. In statistics, if a sequence of statistics converges in probability to the population value as the sample size goes to infinity according to the weak law of large numbers, the statistic is called *consistent*. Convergence in probability implies convergence in distribution, another type of convergence used in the central limit theorem. Moreover the convergence theorems are important because they yield sufficient conditions for the integral to be interchanged with pointwise convergence of random variables.

In Denneberg [5] these different types of convergence of sequences of random variables are considered with respect to a monotone set function instead of a probability measure. To obtain the same relations among different types of convergence some other properties are required for the monotone set function. In particular it has been proven that pointwise convergence (that is a particular case of convergence with probability 1) of a sequence of random variables to a random variable X implies the μ-stochastic convergence (convergence in probability if μ is a probability measure) if either μ is continuous from above or the convergence is uniform. If μ

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is a subadditive monotone set function then μ-stochastic convergence implies convergence in μ-distribution. In [2] μ-stochastic convergence plus the uniform integrability have been proven to imply the convergence in mean for monotone, subadditive, normalized continuous from above set functions; moreover the convergence in mean of a sequence of Borel measurable random variables has been proven to imply the μ-stochastic convergence. In this paper relations among different types of convergences with respect to upper probability defined by Hausdorff outer measures are investigated. The necessity to introduce Hausdorff (outer) measures as new tool to assess (upper) conditional probability is due to some problems related to the axiomatic definition of regular conditional probability [8]. In fact every time that the σ-field of the conditioning events is not countably generated, conditional probability, defined by the Radon-Nikodym derivative may be not separately coherent as required in the Walley's approach [9].

The paper is organized as follows. In Section 2 different types of convergence with respect to a monotone set function and they relations are recalled. Upper conditional previsions defined with respect to Hausdorff outer measures are proposed in Section 3 and they are proven to be the upper envelopes of all linear previsions defined on the class of all bounded random variables and agree with them on the Borel σ-field. Upper conditional probabilities are obtained if all random variables are 0-1 valued and the unconditional probability is obtained when the conditioning event is Ω. In Section 4, relations among different types of convergence with respect to the given upper conditional probability are proven. Moreover the upper conditional previsions are proven to satisfy the Monotone Convergence Theorem so that they are continuous from below with respect to the pointwise convergence. In Section 5 convergence in distribution with respect to the given upper conditional probability and for Borel-measurable random variables is proven to be equivalent to the pointwise convergence of the expectation functionals on all bounded continuous functions.

2 Convergences of Random Variables with Respect to a Monotone Set Function

In [5] different kinds of convergence of a sequence of random variables with respect to a monotone set function are introduced and their relations have been proven.

Given a non-empty set Ω and denoted by S a set system, containing the empty set and properly contained in $\wp(\Omega)$, the family of all subsets of Ω, a monotone set function μ: $S \to \overline{\Re}_+ = \Re_+ \cup \{+\infty\}$ is such that $\mu(\oslash)$=0 and if $A, B \in S$ with $A \subset B$ then $\mu(A) \leq \mu(B)$. A monotone set function on S is *continuous from below* if for each increasing sequence of sets A_n of S such that A=$\bigcup_{n=1}^{\infty} A_n$ belongs to S we have $lim_{n\to\infty}\mu(A_n) = \mu(A)$. A monotone set function on S is *continuous from above* if for each decreasing sequence of sets A_n of S such that A=$\bigcap_{n=1}^{\infty} A_n$ belongs to S we have $lim_{n\to\infty}\mu(A_n) = \mu(A)$. If μ is a finite, additive monotone set function on S, which is closed under set difference then the following properties are equivalent

(i) μ is continuous from below;

(ii) μ is continuous from above

(iii) μ is continuous from above at the empty-set.

If S is a σ-field then μ is σ-additive if and only if it is additive and continuous from below.

Given a monotone set function μ on S the *outer set function* of μ is the set function defined on the whole power set $\wp(\Omega)$ by

$$\mu^*(A) = inf\Big\{\mu(B)|A \subset B \in S\Big\}\ ,\ A \in P(\Omega).$$

The inner set function of μ is the set function defined on the whole power set $\wp(\Omega)$ by

$$\mu_*(A) = sup\Big\{\mu(B)|B \subset A; B \in S\Big\}\ ,\ A \in P(\Omega).$$

We recall the definitions of different types of convergence with respect to a monotone set function and their implications given in [5]. Let μ be a monotone set function defined on S properly contained in P(Ω) and $X : \Omega \to \overline{\Re} = \Re \cup \Big\{-\infty, \infty\Big\}$ an arbitrary function on Ω then the set function

$$G_{\mu,X}(x) = \mu\Big\{\omega \in \Omega : X(\omega) > x\Big\}$$

is decreasing and it is called *decreasing distribution function* of X with respect to μ. If μ is continuous from below then $G_{\mu,X}(x)$ is right continuous.

A function $X : \Omega \to \overline{\Re}$ is called upper μ-measurable if $G_{\mu^*,X}(x) = G_{\mu_*,X}(x)$.

Given a monotone set function μ defined on a field S and a sequence $X_n : \Omega \to \overline{\Re}$ of upper μ-measurable functions we say that X_n *converges in μ-distribution* to an upper μ measurable function $X : \Omega \to \overline{\Re}$ if $lim_{n\to\infty} G_{\mu,X_n} = G_{\mu,X}$ except at on at most countable set. Since a monotone set function is continuous except on an at most countable set the previous condition is equivalent to $lim_{n\to\infty} G_{\mu,X_n}(x) = G_{\mu,X}(x)$ for all continuity points x of $G_{\mu,X}$.

A sequence of random variables X_n *converges μ-stochastically* (or converges in probability if μ is a probability) to a random variable X if $|X_n$-$X|$ converges in μ^*-distribution to the null function $G_{\mu,0}(x)$, where $G_{\mu,0}(x) = \mu(\Omega)$ if $x \leq 0$ and $G_{\mu,0}(x) = 0$if $x > 0$.

If μ is monotone and subadditive and X_n converges μ-stochastically to X then X_n converges in μ-distribution to X (Proposition 8.5 of [5]). If μ is monotone and continuous from above and X_n converges pointwise to X then X_n converges μ-stochastically to X (Proposition 8.8 of [5]).

Given an upper μ-measurable function $X : \Omega \to \overline{\Re}$ with decreasing distribution function $G_{\mu,X}(x)$, the Choquet integral of X with respect to μ is defined if $\mu(\Omega) < \infty$ through

$$\int X d\mu = \int_{-\infty}^{0} (G_{\mu,X}(x) - \mu(\Omega))dx + \int_{0}^{\infty} G_{\mu,X}(x)dx.$$

The integral is in $\Re$ or can assume the values $-\infty$, ∞ and 'non-existing'. If $X \geq 0$ or $X \leq 0$ the integral always exists.

Let μ be a monotone set function and let X_n be a sequence of random variables such that $Y \leq X_n \leq Z$ for every n $\in N$ and Y and Z have finite Choquet integral with respect to μ; if X_n converges in μ-distribution to X then $lim_{n\to\infty} \int X_n d\mu = \int X d\mu$ (General Dominated Convergence Theorem, Proposition 8.9 of [5]).

3 Upper Conditional Previsions Defined with Respect to Hausdorff Outer Measures

Let (Ω, d) be a metric space, a bounded random variable X is a bounded function from Ω to $\Re$ and let $\mathbf{L}(\Omega)$ be the set of all bounded random variable on Ω. When $\mathbf{K}$ is a linear space of bounded random variables a coherent upper prevision is a real function $\overline{P}$ defined on $\boldsymbol{K}$, such that the following conditions hold for every X and Y in $\boldsymbol{K}$:

(1) $\overline{P}\ (X) \leq \sup(X)$;

(2) $\overline{P}(\lambda\ X) = \lambda\ \overline{P}(X)$ for each positive constant λ;

(3) $\overline{P}(X{+}Y) \leq \overline{P}(X) + \overline{P}(Y)$.

Suppose that $\overline{P}$ is an upper prevision defined on a linear space $\mathbf{K}$, its conjugate lower prevision $\underline{P}$ is defined on the same domain $\mathbf{K}$ by $\underline{P}$(-X) = - $\overline{P}$ (X). If for every X belonging to $\mathbf{K}$ we have $P(X) = \overline{P}(X) = \underline{P}(X)$, then P is called a *linear* prevision.

When $\boldsymbol{K}$ is a linear space of events, that can be regarded as a class of 0-1 valued gambles then $\overline{P}(X)$ is called an upper coherent probability and $\underline{P}$ (X) is a lower coherent probability.

A necessary and sufficient condition for an upper probability $\overline{P}$ to be coherent is to be the *upper envelope* of linear previsions, i.e. there is a class M of linear previsions such that $\overline{P}$ =sup$\{P(X) : P \in M\}$ ([9] 3.3.3).

Given a coherent upper (lower) prevision $\overline{P}$ defined on a domain $\mathbf{K}$ the maximal (minimal) coherent extension of $\overline{P}$ to the class of all bounded random variables is called [9] *natural extension* of $\overline{P}$ ([9] 3.1.1).

The linear extension theorem ([9] 3.4.2) assures that the class of all linear extensions to the class of all bounded random variables of a linear prevision P defined on a linear space $\mathbf{K}$ is the class $M(P)$ of all linear previsions that are dominated by P on $\mathbf{K}$. Moreover the upper and lower envelopes of $M(P)$ are the natural extensions of P ([9] Corollary 3.4.3).

We recall the notion of separately coherent conditional upper previsions.

Let $\boldsymbol{B}$ denote a partition of Ω, which is a class of non-empty, pair wise-disjoint subsets whose union is Ω. For B in $\boldsymbol{B}$ let $\boldsymbol{H}(B)$ be the set of all random variables defined on B. An upper conditional prevision $\overline{P}(X|B)$ is a real function defined on $\boldsymbol{H}(B)$. Upper conditional previsions $\overline{P}(X|B)$, defined for B in $\boldsymbol{B}$ and X in $\boldsymbol{H}(B)$

are required [9] to be *separately coherent*, that is for every conditioning event B $\overline{P}(\cdot|B)$ is a coherent upper prevision on the domain $\boldsymbol{H}(B)$ and $\overline{P}(B|B) = 1$. Upper and lower conditional probabilities are particular kinds of upper and lower conditional previsions obtained when $\boldsymbol{H}(B)$ is a class of events.

3.1 Hausdorff Outer Measures

Given a non-empty set Ω an *outer measure* is a function $\mu^* : \wp(\Omega) \rightarrow [0, \infty]$ such that $\mu^*(\oslash) = 0$, $\mu^*(A) \leq \mu^*(A')$ if $A \subseteq A'$ and $\mu^*(\bigcup_{i=1}^{\infty} A_i) \leq \sum_{i=1}^{\infty} \mu^*(A_i)$.

Examples of outer set functions or outer measures are the Hausdorff outer measures.

Let (Ω, d) be a metric space. A topology, called the *metric topology*, can be introduced into any metric space by defining the open sets of the space as the sets G with the property:

if x is a point of G, then for some $r > 0$ all points y with $d(x, y) < r$ also belong to G.

It is easy to verify that the open sets defined in this way satisfy the standard axioms of the system of open sets belonging to a topology ([7] p.26).

If we assume that (Ω, d) is the Euclidean metric space where Ω is a subset of $\Re^n$, then the topology is the metric topology defined into the Euclidean metric space (Ω,d), that is the *natural* topology. If we assume that d is the discrete metric then the metric topology is $\wp(\Omega)$. The following results about Hausdorff outer measures are valid in a general metric space setting.

The diameter of a non empty set U of Ω is defined as $|U|= \sup\left\{d(x, y) : x, y \in U\right\}$ and if a subset A of Ω is such that $A\subset \bigcup_i U_i$ and $0< |U_i| < \delta$ for each i, the class $\left\{U_i\right\}$ is called a δ-cover of A.

Let s be a non-negative number. For $\delta >0$ we define $h_{s,\delta}$ $(A) = \inf \sum_{i=1}^{\infty} |U_i|^s$, where the infimum is over all δ-covers $\left\{U_i\right\}$.

The *Hausdorff s-dimensional outer measure* of A, denoted by $h^s(A)$, is defined as

$$h^s(A) = lim_{\delta\rightarrow 0}\ h_{s,\delta}\ (A).$$

This limit exists, but may be infinite, since $h_{s,\delta}(A)$ increases as δ decreases. The *Hausdorff dimension* of a set A, $dim_H(A)$, is defined as the unique value, such that

$$h^s(A) = \infty \text{ if } 0 \leq s < dim_H(A),$$
$$h^s(A) = 0 \text{ if } dim_H(A) < s < \infty.$$

We can observe that if $0 < h^s(A) < \infty$ then $dim_H(A) = s$, but the converse is not true. We assume that the Hausdorff dimension of the empty set is equal to -1 so that no event has Hausdorff dimension equal to the empty set. Denote by t the Hausdorff dimension of Ω , if an event A is such that $dim_H(A) = s < t$ then the Hausdorff dimension of the complementary set is equal to t since the following relation holds:

$$dim_H(A \cup B){=}\max\left\{dim_H(A), dim_H(B)\right\}.$$

Hausdorff outer measures are *metric* outer measures, that is $h^s(E \cup F) = h^s(E) + h^s(F)$ whenever E and F are *positively separated*, i.e. $d(E, F) = inf\left\{d(x, y) : x \in E, y \in F\right\} > 0$.

A subset A of Ω is called *measurable* with respect to the outer measure h^s if it decomposes every subset of Ω additively, that is if $h^s(E) = h^s(A \cap E) + h^s(E - A)$ for all sets $E \subseteq \Omega$.

All Borel subsets of Ω are measurable with respect to a metric outer measure ([6] Theorem 1.5). So every Borel subset of Ω is measurable with respect to every Hausdorff outer measure h^s since Hausdorff outer measures are metric.

The restriction of h^s to the σ-field of h^s-measurable sets, containing the σ-field of the Borel sets, is called Hausdorff s-dimensional measure. The Borel σ-field is the σ-field generated by all open sets. The Borel sets include the closed sets (as complement of the open sets), the F_σ-sets (countable unions of closed sets) and the G_σ-sets (countable intersections of open sets), etc.

In particular the Hausdorff 0-dimensional measure is the counting measure and the Hausdorff 1-dimensional measure is the Lebesgue measure.

The Hausdorff s-dimensional measures are *modular* on the Borel σ-field, that is $h^s(A \cup B) + h^s(A \cap B) = h^s(A) + h^s(B)$ for every pair of Borelian sets A and B; so that (Proposition 2.4 of [5]) the Hausdorff outer measures are *submodular* ($h^s(A \cup B) + h^s(A \cap B) \leq h^s(A) + h^s(B)$) and the Hausdorff inner measures are *supermodular* or *2-monotone* ($h^s(A \cup B) + h^s(A \cap B) \geq h^s(A) + h^s(B)$).

An important property of Hausdorff outer measures is that they are *regular* ([6] Theorem 1.6), that is if for every set A there is an h^s-measurable set E containing A with $h^s(E) = h^s(A)$. In Theorem 1.6 (a) of [6] it has been proven that if A is any subset of $\Re^n$ there is a G_σ-set G containing A with $h^s(A) = h^s(G)$. In particular h^s is a *outer regular* measure.

Moreover Hausdorff outer measures are *continuous from below* ([6] Lemma 1.3), that is for any increasing sequences of sets $\left\{A_i\right\}$ we have $\lim_{i\to\infty} h^s(A_i) = h^s(\lim_{i\to\infty} A_i)$.

A useful consequence of regularity of Hausdorff outer measures is that h^s-measurable sets with finite Hausdorff outer measure can be approximated from below by closed subsets ([6] Theorem 1.6 (b)) or equally that the restriction of every Hausdorff outer measure h^s to the class of all h^s-measurable sets with finite Hausdorff outer measure is *inner regular* on the class of all closed subsets of Ω.

3.2 A New Model of Separately Coherent Upper Conditional Previsions

Upper conditional previsions for bounded random variables are defined by the Choquet integral with respect to Hausdorff outer measures if the conditioning event has positive and finite Hausdorff outer measure in its dimension; otherwise, they are defined by a 0-1 valued finitely, but not countably, additive probability.

Theorem 1 *Let (Ω, d) be a metric space and let $\boldsymbol{B}$ be a partition of Ω. For every $B \in \boldsymbol{B}$ denote by s the Hausdorff dimension of the conditioning event B and by h^s the Hausdorff s-dimensional outer measure. Let $\boldsymbol{L}(B)$ be the class of all bounded random variables on B. Moreover, let m be a 0-1 valued finitely additive, but not countably additive, probability. Then for every B in $\boldsymbol{B}$ the functionals $\overline{P}(X|B)$ defined on $\boldsymbol{L}(B)$ by*

$$\overline{P}(X|B) = \frac{1}{h^s(B)} \int_B X dh^s \text{ if } 0 < h^s(B) < \infty$$

and by

$$\overline{P}(X|B) = m(XB) \text{ if } h^s(B) = 0, \infty$$

are separately coherent upper conditional previsions.

Proof: Since $\mathbf{L}(B)$ is a linear space we have to prove that, for every $B \in \mathbf{B}$ $\overline{P}(X|B)$ satisfy conditions 1), 2), 3) and the condition $\overline{P}(B|B) = 1$.

From the definition of $\overline{P}(X|B)$ we have that for every conditioning event B the upper conditional prevision $\overline{P}(\cdot|B)$ satisfies properties 1) and 2). Moreover property 3) follows from the given definition in the case where B has Hausdorff measure equal to zero or infinity. If B has finite and positive Hausdorff outer measure in its dimension then property 3), follows from the Subadditivity Theorem ([5] Theorem 6.3) since Hausdorff outer measures are monotone, submodular and continuous from below. From the definition we also have $\overline{P}(B|B) = 1$.

Upper (lower) conditional probabilities can be obtained from the previous definition in the case where $\boldsymbol{L}(B)$ is the class of all 0-1 valued random variables of B. In [4] they are proven to be separately coherent.

Theorem 2 *Let (Ω, d) be a metric space and let $\boldsymbol{B}$ be a partition of Ω. Denote by s the Hausdorff dimension of the conditioning event B and by h^s the Hausdorff s-dimensional outer (inner) measure. Let $\boldsymbol{F}$ be the class of all subsets of Ω. Moreover, let m be a 0-1 valued finitely additive, but not countably additive, probability. Then the functions defined on $\boldsymbol{F} \times \boldsymbol{B}$ by*

$$\overline{P}(A|B) = \frac{h^s(AB)}{h^s(B)} \text{ if } 0 < h^s(B) < \infty$$

and by

$$\overline{P}(A|B) = m(AB) \text{ if } h^s(B) = 0, \infty$$

are separately coherent upper (lower) conditional probabilities.

For every $B \in \mathbf{B}$ let $P(\cdot|B)$ be the restriction to the class of all bounded Borel-measurable random variables of the upper and lower conditional previsions defined in Theorem 1.

In the next theorem we prove that the upper and lower conditional previsions defined as in Theorem 1 are the upper and lower envelopes of all linear extensions of $P(\cdot|B)$ to the class of $\boldsymbol{L}(B)$.

Theorem 3 *Let (Ω, d) be a metric space and let $\boldsymbol{B}$ be a partition of Ω. For every $B \in \boldsymbol{B}$ let $P(\cdot|B)$ be the restriction to the class of all bounded Borel-measurable random variables of the upper and lower conditional previsions defined in Theorem 1. For every conditioning event $B \in \boldsymbol{B}$ let $\boldsymbol{L}(B)$ be the class of all bounded random variables defined on B; then the upper and lower conditional previsions defined on $\boldsymbol{L}(B)$ as in Theorem 1 are the upper and lower envelopes of all linear extensions of $P(\cdot|B)$ to the class $\boldsymbol{L}(B)$.*

Proof: Since $P(\cdot|B)$ is a linear conditional prevision on the σ-field of all bounded Borel-measurable random variables, which is a linear space, for the linear extension theorem we have that the class of all linear extensions of $P(\cdot|B)$ to the class of all bounded random variables defined on B is $M(P)$, i.e. the class of all linear conditional previsions dominated by $P(\cdot|B)$ on the σ-field of all bounded Borel-measurable random variables. Upper and lower envelopes of all such extensions are the natural extensions of $P(\cdot|B)$ ([9] Corollary 3.4.3). If the conditioning event B has positive and finite Hausdorff measure in its dimension the natural extensions of $P(\cdot|B)$ are the upper and lower conditional previsions $\overline{P}(\cdot|B)$ and $\underline{P}(\cdot|B)$ defined as in Theorem 1. In fact the natural extensions of a countably additive probability defined on a σ-field are the outer and inner measures generated by it ([9] Theorem 3.1.5). Moreover since Hausdorff outer (inner) measures are submodular (supermodular or 2-monotone) the natural extensions to the class of all bounded random variables on B of the upper (lower) conditional probabilities defined as in Theorem 1 are given by the Choquet integral with respect to these upper (lower) probabilities [3]. If the conditioning event B has Hausdorff outer measure equal to zero or infinity then $P(\cdot|B)$ is a 0-1 valued additive probability on the Borel σ-field and so it is the upper (lower) envelope of all probability measures defined on the Borel σ-field and dominated by $P(\cdot|B)$.

4 Relations among Different Types of Convergence

In the next theorem we prove that when B has finite and positive Hausdorff outer measure in its dimension and upper conditional probabilities are defined as in Theorem 2, then μ-stochastic convergence implies convergence in μ-distribution.

Theorem 4 *Let $(\Omega,\ d)$ be a metric space and let $\boldsymbol{B}$ be a partition of Ω. For every $B \in \boldsymbol{B}$ such that B has finite and positive Hausdorff outer measure in its dimension denote by $\mu = \overline{P}(A|B)$ the upper conditional probability defined as in Theorem 2. Let X_n be a sequence of random variables on B; if X_n converges to a random variable X μ-stochastically then X_n converges in μ-distribution to X.*

Proof: Since B has finite and positive Hausdorff measure in its dimension we have $\mu = \overline{P}(A|B) = \frac{h^s(AB)}{h^s(B)}$. Moreover every outer Hausdorff measure is subadditive so we obtain that μ-stochastic convergence implies convergence in μ-distribution.

An important consequence of Theorem 4 is that upper conditional probabilities defined as in Theorem 2, satisfy the General Dominated Convergence Theorem.

We prove that when B is a non-empty set with positive and finite Hausdorff outer measure in its dimension, the upper conditional prevision defined as in Theorem 1 satisfies the following Monotone Convergence Theorem for monotone set functions ([5] Theorem 8.1).

Theorem 5 *Let μ be a monotone set function on a σ-algebra $\boldsymbol{F}$ properly contained in $P(\Omega)$, which is continuous from below. For an increasing sequence of non negative, $\boldsymbol{F}$-measurable random variables X_n the limit function $X = lim_{n\to\infty} X_n$ is $\boldsymbol{F}$-measurable and $lim_{n\to\infty} \int X_n d\mu = \int X d\mu$.*

We can observe that if the monotone set function is defined on the power set then we do not need to impose any measurability condition since the Choquet integral is defined for every random variable. Moreover the Monotone Convergence Theorem involves a sequence of non negative, measurable random variables but this is not a restriction since for any measurable random variable f there exist two measurable, non negative random variables $f^+(\omega) = max\{f(\omega), 0\}$ and $f^-(\omega) = -min\{f(\omega), 0\}$ such that $f = f^+ - f^-$([1] p.203).

Theorem 6 *Let $(\Omega,\ d)$ be a metric space and let $\boldsymbol{B}$ be a partition of Ω. For every $B \in \boldsymbol{B}$ let $\boldsymbol{L}(B)$ be the class of all bounded random variables on B. If B has positive and finite Hausdorff outer measure in its dimension then the coherent upper previsions defined as in Theorem 1 are continuous from below, that is given an increasing sequence of non negative random variables X_n converging pointwise to the random variable X we have that $lim_{n\to\infty}\overline{P}(X_n|B) = \overline{P}(X|B)$.*

Proof: If B has positive and finite Hausdorff outer measure in its dimension we have that $\overline{P}(X|B) = \frac{1}{h^s(B)} \int_B X dh^s$. Since each s-dimensional Hausdorff outer measure is continuous from below then by the Monotone Convergence Theorem it follows that the given upper conditional prevision is continuous from below.

When B has positive and finite Hausdorff outer measure in its dimension, the upper conditional probabilities μ defined as in Theorem 2, are monotone, subadditive and continuous from below so they satisfy Proposition 3.7 of [2], that is a sequence of random variables μ-uniformly integrable and μ-stochastically converging to a random variable X, converges in μ-mean to X. Moreover from Proposition 3.8 of [2] we obtain that if μ is equal to the upper conditional probability defined as in Theorem 2 then a sequence of Borel measurable random variables converging in μ-mean to X is μ-stochastically converging. In the next theorem we prove that when B has finite and positive Hausdorff outer measure in its dimension and we consider the restriction to the Borel σ-field of the upper conditional probabilities defined as in Theorem 2, then pointwise convergence implies μ-stochastic convergence.

Theorem 7 *Let $(\Omega,\ d)$ be a metric space and let $\boldsymbol{B}$ be a partition of Ω. Let $\boldsymbol{F}$ be the σ-field of all Borel subsets of Ω. Moreover, let m be a 0-1 valued finitely additive, but not countably additive, probability. For every $B \in \boldsymbol{B}$ such that B has finite and positive Hausdorff outer measure in its dimension denote by μ be the (upper) conditional probabilities defined on $\boldsymbol{F}$ as in Theorem 2, that is $\mu = \overline{P}(A|B)$. Let X_n be a sequence of Borel measurable random variables on B converging pointwise to a random variable X. Then X_n converges to X μ-stochastically.*

Proof: Since B has finite and positive Hausdorff measure in its dimension we have $\mu = \overline{P}(A|B) = \frac{h^s(AB)}{h^s(B)}$. Moreover every outer Hausdorff measure is continuous from below and countably additive on the Borel σ-field. So every (outer) Hausdorff measure is continuous from above on the Borel σ-field. Then the pointwise convergence of a sequence of random variables X_n to X implies the μ-stochastic convergence of X_n to X.

Remark 1: In general a coherent upper probability is not continuous from below and continuous from above; for example if we consider a coherent upper probability defined as natural extension of a merely finitely additive probability on a σ-field, then it is not continuous from above and continuous from below since an additive measure on a σ-field is continuous from above and continuous from below if and only if it is σ-additive. As a consequence we have that the pointwise convergence does not imply stochastic convergence with respect to this upper probability and the Monotone Convergence Theorem cannot always be applied. Hausdorff outer measures satisfy Theorem 6 and Theorem 7 because they are Borel regular outer measures.

5 Convergence in μ-distribution

In probability theory convergence in distribution has been proven to be equivalent to the pointwise convergence of the expectation functionals on all bounded continuous functions (see for example Theorem 29.1 of [1]).

If μ is a probability measure on a probability space and X is a measurable random variable then to every decreasing distribution function $G_{\mu,X}(x)$ corrisponds a probability measure on ($\Re$,**F**), where **F** is the Borel σ-field. If μ_n and μ are the probability measures on ($\Re$,$\boldsymbol{F}$) corresponding to $G_{\mu,X_n}(x)$ and $G_{\mu,X}(x)$ then X_n converges in μ-distribution to X if and only if $lim_{n\to\infty}\mu_n(A)$=$\mu(A)$ for every A=(x,∞).

This last condition is equivalent to the pointwise convergence of expectation functionals on all bounded and continuous function f (Theorem 29.1 of [1]), that is $lim_{n\to\infty} \int f d\mu_n = \int f d\mu$.

The notion of upper probability induced by a random variable X is proposed.

Definition 1 *Let $(\Omega,\ d)$ be a metric space and let $\boldsymbol{B}$ be a partition of Ω. For every $B \in \boldsymbol{B}$ with positive and finite Hausdorff outer measure in its dimension denote by $\mu = \overline{P}(A|B)$ the upper conditional probabilities defined as in Theorem 2. Given a random variable X on B then the upper probability μ_X induced by X on ($\Re$,$\boldsymbol{F}$), where $\boldsymbol{F}$ is the Borel σ-field, is defined by $\mu_X(B) = \overline{P}(\omega \in B : \omega\ \in X^{-1}(H)) = \frac{h^s(X^{-1}(H))}{h^s(B)}$ for H belonging to $\boldsymbol{F}$.*

We have that the equivalence between convergence in μ-distribution for Borel-measurables random variables and the pointwise convergence of expectation functionals on all bounded and continuous function f remains valid when upper probabilities are defined with respect to Hausdorff outer measures.

Theorem 8 *Let $(\Omega,\ d)$ be a metric space and let $\boldsymbol{B}$ be a partition of Ω. For every $B \in \boldsymbol{B}$ with positive and finite Hausdorff outer measure in its dimension denote by $\mu = P(A|B)$ the restriction to the Borel σ-field of the upper conditional probability defined as in Theorem 2. Let $L^*(B)$ be the class of all Borel measurable random variables on B. Then the convergence in μ-distribution of a sequence of random variables of $L^*(B)$ to a random variable X is equivalent to the pointwise convergence of expectation functionals on all bounded and continuous function f that is $lim_{n\to\infty} \int f d\mu_n = \int f d\mu$.*

Proof: If X_n and X are Borel-measurable random variables and H is a Borelian set then the sets $X_n^{-1}(H)$ and $X^{-1}(H)$ are also Borelian sets; moreover since every Hausdorff s-dimensional outer measure is countably additive on the Borel σ-field then the (upper) conditional probabilities μ_n and μ induced respectively by X_n and X on $(\Re,\boldsymbol{F})$ are probability measures. Then convergence in μ-distribution is equivalent to the pointwise convergence of expectation functionals on all bounded and continuous function f.

6 Conclusions

This paper investigates the relations among different types of convergence for random variables when they are based on an upper probability approach where conditional upper expectations with respect to Hausdorff outer measures are used whenever we have to condition on a set with probability zero.

Upper (lower) conditional previsions defined with respect to Hausdorff outer measures are proven to be the upper (lower) envelopes of all linear extensions to the class of all random variables of the restriction to the Borel-measurable random variables of the given upper conditional previsions.

It is proven that the relations among different types of convergences of random variables defined with respect to upper conditional probability defined by Hausdorff outer measures are the same that hold if convergences are defined with respect to a probability measure. When the conditioning event has finite Hausdorff outer measure in its dimension these results are obtained because Hausdorff outer measures are Borel regular outer measures and so continuous from below and continuous from above on the Borel σ-field. In general if upper conditional probability is defined as natural extension of a coherent merely finitely additive probability defined on a σ-field we have that μ-stochastically convergence does not imply convergence in μ-distribution since in this case the upper conditional probability is not continuous from above.

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