Journal of Uncertain Systems
Vol.4, No.1, pp.4-13, 2010
Online at: www.jus.org.uk

Disjoint Programming in Computational Decision Analysis

Xiaosong Ding[1,*], Mats Danielson[2], Love Ekenberg[2]
[1]*International Business School, Beijing Foreign Studies University 100089, Beijing, People's Republic of China*
[2]*Department of Computer and Systems Sciences, Stockholm University and Royal Institute of Technology*
Forum 100, Kista, SE-164 40, Sweden

Received 8 December 2008; Revised 10 May 2009

Abstract

This paper discusses a series of imprecise decision models and their corresponding computational aspects arising in computational decision analysis. The imprecise decision models relax the traditional point estimates into intervals and incorporate various types of vague information represented as linear constraints from a decision-maker. When the principle of maximizing expected utility is applied as the decision rule, the evaluations of these models become non-convex optimization problems and require some global optimization strategies. This paper presents a class of global optimization algorithms for solving such non-convex programs. We take advantage of polar cuts and the disjoint structural property of the imprecise decision models to develop generalized cutting plane methods that are different from the traditional class of branch and bound approaches.

Keywords: global optimization, disjoint programming, cutting plane, imprecise decision analysis

1 Introduction

Most classical decision analysis approaches consist of a set of straightforward decision rules applied to precise estimates of weights, probabilities, and/or utilities no matter how unsure a decision-maker is of his estimates. The requirement for numerically precise data has been considered unrealistic by an increasing number of researchers and decision-makers. Techniques allowing imprecision have been suggested, among which the interval methods have been widely utilized to strengthen other decision models that are built upon the principle of maximizing the expected utility (PMEU), e.g., [7, 8, 9, 13, 16, 17]. When PMEU is applied, disjoint bilinear programming (DBLP) programs such as $\sum \alpha_{ij} p_{ij} u_{ij}$ or $\sum \alpha_{ki} w_k u_{ki}$ where α_{ij} or α_{ki} are coefficients, are obtained while searching for the best alternatives if both interval probabilities or weights and interval utilities are represented by variables. When modeling multi-criteria problems that also have uncertain events, combining these two kinds of decision models gives rise to multi-criteria probabilistic models with trilinear expressions such as $\sum \alpha_{kij} w_k (\sum p_{kij} u_{kij})$, and yields computationally hard disjoint trilinear programming (DTLP) programs together with constraints on weights, probabilities, and utility values. Moreover, for decision situations that possess disjoint probability chain property, adopting PMEU may even lead to disjoint multi-linear programming (DMLP) programs.

Although optimization techniques have been developed rapidly in recent years, much of the area of optimization has been devoted to solving large problems. Nevertheless, the particular programs arising in imprecise computational decision analysis require approaches for solving sequences of related smaller global optimization problems in interactive time because in mathematical programming, DBLP, DTLP and DMLP all exhibit non-convexity. Therefore, some global optimization strategies that iterate between a global phase and a local phase in search of the global optimum are necessary. Two main branches of deterministic approaches for handling such non-convex programs are cutting plane methods [11, 12, 19, 20, 21], and branch and bound methods [1, 2, 4, 14, 15, 18]. Combined approaches have also been suggested [3, 10].

This paper intends to discuss a series of imprecise decision models and their corresponding computational aspects. A class of generalized cutting plane methods is developed by taking advantage of polar cuts and the disjoint structural property of the imprecise decision models. The following section presents an imprecise decision framework

[*] Corresponding author. Email: xiaosong.ding@gmail.com (X. Ding).

that can be readily transformed into DBLP and the fundamental technique for solving DBLP. Thereafter, we extend the decision framework into an imprecise probabilistic multi-criteria decision model, transform it into DTLP, and discuss the corresponding global optimization algorithm in great detail. The fourth section presents an imprecise multilevel probability chain decision model that can be transformed into DMLP, and discusses the global optimization algorithm extended from the third section. The final section concludes this paper and indicates further research topics.

2 Disjoint Bilinear Programming

2.1 A Bilinear Imprecise Decision Model

In interval decision analysis, a decision-maker is often encouraged to be deliberately imprecise in his subjective judgements. Input sentences in decision problems consisting of discrete alternatives with events and consequences, e.g., interval estimates and qualitative information, are translated into linear constraints. For instance, the probability (or utility) of a consequence c_{ij} being between the numbers a_k and b_k is expressed as $p_{ij} \in [a_k, b_k]$ (or $u_{ij} \in [a_k, b_k]$). Relations can be handled similarly: a measure of c_{ij} is greater than a measure of c_{kl} is expressed as $p_{ij} \ge p_{kl}$ (or analogously $u_{ij} \ge u_{kl}$). In this way, each statement is represented by one or more constraints.

Definition 1: An *information frame* represents a decision situation with *m* alternatives, each alternative having $m_i, i = 1,\dots,m$ consequences, as a structure $\langle C,P,U \rangle = \langle \{\{c_{ij}\}_{m_i}\}_m, P, U \rangle, j \in 1,\dots,m_i, i \in 1,\dots,m,$ where each c_{ij} denotes a consequence, *P* is a finite list of linear constraints over p_{ij}-variables denoting probabilities over the consequences c_{ij}, and *U* is a finite list of linear constraints over u_{ij}-variables denoting utilities over the consequences c_{ij}.

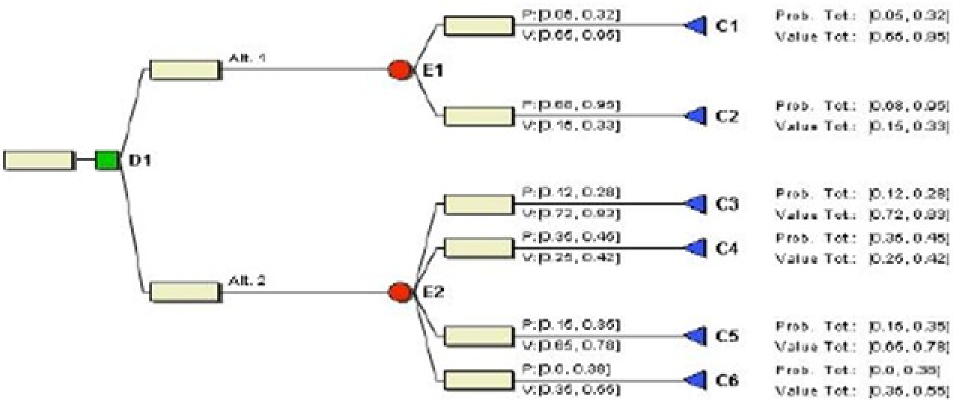

Figure 1: A bilinear imprecise decision model

Suppose we have a decision situation as shown in Figure 1 where *D1* is a decision node, *E1* and *E2* are probability nodes, representing indeterminism, with associated probability distributions, and the leaves are consequence nodes with convex sets of associated value or utility functions. This decision situation can be mapped into an information frame $\langle C,P,U \rangle = \langle \{\{c_{ij}\}_{m_i}\}_m, P, U \rangle$ with $m = 2, m_1 = 2$ and $m_2 = 4$.

This structure is then populated with user statements represented as linear inequalities. Since a vector in the polytope can be considered to represent a distribution, a probability base *P* can be interpreted as constraints defining the set of all possible probability measures over the consequences. Analogously, a utility base *U* consists of constraints defining the set of all possible utility functions over the consequences. These two bases together with a tree structure constitute the information frame.

Definition 2: Given an information frame $\langle \{\{c_{ij}\}_{m_i}\}_m, P, U \rangle$, the *expected utility* of an alternative A_i is $E(A_i) = \sum_{j \in m_i} p_{ij} u_{ij}$, where p_{ij} and u_{ij} are variables in *P* and *U*, respectively; p_{ij} denotes the probability of c_{ij} occurring given that alternative A_i is chosen; and u_{ij} denotes the corresponding utility.

Using precise numbers, evaluating the expected utility of an alternative is rather straightforward. Nevertheless, when numerically imprecise information is involved, the expected utility has to be evaluated with respect to *P* and *U*.

By considering all possible solution vectors, a range of expected utilities for each alternative can be received. Once we have a well-defined information frame and apply PMEU as the decision rule, we need to calculate the minimal and maximal expected utilities for each alternative.

Firstly, when imprecise information represented as linear constraints exists within either *P* or *U*, to evaluate the decision model is reduced to linear programming (LP) problems that can be readily solved with modern optimization techniques [8]. Secondly, as imprecise information exists in both *P* and *U* concerning certain criterion, the evaluation of a decision model under this criterion based on PMEU results in a simplified DBLP program in the sense that it only contains cross product terms such as $\sum pv$ but no linear terms. For example, if we need to evaluate the second alternative in Figure 1, we have to calculate $\max/\min \sum_{j=1}^{4} p_{2j}u_{2j}$ subject to disjoint linear constraints concerning *P* and *U*, respectively. It should be noted that for this bilinear imprecise decision model, we only allow a decision-maker to issue his imprecise statements with respect to each alternative such as $0.2 \le p_{11} + p_{12} \le 0.3$, but exclude the cross linear constraints among different alternatives such as $0.2 \le p_{11} + p_{21} \le 0.3$. Then, this decision model can be transformed into DBLP that is mathematically stated as

$$\begin{aligned} &\min\ f(x,y) = (c_x^t x + d_x)(c_y^t y + d_y) \Leftrightarrow \min\ f(x,y) = c^t x + d^t y + x^t C y \\ &s.t. X_0 = \{x \in R^{n_1} : A_1 x \le b_1, x \ge 0\}, Y_0 = \{y \in R^{n_2} : A_2 y \le b_2, y \ge 0\}. \end{aligned} \tag{1}$$

In (1), $A_1 x \le b_1$ and $A_2 y \le b_2$ represent the linear constraints populated in *P* and *U*, respectively.

2.2 Evaluation of DBLP

2.2.1 Local Optimization

The most important property of DBLP is that, even though $f(x,y)$ may not be quasi-concave, there exists an extreme point $\bar{x} \in X_0$ and an extreme point $\bar{y} \in Y_0$ such that $(\bar{x},\bar{y})$ is an optimal solution of DBLP [1]. This solution property and the structure of DBLP itself suggest an LP based vertex following algorithm that converges to a Karush-Kuhn-Tucker point [12].

Definition 3: Consider $P: \min f(x)$ subject to $x \in S$, where S is a compact polyhedral set and f is non-convex. A local star minimizer (LSM) of P is defined as a point $\bar{x}$ such that $f(\bar{x}) \le f(x)$ for each $x \in N_S(\bar{x})$, where $N_S(\bar{x})$ denotes the set of extreme points in S that are adjacent to $\bar{x}$.

For DBLP, an extreme point is adjacent to $(\bar{x},\bar{y})$ if and only if it is of the form either $(x^i,\bar{y})$ or $(\bar{x},y^i)$, where $x^i \in N_{X_0}(\bar{x})$ and $y^i \in N_{Y_0}(\bar{y})$.

Definition 4: An extreme point $(\bar{x},\bar{y})$ is called a pseudo-global minimizer (PGM) if $f(\bar{x},\bar{y}) \le f(x,y)$ for each $x \in B_\delta(\bar{x}) \cap X_0$ and for each $y \in Y_0$, where $B_\delta(\bar{x})$ is a δ neighborhood around $\bar{x}$.

With the above property and definition, we can have an LP based mountain climbing procedure to obtain a PGM of DBLP.

Algorithm 1:

(a) Find a feasible extreme point $\tilde{x}^1 \in X_0^i$, where X_0^i represents the reduced feasible region in i^{th} iteration after all cuts have been added.

(b) (1) Solve: $\min\{f(\tilde{x}^1, y) \mid y \in Y_0\}$ to yield an optimal $\tilde{y}^1$;

(2) Solve: $\min\{f(x,\tilde{y}^1) \mid x \in X_0^i\}$ to yield an optimal $\tilde{x}^2$;

Set $\tilde{x}^1 \leftarrow \tilde{x}^2$ and repeat step (b) until it converges to an LSM $(\bar{x},\bar{y})$.

(c) Suppose $\bar{x}$ is non-degenerate and let $\hat{x} \in N(\bar{x})$ be such that

$$f(\hat{x},\hat{y}) = \min_{y\in Y_0} f(\hat{x},y) < \min_{y\in Y_0} f(\bar{x},y) = f(\bar{x},\bar{y}).$$

Go to step (b(2)) with $\tilde{y}^1 \leftarrow \hat{y}$.

(d) Terminate with $(\bar{x},\bar{y})$ as a PGM.

Algorithm 1 is relatively easy to implement and there are other approaches concentrating on the location of different types of extreme point [10, 19]. However, in this paper, we will not step into the details of other local optimization methods since we focus on the utilization and generalization of such local algorithms in search of the global optimum. The incorporation of other approaches is supposed to be similar.

2.2.2 Generation of a Cutting Plane

Given a PGM $(\bar{x}, \bar{y})$ located by Algorithm 1, we employ polar cuts to cut off local vertex solutions. Assume $\bar{x}$ is a non-degenerate extreme point of X_0; let $x_j, j \in \bar{N}$, be the n_1 nonbasic variables at $\bar{x}$, where $\bar{N}$ is the index set for the nonbasic variables. Then X_0 has precisely n_1 distinct edges incident to $\bar{x}$. Each half line $\xi^j = \{x : x = \bar{x} - \bar{a}^j \lambda_j, \lambda_j \geq 0\}, j \in \bar{N}$ contains exactly one such edge [5].

Definition 5: The generalized reverse polar of Y_0 for a given scalar α is given by $Y_0(\alpha) = \{x : f(x, y) \geq \alpha\}$ for all $y \in Y_0$.

Let $(\bar{x}, \bar{y})$ be a PGM, let the rays ξ^j be defined as above, let α be the current best objective value (CBOV) of $f(x, y)$, and let $\bar{\lambda}_j$ be defined by

$$\bar{\lambda}_j = \max\{\lambda_j : f(\bar{x} - \bar{a}^j \lambda_j, y) \geq \alpha \text{ for all } y \in Y_0\} \text{ if } \xi^j \not\subset Y_0(\alpha). \tag{2}$$

As for the case when $\xi^j \subset Y_0(\alpha)$, we simply set $\bar{\lambda}_j$ to ∞ rather than employing the negative extension of polar cuts [19] because in that case, the following approach to compute $\bar{\lambda}_j$ cannot be used any longer. The inequality $\sum_{j \in \bar{N}} x_j / \bar{\lambda}_j \geq 1$ determines a valid cutting plane. Each $\bar{\lambda}_j$ can be computed by an approach on the basis of LP duality theory that costs only one LP iteration. Consider (1) and (2), in which we need to obtain

$$\max\{\lambda_j : f(\bar{x} - \bar{a}^j \lambda_j, y) \geq \alpha \mid A_2 y \leq b_2, y \geq 0\} =$$
$$\max\{\lambda_j : \min_y [c^t(\bar{x} - \bar{a}^j \lambda_j) + d^t y + (\bar{x} - \bar{a}^j \lambda_j)^t C y] \geq \alpha \mid A_2 y \leq b_2, y \geq 0\}.$$

Using LP duality theory, the foregoing can be rewritten as

$$\max\{\lambda_j : \max_u [c^t(\bar{x} - \bar{a}^j \lambda_j) + b_2^t u] \geq \alpha \mid A_2^t u \leq d + C^t(\bar{x} - \bar{a}^j \lambda_j), u \leq 0\}$$
$$= \max\{\lambda_j : \min_u [c^t \bar{a}^j \lambda_j - b_2^t u] \leq c^t \bar{x} - \alpha \mid C^t \bar{a}^j \lambda_j + A_2^t u \leq C^t \bar{x} + d, u \leq 0\}.$$

In matrix form, this expression asks for the maximal λ_j in

$$\max_{\lambda_j, u} \lambda_j$$
$$s.t. \begin{bmatrix} c^t \bar{a}^j & -b_2^t \\ C^t \bar{a}^j & A_2^t \end{bmatrix} \begin{bmatrix} \lambda_j \\ u \end{bmatrix} \leq \begin{bmatrix} c^t \bar{x} - \alpha \\ C^t \bar{x} + d \end{bmatrix}, u \leq 0. \tag{3}$$

Hence, we can obtain $\bar{\lambda}_j$ when $\xi^j \not\subset Y_0(\alpha)$ by solving just one LP program.

2.2.2 Global Optimization

At this stage, we are able to present a global optimization algorithm for solving DBLP by using **Algorithm 1** and polar cuts.

Algorithm 2:

(a) Let CBOV, $obj_0 = +\infty$; let the initial best feasible solution $\{(\hat{x}^0, \hat{y}^0)\} = \phi$. Set $i = 1$ and $X_0^i = X_0$.

(b) If $X_0^i = \phi$, terminate with obj_{i-1} as the global minimum and $(\hat{x}^{i-1}, \hat{y}^{i-1})$ as its corresponding global minimizer.

(c) Find a PGM $(\bar{x}^i, \bar{y}^i)$ by using Algorithm 1, and set the corresponding

$$obj_i = \min\{obj_{i-1}, f(\bar{x}^i, \bar{y}^i)\}, (\hat{x}^i, \hat{y}^i) = \arg\min\{obj_{i-1}, f(\bar{x}^i, \bar{y}^i)\}.$$

(d) Compute $\bar{\lambda}_j$s by solving LP programs if $\xi^j \not\subset Y_0(\alpha)$, and set $\bar{\lambda}_j$s to ∞ if $\xi^j \subset Y_0(\alpha)$. Generate a polar cut and define $X_0^{i+1} = X_0^i \cap H^+(\bar{x}^i)$.

(e) If there exists no $\bar{\lambda}_j$ such that $\xi^j \not\subset Y_0(\alpha)$, terminate with obj_i as the global minimum and $(\hat{x}^i, \hat{y}^i)$ as its corresponding global minimizer.

(f) Set $i \leftarrow i + 1$, and return to (b).

According to the stopping rules, Algorithm 2 yields an exact global minimum for DBLP.

Convergence Proof: First, note that Algorithm 1 is finite so step (c) in Algorithm 2 yields exact solutions. Consider the sequence of PGMs $\{(\bar{x}^i, \bar{y}^i)\}$ generated and let $H(\bar{x}^i)$ be the cutting plane that eliminates $\bar{x}^i$. In step (d) of i^{th} iteration, the algorithm is terminated as the consequence of introducing polar cuts, and the algorithm is terminated if $X_0^i \cap H^+(\bar{x}^i) = \phi$. Otherwise, the cut $H(\bar{x}^i)$ is applied and a new PGM $(\bar{x}^{i+1}, \bar{y}^{i+1})$ is found where $\bar{x}^{i+1} \in X_0^i \cap H^+(\bar{x}^i)$ and $\bar{x}^i \notin H^+(\bar{x}^i)$.

It is possible for the process not to terminate by any of the stopping rules in Algorithm 2. An infinite sequence would then be generated, and we need to show that the sequence $\{\bar{x}^i\}$ has a limit point x^* such that $\lim_{i\to\infty} X_0^i \cap H^+(\bar{x}^i) = \phi$.

Since X_0 is a compact set, there exists a limit point x^* such that for a given $\varepsilon \geq 0$ and a positive integer ν, $\left\|\bar{x}^i - x^*\right\| \leq \varepsilon$ for infinitely many $i \geq \nu$. If $X_0^i \cap H^+(\bar{x}^i) \neq \phi$ for all $i \geq \nu$, then all subsequent PGMs $(\bar{x}^l, \bar{y}^l)$ generated will satisfy the condition $\bar{x}^l \in H^+(\bar{x}^\nu)$ for all $l > \nu$. From the definition of a PGM, $x^* \in B_\delta(x^*) \cap X_0$ and $\bar{x}^l \notin B_\delta(x^*)$ for some $\delta > 0$. Hence, $\left\|\bar{x}^l - x^*\right\| \geq \delta$ for all $l > \nu$. This contradicts the statement that x^* is a limit point. Therefore, $\lim_{i\to\infty} X_0^i \cap H^+(\bar{x}^i) = \phi$ and the cutting plane algorithm is terminated.

3 Disjoint Trilinear Programming

3.1 A Probabilistic Multi-Criteria Decision Model

The imprecise bilinear decision model handles a decision situation with respect to only one criterion. However, in many decision situations it is necessary to study a decision problem from more than one perspective using multiple criteria. There are several multi-criteria models, ranging from value function based methods over goal and reference point methods to outranking methods. For this PMEU-based model, an additive multi-attribute utility theory (MAUT) approach is employed, in which the expected utilities of each alternative are weighted together according to importance weights.

Consider n information frames under n criteria and suppose that the imprecise statements concerning importance weights are collected in a set W with w_k-variables.

Definition 6: An extended information frame represents a decision situation with m alternatives. Each alternative is assessed under n criteria $k \in 1,...,n$, and each criterion has an information frame $I_k = \left\langle \{\{c_{kij}\}_{m_{ki}}\}_{m_k}, P_k, U_k \right\rangle$, as a structure $\left\langle W, \{I_k\} \right\rangle$ where W is a finite list of linear constraints over w_k-variables denoting weights over the criteria.

Since the additive model is PMEU-based, evaluation rules are based on an additive extension of the expected utility.

Definition 7: Given an extended information frame $\left\langle W, \{I_k\} \right\rangle, k \in 1,...,n$, the weighted expected utility of an alternative A_i is $G(A_i) = \sum_{k=1}^{n} w_k (\sum_{j=1}^{m_i} p_{kij} u_{kij})$, where p_{kij} and u_{kij} are variables in P_k and U_k, respectively; p_{kij} denotes the probability of c_{kij} occurring given that A_i is taken; and u_{kij} denotes the corresponding utility valued under criterion k.

When imprecise information exists in weights, probabilities, and utilities, the evaluation of weighted expected utility in an extended information frame results in DTLP that is mathematically stated as:

$$
\begin{aligned}
&\min f(x, y, z) = (c_x^t x + d_x)(c_y^t y + d_y)(c_z^t z + d_z) \\
&s.t. X_0 = \{x \in R^{n_1} : A_1 x \leq b_1, x \geq 0\} \\
&\quad Y_0 = \{y \in R^{n_2} : A_2 y \leq b_2, y \geq 0\} \\
&\quad Z_0 = \{z \in R^{n_3} : A_3 z \leq b_3, z \geq 0\}.
\end{aligned}
\tag{4}
$$

In (4), $A_1 x \leq b_1, A_2 y \leq b_2$ and $A_3 z \leq b_3$ represent the linear constraints populated in P, U and W, respectively. To receive a range of the weighted expected utilities for each alternative under n criteria, we need to calculate both minimal and maximal values. When the received ranges for two alternatives overlap, a decision-maker is suggested to provide more information for the ranges to become separated, and thus indicate a preference order.

3.2 Evaluation of DTLP

3.2.1 Local Optimization

For DTLP, because of the disjoint constraint sets concerning P, U and W, respectively, the solution property of DBLP can be extended.

Theorem 1: If X_0, Y_0 and Z_0 are nonempty and bounded, then DTLP has an optimal solution $(x^*, y^*, z^*,)$ in which x^*, y^* and z^* are basic feasible solutions of X_0, Y_0 and Z_0, respectively.

Proof: Let $\bar{x}, \bar{y}, \bar{z}$ be an optimal solution. First, consider the LP problem concerning x, $\min\{f(x, \bar{y}, \bar{z}) \mid x \in X_0\}$, and let x^* be its optimal basic solution. Then we have $f(x^*, \bar{y}, \bar{z}) \le f(\bar{x}, \bar{y}, \bar{z})$. Next, consider the LP problem concerning y, $\min\{f(x^*, y, \bar{z}) \mid y \in Y_0\}$, and let y^* be its optimal basic solution. By the same argument, we have $f(x^*, y^*, \bar{z}) \le f(x^*, \bar{y}, \bar{z})$. Finally, consider the LP problem concerning z, $\min\{f(x^*, y^*, z) \mid z \in Z_0\}$, and let z^* be its optimal basic solution. Then we have $f(x^*, y^*, z^*) \le f(x^*, y^*, \bar{z})$. Therefore, x^*, y^* and z^* are basic feasible solutions of X_0, Y_0 and Z_0, respectively.

Based on this solution property and the knowledge in DBLP, we can have the following local optimization algorithm to locate a PGM of DTLP.

Algorithm 3:

(a) Find feasible extreme points $\tilde{x}^1 \in X_0^i$ and $\tilde{y}^1 \in Y_0$.

(b) (1)Solve: $\min\{f(\tilde{x}^1, \tilde{y}^1, z) \mid z \in Z_0\}$, to yield an optimal $\tilde{z}^1$;

(2) Solve: $\min\{f(x, \tilde{y}^1, \tilde{z}^1) \mid x \in X_0^i\}$, to yield an optimal $\tilde{x}^2$;

(3)Solve: $\min\{f(\tilde{x}^2, y, \tilde{z}^1) \mid y \in Y_0\}$, to yield an optimal $\tilde{y}^2$;

Set $\tilde{x}^1 \leftarrow \tilde{x}^2, \tilde{y}^1 \leftarrow \tilde{y}^2$ and repeat (b) until it converges to an LSM $(\bar{x}, \bar{y}, \bar{z})$.

(c) Suppose $\bar{x}$ is non-degenerate and let $\hat{x} \in N(\bar{x})$ be such that

$$f(\hat{x}, \hat{y}, \bar{z}) = \min_{y \in Y_0} f(\hat{x}, y, \bar{z}) < f(\bar{x}, \bar{y}, \bar{z}).$$

Go to (b(2)) with $\tilde{y}^1 \leftarrow \hat{y}, \tilde{z}^1 \leftarrow \bar{z}$.

(d) Suppose $\bar{x}$ is non-degenerate and let $\hat{x} \in N(\bar{x})$ be such that

$$f(\hat{x}, \bar{y}, \hat{z}) = \min_{z \in Z_0} f(\hat{x}, \bar{y}, z) < f(\bar{x}, \bar{y}, \bar{z}).$$

Go to (b(2)) with $\tilde{y}^1 \leftarrow \bar{y}, \tilde{z}^1 \leftarrow \hat{z}$.

(e) Terminate with $(\bar{x}, \bar{y}, \bar{z})$ as a PGM.

3.2.2 Global Optimization

Given a PGM $(\bar{x}, \bar{y}, \bar{z})$ of DTLP, we need to develop a generalized cutting plane method to obtain the global optimum. As a result of the independency between X_0, Y_0 and Z_0, we can develop a generalized cutting plane method solving DTLP.

Algorithm 4:

(a) Let CBOV, $obj_0 = +\infty$, let the initial best feasible solution $\{(\hat{x}^0, \hat{y}^0, \hat{z}^0)\} = \phi$. Set $i = 1$ and $X_0^i = X_0$.

(b) If $X_0^i = \phi$, terminate with obj_{i-1} as the global minimum and the solution $(\hat{x}^{i-1}, \hat{y}^{i-1}, \hat{z}^{i-1})$ as its corresponding global minimizer.

(c) Find a PGM in X_0^i by using Algorithm 3, and set the corresponding

$$obj_i = \min\{obj_{i-1}, f(\bar{x}^i, \bar{y}^i, \bar{z}^i)\}, (\hat{x}^i, \hat{y}^i, \hat{z}^i) = \arg\min\{obj_{i-1}, f(\bar{x}^i, \bar{y}^i, \bar{z}^i)\}.$$

(d) Compute $\bar{\lambda}_{jY}$ concerning Y_0, and compute $\bar{\lambda}_{jZ}$ concerning Z_0 for all $j \in \bar{N}$.

(e) If either there exists no $\bar{\lambda}_{jY}$ such that $\xi^j \not\subset Y_0(\alpha)$ or $\bar{\lambda}_{jZ}$ such that $\xi^j \not\subset Z_0(\alpha)$, terminate with obj_i as the global minimum and $(\hat{x}^i, \hat{y}^i, \hat{z}^i)$ as its corresponding global minimizer.

(f) Let $\bar{\lambda}_j = \min\{\bar{\lambda}_{jY}, \bar{\lambda}_{jZ}\}$ for all $j \in \bar{N}$, generate a polar cut, and let $X_0^{i+1} = X_0^i \cap H^+(\bar{x}^i)$.

(g) Set $i = i + 1$, and return to (b).

The fundamental idea in Algorithm 4 is that at a local minimizer in X_0^i, we compute $\bar{\lambda}_{jY}$ and $\bar{\lambda}_{jZ}$ concerning Y_0 and Z_0, respectively, and choose the smaller one as $\bar{\lambda}^j$ for each edge. This will guarantee the global optimality as what we have done in DBLP. Convergence of Algorithm 4 can be proved analogously to the previous work.

Proposition: Given an extended information frame, $\max G(A_i)$ and $\min G(A_i)$ can be computed by DTLP optimization. This way, weighted expected utility ranges can be obtained for each alternative.

This follows from the observation that the problem structure of evaluating PMEU with respect to an extended information frame coincides with the formulation of (4).

4 Disjoint Multi-Linear Programming

4.1 A Multilevel Decision Model

With appropriate modifications to an information frame, a well defined decision structure as shown in Figure 2 can be obtained, in which we have a sequence of chance nodes representing uncertainty. Nevertheless, P is now broken into several disjoint parts, e.g., P_1, P_2 and P_3, with respect to each level.

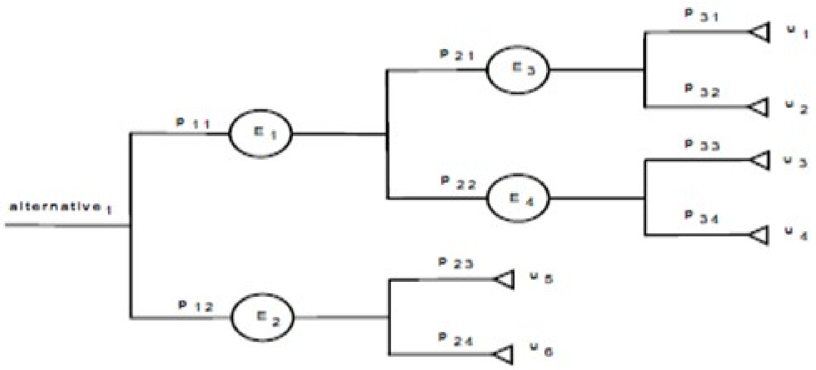

Figure 2: A multi-linear decision model

To calculate the expected utility, we have

$$\begin{aligned} E(A_1) &= p_{11}(p_{21}(p_{31}u_1 + p_{32}u_2) + p_{22}(p_{33}u_3 + p_{34}u_4)) + p_{12}(p_{23}u_5 + p_{24}u_6)) \\ &= p_{11}p_{21}p_{31}u_1 + p_{11}p_{21}p_{32}u_2 + p_{11}p_{22}p_{33}u_3 + p_{11}p_{22}p_{34}u_4 + p_{12}p_{23}u_5 + p_{12}p_{24}u_6. \end{aligned} \quad (5)$$

As imprecise information prevails, we translate it into linear constraints within each level. It should be noted that currently we only allow constraints from the same level rather than different levels. For example, the interval statement $0.2 \le p_{21} + p_{23} \le 0.3$ is considered proper, while the interval statement $0.2 \le p_{11} + p_{31} \le 0.3$ is considered improper since p_{11} and p_{31} are from level 1 and level 2, respectively, and this will destroy the disjoint structural property. In other words, imprecise statements are supposed to be confined within each level.

To evaluate the alternative by using PMEU with imprecise information, we have to compute two extreme values, i.e., maximum and minimum, of (5) subject to disjoint linear constraints in order to receive a range of expected utility for the alternative. This will result in a special case of DMLP that is intrinsically hard to solve.

If we take $x_i = p_i = (p_{i1},\dots,p_{in_i})^t, i = 1,\dots,n$, and $x_{n+1} = (u_1,\dots,u_{n_u})^t$, then the imprecise multilevel decision model (5) can be transformed into the following DMLP model as

$$\begin{aligned} &\min\ f(x_1,\dots,x_{n+1}) = \sum_{t=1}^{T} \prod_{j \in J_t} x_j \\ &s.t. X_i = \{x_i \in R^{n_i} : A_i x_i \le b_i, x_i \ge 0\}, i = 1,\dots,n+1. \end{aligned} \quad (6)$$

In (6), J_t denotes the index set, and we can have at most one decision variable from x_i for each J_t. Taking (5) as an example, there exists at most one decision variable from each level within each product term. This property of the objective function and the disjoint linear constraint sets, X_i s, demonstrate the disjoint property of DMLP, and therefore, its solution property is similar to that of DTLP.

4.2 Evaluation of DMLP

4.2.1 Local Optimization

Based on the solution property of DMLP and the knowledge in DBLP and DTLP, we can have the following local optimization algorithm to locate a PGM of DMLP, in which X_{n+1}^k denotes the feasible region of X_{n+1} in k^{th} iteration; and $N(\bar{x}_{n+1})$ denotes the set of extreme points in X_{n+1}^k that are adjacent to $\bar{x}_{n+1}$.

Algorithm 5:

(a) Find feasible extreme points $\tilde{x}_i^1 \in X_i, i = 1,...,n$.

(b) (1) Solve: $\min\{f(\tilde{x}_1^1,...,\tilde{x}_n^1, x_{n+1}) \mid x_{n+1} \in X_{n+1}^k\}$ to yield $\tilde{x}_{n+1}^1$.

(2) Solve: $\min\{f(x_1, \tilde{x}_2^1,...,\tilde{x}_{n+1}^1,) \mid x_1 \in X_1\}$ to yield $\tilde{x}_1^2$.

(3) Solve: $\min\{f(\tilde{x}_1^2, x_2, \tilde{x}_3^1,...,\tilde{x}_{n+1}^1,) \mid x_2 \in X_2\}$ to yield $\tilde{x}_2^2$.

$\vdots$

(n+1) Solve: $\min\{f(\tilde{x}_1^2,...,\tilde{x}_{n-1}^2, x_n, \tilde{x}_{n+1}^1,) \mid x_n \in X_n\}$ to yield $\tilde{x}_n^2$.

Set $\tilde{x}_i^1 \leftarrow \tilde{x}_i^2, i = 1,...,n$, and repeat (b) until the solution converges to an LSM $(\bar{x}_1,...,\bar{x}_{n+1})$.

(c) Suppose $\bar{x}_{n+1}$ is non-degenerate, and for each $x_i, i = 1,...,n$, let $\hat{x}_{n+1} \in N(\bar{x}_{n+1})$ be such that

$$f(\bar{x}_1,...,\bar{x}_{i-1}, \hat{x}_i, \bar{x}_{i+1},...,\bar{x}_n, \hat{x}_{n+1}) = \min_{x \in X_i} f(\bar{x}_1,...,\bar{x}_{i-1}, x_i, \bar{x}_{i+1},...,\bar{x}_n, \hat{x}_{n+1}) < f(\bar{x}_1,...,\bar{x}_{n+1}).$$

Go to (b(1)) with $\tilde{x}_i^1 \leftarrow \bar{x}_i, i = 1,...,i-1, i+1,...,n$ and $\tilde{x}_i^1 \leftarrow \hat{x}_i$.

(d) Terminate with $(\bar{x}_1,...,\bar{x}_{n+1})$ as a PGM.

4.2.2 Global Optimization

Now we can develop the generalized cutting plane method for solving DMLP by taking advantage of its disjoint property. As before, for the generalized reverse polar, if $\xi^j \subset X_i(\alpha), i = 1,...,n, j \in \bar{N}$, we simply set $\bar{\lambda}_j = \infty$ other than employing the negative extension of polar cuts.

Algorithm 6:

(a) Let CBOV, $obj_0 = +\infty$, let the initial best feasible solution $\{(\hat{x}_1^0,...,\hat{x}_{n+1}^0)\} = \phi$, and set $k = 1$.

(b) If $X_{n+1}^k = \phi$, terminate with obj_{k-1} as the global minimum and $(\hat{x}_1^{k-1},...,\hat{x}_{n+1}^{k-1})$ as its corresponding global solution.

(c) Find a PGM in X_{n+1}^k by using Algorithm 5, and set the corresponding

$$obj_k = \min\{obj_{k-1}, f(\bar{x}_1^k,...,\bar{x}_{n+1}^k)\}, (\hat{x}_1^k,...,\hat{x}_{n+1}^k) = \arg\min\{obj_{k-1}, f(\bar{x}_1^k,...,\bar{x}_{n+1}^k)\}.$$

(d) Compute $\bar{\lambda}_{jX_i}$ with respect to X_i for all $j \in \bar{N}, i = 1,...,n$.

(e) If there exists no $\bar{\lambda}_{jX_i}$ such that $\xi^j \not\subset X_i(obj_k), i = 1,...,n$, terminate with obj_k as the global minimum and $(\hat{x}_1^k,...,\hat{x}_{n+1}^k)$ as its corresponding global minimizer.

(f) Let $\bar{\lambda}_j = \min\{\bar{\lambda}_{jX_i}\}$ for all $j \in \bar{N}, i = 1,...,n$, generate a polar cut, and let $X_{n+1}^{k+1} = X_{n+1}^k \cap H^+(\bar{x}_{n+1}^k)$.

(g) Set $k = k+1$, and return to (b).

The basic idea in Algorithm 6 is similar to those explained in Algorithm 4 and its convergence can be proved analogously to the previous work.

5 Conclusions and Further Research

In this paper, we have discussed three types of imprecise decision models arising in computational decision analysis, the corresponding solution property of DBLP, DTLP and DMLP, and their global optimization algorithms. Being based on only LP operations, they do not contain traditional nonlinear computational elements. Even though we have gained some computational experience in Algorithm 2; see [3, 4, 10, 12], Algorithm 4 and Algorithm 6 still need further investigations and modifications with respect to their computational performance benchmark against the general-purpose designed branch and bound methods [14, 15].

As for the optimization algorithms themselves, there are several aspects for further development and improvement.

Firstly, it is possible to incorporate certain lower bounding techniques in branch and bound procedures to fathom the rest of the feasible region and achieve faster convergence [10]. For example, we are able to obtain an interval such as $x^L \le x \le x^U$ for each variable. Then using the arithmetic intervals [1, 2, 14], the convex envelope of a trilinear term xyz subject to box constraints can be calculated as

$$xyz \ge \max \left\{ \begin{array}{c} xyz^L + xy^L z + x^L yz \\ -xy^L z^L - x^L yz^L - x^L y^L z + x^L y^L z^L \\ xyz^L + xy^U z + x^U yz \\ -xy^U z^L - x^U yz^L - x^U y^U z + x^U y^U z^L \\ xyz^U + xy^L z + x^U yz \\ -xy^L z^U - x^U yz^U - x^U y^L z + x^U y^L z^U \\ xyz^U + xy^U z + x^L yz \\ -xy^U z^U - x^L yz^U - x^L y^U z + x^L y^U z^U \end{array} \right\}.$$

Each bilinear term can be further lower bounded by the maximum of two linear constraints, and we need to compute the maximum of 32 linear constraints to obtain an underestimate of xyz. In that case, both algorithms may be further improved to achieve fast convergence.

Secondly, the generation of a cutting plane at a degenerate extreme point as in Algorithms 2, 4 and 6 can be further investigated since it is a common situation in computational decision analysis [5, 6].

Finally, we may incorporate the negative extension of polar cuts when $\xi^j \subset Y_0(\alpha)$ in order to generate more efficient cutting planes, and focus on the extreme points belonging to the original feasible set that are candidates for the global solution rather than those induced by the generated cutting planes [19].

6 Acknowledgements

This work is partially supported by National Natural Science Foundation of China under Grant #70771013 and Project 211.

References

[1] Al-Khayyal, F.A., Jointly constrained bilinear programs and related problems: An overview, *Computers & Mathematics with Application*, vol.19, no.11, pp.53–62, 1990.

[2] Al-Khayyal, F.A., and J.E. Falk, Jointly constrained biconvex programming, *Mathematics of Operations Research*, vol.8, pp.273–286, 1983.

[3] Alarie, S., C. Audet, B. Jaumard, and G. Savard, Concavity cuts for disjoint bilinear programming, *Mathematical Programming*, vol.90, no.2, pp.373–398, 2001.

[4] Audet, C., P. Hansen, B. Jaumard, and G. Savard, A symmetrical linear maxmin approach to disjoint bilinear programming, *Mathematical Programming*, vol.85, no.3, pp.573–592, 1999.

[5] Balas, E., Intersection cuts – A new type of cutting planes for integer programming, *Operations Research*, vol.19, pp.19–39, 1971.

[6] Carvajal-Moreno, R., Minimization of concave functions subject to linear constraints, Operations Research Center Report, University of Berkeley, Calfornia, 1972.

[7] Danielson, M., Generalized evaluation in decision analysis, *European Journal of Operational Research*, vol.162, no.2, pp.442–449, 2005.

[8] Danielson, M., and L. Ekenberg, A framework for analysing decisions under risk, *European Journal of Operational Research*, vol.104, no.3, pp.474–484, 1998.

[9] Danielson, M., and L. Ekenberg, Computing upper and lower bounds in interval decision trees, *European Journal of Operational Research*, vol.181, no.2, pp.808–816, 2007.

[10] Ding, X., and F.A. Al-Khayyal, Accelerating convergence of cutting plane algorithms for disjoint bilinear programming, *Journal of Global Optimization*, vol.38, no.3, pp.421–436, 2007.

[11] Gallo, G., and A. ülkücü, Bilinear programming: An exact algorithm, *Mathematical Programming*, vol.12, pp.173–194, 1977.

[12] Konno, H., A cutting plane algorithm for solving bilinear programs, *Mathematical Programming*, vol.11, pp.14–27, 1976.

[13] Rios Insua, D., and S. French, A framework for sensitivity analysis in discrete multi-objective decision making, *European Journal of Operational Research*, vol.54, pp.176–190, 1991.

[14] Ryoo, H.S., and N.V. Sahinidis, Analysis of bounds for multilinear functions, *Journal of Global Optimization*, vol.19. 4, pp.403–424, 2001.

[15] Ryoo, H.S., and N.V. Sahinidis, Global optimization of multiplicative programs, *Journal of Global Optimization*, vol.26. 4, pp.387–418, 2003.

[16] Sage, A.P., and C.C. White, ARIADNE: A knowledge-based interactive system for planning and decision support, *IEEE Transactions on Systems, Man and Cybernetics*, vol.14. 1, pp.35–47, 1984.

[17] Salo, A., and R.P. Hamalainen, Preference programming through approximate ratio comparisons, *European Journal of Operational Research*, vol.82, no.3, pp.458–475, 1995.

[18] Sherali H.D., and A. Alameddine, A new reformulation linearization algorithm for bilinear programming problems, *Journal of Global Optimization*, vol.2, pp.379–410, 1992.

[19] Sherali, H.D., and C.M. Shetty, A finitely convergent algorithm for bilinear programming problems using polar cuts and disjunctive face cuts, *Mathematical Programming*, vol.19, pp.14–31, 1980.

[20] Tuy, H., Concave programming under linear constraints, *Soviet Math Dokl.*, vol.5, pp.1437–1440, 1964.

[21] Vaish, H., and C.M. Shetty, A cutting plane algorithm for the bilinear programming problem, *Naval Research Logistics Quarterly*, vol.24, pp.83–94, 1977.

Theorem 8 *Let* $(\Omega,\ d)$ *be a metric space and let* $\boldsymbol{B}$ *be a partition of* Ω*. For every* $B \in \boldsymbol{B}$ *with positive and finite Hausdorff outer measure in its dimension denote by* $\mu = P(A|B)$ *the restriction to the Borel* σ*-field of the upper conditional probability defined as in Theorem 2. Let* $L^*(B)$ *be the class of all Borel measurable random variables on* B*. Then the convergence in* μ*-distribution of a sequence of random variables of* $L^*(B)$ *to a random variable* X *is equivalent to the pointwise convergence of expectation functionals on all bounded and continuous function* f *that is* $lim_{n\to\infty} \int f d\mu_n = \int f d\mu$.

Proof: If X_n and X are Borel-measurable random variables and H is a Borelian set then the sets $X_n^{-1}(H)$ and $X^{-1}(H)$ are also Borelian sets; moreover since every Hausdorff s-dimensional outer measure is countably additive on the Borel σ-field then the (upper) conditional probabilities μ_n and μ induced respectively by X_n and X on $(\Re, \boldsymbol{F})$ are probability measures. Then convergence in μ-distribution is equivalent to the pointwise convergence of expectation functionals on all bounded and continuous function f.

6 Conclusions

This paper investigates the relations among different types of convergence for random variables when they are based on an upper probability approach where conditional upper expectations with respect to Hausdorff outer measures are used whenever we have to condition on a set with probability zero.

Upper (lower) conditional previsions defined with respect to Hausdorff outer measures are proven to be the upper (lower) envelopes of all linear extensions to the class of all random variables of the restriction to the Borel-measurable random variables of the given upper conditional previsions.

It is proven that the relations among different types of convergences of random variables defined with respect to upper conditional probability defined by Hausdorff outer measures are the same that hold if convergences are defined with respect to a probability measure. When the conditioning event has finite Hausdorff outer measure in its dimension these results are obtained because Hausdorff outer measures are Borel regular outer measures and so continuous from below and continuous from above on the Borel σ-field. In general if upper conditional probability is defined as natural extension of a coherent merely finitely additive probability defined on a σ-field we have that μ-stochastically convergence does not imply convergence in μ-distribution since in this case the upper conditional probability is not continuous from above.

References

[1] Billingsley, P., *Probability and Measure*, Wiley, 1986.

[2] Couso, I., S. Montes, and P. Gil, Stochastic convergence, uniform integrability and convergence in mean on fuzzy measure spases, *Fuzzy Sets and Systems*, vol.129, pp.95–104, 2002.

[3] de Cooman, G., M.C.M. Troffaes, and E. Miranda, n-Monotone lower previsions and lower integrals, *Proceedings of the Fourth International Symposium on Imprecise Probability: Theories and Applications*, pp.145–154, 2005.

[4] Doria, S., Probabilistic independence with respect to upper and lower conditional probabilities assigned by Hausdorff outer and inner measures, *International Journal of Approximate Reasoning*, vol.46, pp.617–635, 2007.

[5] Denneberg, D., *Non-additive Measure and Integral*, Kluwer Academic Publishers, 1994.

[6] Falconer, K.J., *The Geometry of Fractals Sets*, Cambridge University Press, 1986.

[7] Rogers, C.A., *Hausdorff Measures*, Cambridge University Press, 1998.

[8] Seidenfeld, T., M. Schervish, and J.B. Kadane, Improper regular conditional distributions, *The Annals of Probability*, vol.29, no.4, pp.1612–1624, 2001.

[9] Walley, P., *Statistical Reasoning with Imprecise Probabilities*, Chapman and Hall, London, 1991.