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A Numerical Scheme for Fuzzy Cauchy Problems

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Abstract

In this paper, we use power series method to solve fuzzy Cauchy differential equations of first order. Theoretical consideration is discussed and some examples are presented to show the ability of the method for fuzzy Cauchy differential equations. We use *Matlab* for numerical calculations.

Keywords: fuzzy Cauchy differential equations, power series, numerical method

1 Introduction

Fuzzy differential equations (FDEs) have been applied extensively in recent years to model uncertainty in mathematical models. First-order ordinary differential equations, in particular Fuzzy Cauchy differential equations, are one of the simplest FDEs which may appear in many applications. Finding a solution to FDE which satisfies necessary existence and uniqueness conditions is our main goal in this paper. Since it is too complicated to find an exact solution, numerical methods are more suitable for mentioned problems.

The concept of a fuzzy derivative was first introduced by Chang and Zadeh [7] it was followed by Dubois and Prade [8] , who defined and used the extension principle. Kandel and Byatt [10, 15, 16] applied the concept of fuzzy differential equation (FDE) to the analysis of fuzzy dynamical problems. Puri and Ralescu in [20] defined the Concept of H-differentiabillity and Seikkala generalized it in [21]. Kaleva [12, 14], Seikkala [21], He and Yi [11], Kloeden [17], Menda [19] and finally, Friedman, Ma and Kandel [9], concentrated on fuzzy Cauchy problems. Furthermore, some numerical methods for solving FDE are discussed in [18, 2, 1, 9].

In this paper, using power series expansion, a numerical procedure is presented. In section 2, first we briefly introduce preliminary topics such as fuzzy number, fuzzy function and fuzzy derivative, and then a fuzzy Cauchy problem is defined. In order to obtain numerical solutions, a method based upon Power series is explained in section 3. In section 4, the mentioned method has been applied to two examples, finally we compare our method with another one.

2 Preliminaries

2.1 Notations and Definitions

We consider the fuzzy sets with respect to a nonempty base set R^n . To each $x \in \mathrm{R}^n$, we assign a membership value $u(x)$ taking values in [0, 1]. Therefore for nonmembership, we have $u(x) = 0$, $0 < u(x) \leq 1$ corresponding to partial membership and $u(x) = 1$, to full membership.

Assumption 2.1 *In this paper, a fuzzy set $u \in R^n$ is a function $u : R^n \to [0, 1]$ which satisfies the following conditions:*
(i) u is normal; meaning there exists $x_0 \in R^n$ such that $u(x_0) = 1$;
(ii) u is fuzzy convex ; meaning for $x, y \in R^n, 0 \leq \lambda \leq 1$,

$$u(\lambda x + (1-\lambda) y) \geq \min(u(x), u(y));$$

(iii) u is upper semi-continuous;
(iv) $[u]^0 = cl\{x \in R^n; u(x) > 0\}$ is compact.

We denote the set of all subsets of R^n which satisfy conditions (i)-(iv), by E^n.

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Definition 2.2 *Let u be an arbitrary upper semicontinuous normal convex fuzzy number with bounded α-level interval. For $0 < \alpha \leq 1$, we denote α-level set by $[u]^\alpha$ and define it as:*

$$[u]^\alpha = \{x \in R^n; u(x) \geq \alpha\}.$$

Obviously this is a closed and bounded interval $\left[u^1(\alpha), u^2(\alpha)\right]$ where,

$$u^1(\alpha) = \min\{x|x \in [u]^\alpha\},\ u^2(\alpha) = \max\{x|x \in [u]^\alpha\} \tag{1}$$

Remark 2.3 *For any $u, v \in E^n$ and $k \in R$ addition and multiplication by k are defined as follows:*

$$\begin{aligned}[u+v]^\alpha &= [u]^\alpha + [v]^\alpha \\ [ku]^\alpha &= k\,[u]^\alpha\end{aligned}$$

here $0 \leq \alpha \leq 1$.

Definition 2.4 *A triangular fuzzy number is defined as a fuzzy set in E^n which is specified by an ordered triple $(u_1, u_2, u_3) \in R^3$ with $u_1 \leq u_2 \leq u_3$ such that $[u]^0 = [u_1, u_3]$ and $[u]^1 = \{u_2\}$ then for $0 \leq \alpha \leq 1$ we have*

$$[u]^\alpha = [u_2 - (1-\alpha)(u_2 - u_1), u_2 + (1-\alpha)(u_3 - u_2)].$$

Definition 2.5 *A mapping $f : I \to E^n$ for some interval I is called a fuzzy process. Therefore according to (1), its α-level set can be written as follows:*

$$[f(t)]^\alpha = [f_1(t,\alpha), f_2(t,\alpha)].$$

Definition 2.6 *For arbitrary $u, v \in E^n$ the quantity*

$$D(u,v) = \sup_{0 \leq \alpha \leq 1} d([u]^\alpha, [v]^\alpha)$$

is the distance between u and v , where d is the Hausdorff metric in E^n.

Remark 2.7 *It can be shown that (E^n, D) is a complete metric space. See [10].*

2.2 Derivatives

Prior to entering the problem, it is necessary to introduce two types of fuzzy derivatives which will be mainly applied in this paper.

Definition 2.8 *For $u, v \in E^n$, $w \in E^n$ is called the Hukuhara difference of u and v if $u = v + w$, and it is denoted by $w = u\underline{H}v$.*

Therefore using this difference, differentiability of a fuzzy function can be defined as:

Definition 2.9 *A function $F : [a,b] \to E^n$ is differentiable at $t_0 \in (a,b)$ if there exists $F'(t_0) \in E^n$ such that the limits*

$$\lim_{h \to 0^+} \frac{F(t_0+h)\ \underline{H}\ F(t_0)}{h} \quad and \quad \lim_{h \to 0^+} \frac{F(t_0)\ \underline{H}\ F(t_0-h)}{h}$$

exist and are equal to $F'(t_0)$.

The above definition is a straightforward generalization of the Hukuhara differentiability of a set-valued function. So if F is differentiable at $t_0 \in (a,b)$, then all its α-levels $F_\alpha(t) = [F(t)]^\alpha$ are Hukuhara differentiable at t_0 and $[F'(t_0)]^\alpha = DF_\alpha(t_0)$, where DF_α denotes the Hukuhara derivative of F_α, for more details see [12, 13].

Definition 2.10 *For any function $f : [a,b] \to E^n$, the Seikkala derivative (see [21, 6]) $SDf(t)$ is defined by*

$$[SDf(t)]^\alpha = [f_1'(t,\alpha), f_2'(t,\alpha)], \qquad 0 < \alpha \leq 1.$$

Remark 2.11 *For any $t \in [a,b]$, $[SDf(t)]^\alpha$ is a fuzzy number.*

Theorem 2.12 *If* $[f_1'(t,\alpha), f_2'(t,\alpha)]$ *are* α*-levels of a fuzzy function, then* $[SDf(t)]^\alpha$ *exists and* $[SDf(t)]^\alpha = [f_1'(t,\alpha), f_2'(t,\alpha)]$.

Proof. *See [6].*

Theorem 2.13 *If* $\frac{d}{dt}[f(t)]^\alpha$ *exists, then* $[SDf(t)]^\alpha = \frac{d}{dt}[f(t)]^\alpha$ *.*

Proof. *See [6].*

Remark 2.14 *Buckely and Feuring in [Theorem 4.2 [6]] claimed that the Seikkala solution is the most general solution to the fuzzy differential equation. Hence in this paper, we focus on Hukuhara and Seikkala derivatives.*

2.3 A Fuzzy Cauchy Problem

In this paper, we consider the first order fuzzy differential equation $y' = f(t,y)$ where y is a fuzzy function of t, $f(t,y)$ a fuzzy function of crisp variable t and fuzzy variable y , and y' is the fuzzy derivative of y . If an initial value $y(t_0) = y_0 \in E^n$ is given, we obtain a *fuzzy Cauchy problem* of first order,

$$y' = f(t, y(t)), \qquad y(t_0) = y_0 \ . \tag{2}$$

Sufficient conditions for existence of a unique solution to Eq.(2) are:

1. *continuity of* f,
2. *Lipschitz condition which declares*,

$$D(f(t,x), f(t,y)) \le LD(x,y), \quad \text{for some} \quad L > 0 \ .$$

Now, for $y(t)$ to be a solution of fuzzy Cauchy problem, we need that y' exists but also Eq.(2) must hold. To check Eq.(2), first we have to compute $f(t,y)$. $\alpha-$levels of $f(t,y)$ can be found as follows:

$$[f(t,y)]^\alpha = [f_1(t,\alpha), f_2(t,\alpha)]$$

with

$$f_1(t,\alpha) = \min\{f(t,y) \,|\, y \in [y(t)]^\alpha\}$$
$$f_2(t,\alpha) = \max\{f(t,y) \,|\, y \in [y(t)]^\alpha\}$$

for $t \in I$, $\alpha \in [0,1]$.

We say that y is a solution to Eq.(2), if y' exists and

$$y_1'(t,\alpha) = f_1(t,\alpha), \quad y_1(t_0,\alpha) = y_0^1(\alpha), \tag{3}$$

$$y_2'(t,\alpha) = f_2(t,\alpha), \quad y_2(t_0,\alpha) = y_0^2(\alpha), \tag{4}$$

where, $y(t_0,\alpha) = \left[y_0^1(\alpha), y_0^2(\alpha)\right]$.

3 Numerical Method

In this section, we assume that the solutions of Eqs.(3)-(4) can be written as follows,

$$y_i(t,\alpha) = y_i(t_0,\alpha) + e_i^\alpha t, \qquad i = 1,2. \tag{5}$$

here, e_i^α, $i = 1,2$ are scalar functions. Substituting (5) into Eqs.(3)-(4) and neglecting higher order terms, we have the linear equation of $e^\alpha = (e_1^\alpha, e_1^\alpha)$ of the form,

$$Ae^\alpha = B, \tag{6}$$

where, A and B are constant matrices.

Solving Eq.(6), the coefficients of t in (5) can be determined. By repeating the mentioned procedure for higher order terms, power series of the solutions of Eqs.(3)-(4), can be obtained in any arbitrary order.

In other words and more details, we define another type of power series for $f_i(t,\alpha),\ i=1,2$, of the form,

$$f_i(t,\alpha) = f_{i,0}^{\alpha} + f_{i,1}^{\alpha}t + f_{i,2}^{\alpha}t^2 + \cdots + (f_{i,n}^{\alpha} + p_{i,1}e_1^{\alpha} + p_{i,2}e_2^{\alpha})t^n, \tag{7}$$

where, $p_{i,j}, i,j = 1,2$ are constants, e_1^{α} and e_2^{α} are basis of vector e^{α}.

Let y^{α} be a vector with two elements in (5), every element of y^{α} can be represented by the power series in (7),

$$y_i(t,\alpha) = y_{i,0}^{\alpha} + y_{i,1}^{\alpha}t + y_{i,2}^{\alpha}t^2 + \cdots + e_i^{\alpha}t^n, \quad i = 1,2. \tag{8}$$

Substituting (8) into Eq.(2) or equivalently Eqs.(3)-(4), we can get the following,

$$f_i^{\alpha} = \left(f_{i,n}^{\alpha} + p_{i,1}e_1^{\alpha} + p_{i,2}e_2^{\alpha}\right)t^{n-1} + Q(t^n), \quad i = 1,2. \tag{9}$$

From (9) and Eq.(6), we can determine the linear equation in Eq.(6) as follows

$$A_{i,j} = p_{i,j}, \tag{10}$$

$$B_i = -f_{i,n}. \tag{11}$$

After solving this linear equation, e_1^{α} and e_2^{α} are at hand. By substituting e_i^{α} into (8), we obtain y_i^{α} $(i = 1,2)$ which are polynomials of degree n. Repeating this procedure from (8) to (11), we can get the power series of the solution in any arbitrary order for a fuzzy Cauchy problem in Eq.(2).

4 Numerical Examples

In this section, we solve two fuzzy Cauchy Problems in [3, 4, 5],and we compare the obtained results with the results in them.

Example 4.1 *Consider the fuzzy Cauchy problem [3, 4, 5],*

$$y'(t,\alpha) = -y(t,\alpha) + t + 1,$$

$$y(0,\alpha) = (0.96 + 0.04\alpha, 1.01 - 0.01\alpha).$$

The exact Seikkala solution is ([5] Example 1, section 4)

$$Y_1(t,\alpha) = (t + (0.985 + 0.015\alpha)\exp(-t) + (-0.025 + 0.025\alpha)\exp(t)),$$

$$Y_2(t,\alpha) = (t + (0.985 + 0.015\alpha)\exp(-t) + (0.025 - 0.025\alpha)\exp(t))$$

for $0 \le \alpha \le 1$.

We put $y_1(0,\alpha) = 0.96 + 0.04\alpha$ and $y_2(0,\alpha) = 1.01 - 0.01\alpha$. Hence, the following problems must be solved (for more details see[5]),

$$\begin{aligned} y_1'(t,\alpha) &= -y_2(t,\alpha) + t + 1, \\ y_2'(t,\alpha) &= -y_1(t,\alpha) + t + 1, \end{aligned} \tag{12}$$

with initial conditions, respectively

$$\begin{aligned} y_1(0,\alpha) &= 0.96 + 0.04\alpha, \\ y_2(0,\alpha) &= 1.01 - 0.01\alpha. \end{aligned}$$

From boundary condition, the solution of (12) can be supposed as

$$\begin{aligned} y_1(t,\alpha) = y_1(0,\alpha) + e_1^{\alpha}t \ &\Rightarrow\ y_1(t,\alpha) = 0.96 + 0.04\alpha + e_1^{\alpha}t, \\ y_2(t,\alpha) = y_2(0,\alpha) + e_2^{\alpha}t \ &\Rightarrow\ y_2(t,\alpha) = 1.01 - 0.01\alpha + e_2^{\alpha}t. \end{aligned} \tag{13}$$

Substituting (13) into (12) and neglecting higher order terms, we have,

$$\begin{aligned} e_1^\alpha - (-0.01 + 0.01\alpha) + Q(t) = 0, \\ e_2^\alpha - (0.04 - 0.04\alpha) + Q(t) = 0. \end{aligned} \tag{14}$$

These formulas corresponding to (9). The linear equation which corresponds to (6) can be given in the following,

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} e_1^\alpha \\ e_2^\alpha \end{bmatrix} = \begin{bmatrix} -0.01 + 0.01\alpha \\ 0.04 - 0.04\alpha \end{bmatrix}$$

where

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, B = \begin{bmatrix} -0.01 + 0.01\alpha \\ 0.04 - 0.04\alpha \end{bmatrix}, e^\alpha = \begin{bmatrix} e_1^\alpha \\ e_2^\alpha \end{bmatrix}.$$

Solving this equation, we get $e^\alpha = \begin{bmatrix} -0.01 + 0.01\alpha \\ 0.04 - 0.04\alpha \end{bmatrix}$ and

$$\begin{aligned} y_1(t,\alpha) = (0.96 + 0.04\alpha) + (-0.01 + 0.01\alpha)t, \\ y_2(t,\alpha) = (1.01 - 0.01\alpha) + (0.04 - 0.04\alpha)t. \end{aligned} \tag{15}$$

According to (15), the solution of (12) can be supposed as,

$$\begin{aligned} y_1(t,\alpha) = (0.96 + 0.04\alpha) + (-0.01 + 0.01\alpha)t + e_1^\alpha t^2, \\ y_2(t,\alpha) = (1.01 - 0.01\alpha) + (0.04 - 0.04\alpha)t + e_2^\alpha t^2. \end{aligned} \tag{16}$$

In this manner, by substituting (16) into (12) and neglecting higher order terms, we have

$$\begin{aligned} (2e_1^\alpha - 0.96 - 0.04\alpha)t + Q(t^2) = 0, \\ (2e_2^\alpha - 1.01 + 0.01\alpha)t + Q(t^2) = 0, \end{aligned} \tag{17}$$

here,

$$A = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}, B = \begin{bmatrix} 0.96 + 0.04\alpha \\ 1.01 - 0.01\alpha \end{bmatrix}, e^\alpha = \begin{bmatrix} e_1^\alpha \\ e_2^\alpha \end{bmatrix},$$

and (17) can be written in the matrix form

$$\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} e_1^\alpha \\ e_2^\alpha \end{bmatrix} = \begin{bmatrix} 0.96 + 0.04\alpha \\ 1.01 - 0.01\alpha \end{bmatrix}.$$

By solving the linear equation, we obtain

$$e^\alpha = \begin{bmatrix} 0.48 + 0.02\alpha \\ 0.505 - 0.005\alpha \end{bmatrix}.$$

Therefore,

$$\begin{aligned} y_1(t,\alpha) = (0.96 + 0.04\alpha) + (-0.01 + 0.01\alpha)t + (0.48 + 0.02\alpha)t^2, \\ y_2(t,\alpha) = (1.01 - 0.01\alpha) + (0.04 - 0.04\alpha)t + (0.505 - 0.005\alpha)t^2. \end{aligned} \tag{18}$$

Repeating the above procedure, we have

$$\begin{aligned} y_1(t,\alpha) &= (0.96 + 0.04\alpha) + (-0.01 + 0.01\alpha)t + (0.48 + 0.02\alpha)t^2 \\ &\quad + (-0.1683 + 0.0017\alpha)t^3 + (0.04 + 0.0017\alpha)t^4 + \cdots , \\ y_2(t,\alpha) &= (1.01 - 0.01\alpha) + (0.04 - 0.04\alpha)t + (0.505 - 0.005\alpha)t^2 \\ &\quad + (-0.16 - 0.0067\alpha)t^3 + (0.0421 - 0.0004\alpha)t^4 + \cdots . \end{aligned} \tag{19}$$

The Power series solutions of the given Eq.(12) are coinciding with the exact solutions. The results for $t = 0.1$ are illustrated in Tables 1-2 and Figure 1.

Table 1: Comparison of results of the presented method and the method in [4] for $t = 0.1$.

α	y_1 $(n = 5)$	y_1 $(in[4])$	Y_1 $(real\ solution)$	$Error$
0	0.9636355825	0.9617093838	0.9636355838	0.13135289×10^{-8}
0.1	0.9677557659	0.9660225287	0.9677557672	0.13191051×10^{-8}
0.2	0.9718759493	0.9703356737	0.9718759506	0.13246816×10^{-8}
0.3	0.9759961327	0.9746488186	0.9759961340	0.13302580×10^{-8}
0.4	0.9801163161	0.9789619636	0.9801163175	0.13358345×10^{-8}
0.5	0.9842364995	0.9832751085	0.9842365009	0.13414107×10^{-8}
0.6	0.9883566830	0.9875882534	0.9883566843	0.13469871×10^{-8}
0.7	0.9924768664	0.9919013984	0.9924768677	0.13525636×10^{-8}
0.8	0.9965970498	0.9962145433	0.9965970511	0.13581401×10^{-8}
0.9	1.0007172332	1.000527688	1.0007172346	0.13637164×10^{-8}
1	1.0048374166	1.004840833	1.004837418	0.13692929×10^{-8}

Table 2: Comparison of results of the presented method and the method in [4] for $t = 0.1$.

α	y_2 $(n = 5)$	y_2 $(in[4])$	Y_2 $(real\ solution)$	$Error$
0	1.0188941283	1.020827058	1.0188941297	0.13839780×10^{-8}
0.1	1.0174884571	1.019228436	1.0174884585	0.13825094×10^{-8}
0.2	1.0160827860	1.017629813	1.0160827873	0.13810410×10^{-8}
0.3	1.0146771148	1.016031191	1.0146771162	0.13795722×10^{-8}
0.4	1.0132714436	1.014432568	1.0132714450	0.13781038×10^{-8}
0.5	1.0118657725	1.012833946	1.0118657738	0.13766352×10^{-8}
0.6	1.0104601013	1.011235323	1.0104601027	0.13751666×10^{-8}
0.7	1.0090544301	1.009636701	1.0090544315	0.13736982×10^{-8}
0.8	1.0076487590	1.008038078	1.0076487603	0.13722298×10^{-8}
0.9	1.0062430878	1.006439456	1.0062430892	0.13707612×10^{-8}
1	1.0048374166	1.004840834	1.0048374180	0.13692929×10^{-8}

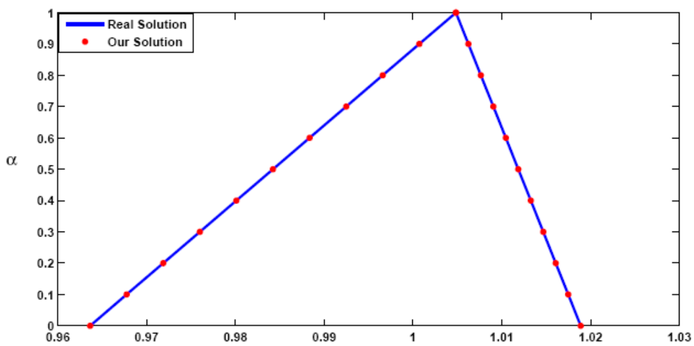

Figure 1: The results of Example 4.1 .

Example 4.2 *Consider the fuzzy Cauchy problem [3, 4, 5],*

$$y'(t, \alpha) = -y(t, \alpha),$$

$$y(0, \alpha) = (0.96 + 0.04\alpha, 1.01 - 0.01\alpha).$$

The exact Seikkala solution is ([5] Example 2, section 4)

$$Y_1(t,\alpha) = ((0.985 + 0.015\alpha)\exp(-t) + (-0.025 + 0.025\alpha)\exp(t)),$$
$$Y_2(t,\alpha) = ((0.985 + 0.015\alpha)\exp(-t) + (0.025 - 0.025\alpha)\exp(t))$$

for $0 \le \alpha \le 1$.

If we apply the presented method to the given equation, the following solution is obtained,

$$y_1(t,\alpha) = \tfrac{24}{25} + \tfrac{1}{25}\alpha + \left(-\tfrac{101}{100} + \tfrac{1}{100}\alpha\right)t + \left(\tfrac{12}{25} + \tfrac{1}{50}\alpha\right)t^2 + \left(-\tfrac{101}{600} + \tfrac{1}{600}\alpha\right)t^3$$
$$+ \left(\tfrac{1}{25} + \tfrac{1}{600}\alpha\right)t^4 + \left(-\tfrac{101}{12000} + \tfrac{1}{12000}\alpha\right)t^5 + \cdots,$$
$$y_2(t,\alpha) = \tfrac{101}{100} - \tfrac{1}{100}\alpha + \left(-\tfrac{24}{25} - \tfrac{1}{25}\alpha\right)t + \left(\tfrac{101}{200} - \tfrac{1}{200}\alpha\right)t^2 + \left(-\tfrac{4}{25} - \tfrac{1}{150}\alpha\right)t^3$$
$$+ \left(\tfrac{101}{2400} - \tfrac{1}{2400}\alpha\right)t^4 + \left(-\tfrac{1}{125} - \tfrac{1}{3000}\alpha\right)t^5 + \cdots.$$

The numerical results are illustrated in Tables 3-4 and Figure 2.

Table 3: Comparison of results of the presented method and the method in [4] for $t = 0.1$.

α	y_1 $(n=5)$	y_1 $(in[4])$	Y_1 *(real solution)*	*Error*
0	0.8636355825	0.8612677593	0.8636355838	0.13135289×10^{-8}
0.1	0.8677557659	0.8655594514	0.8677557672	0.13191052×10^{-8}
0.2	0.8718759493	0.8698511436	0.8718759506	0.13246818×10^{-8}
0.3	0.8759961327	0.8741428358	0.8759961340	0.13302580×10^{-8}
0.4	0.8801163161	0.8784345279	0.8801163175	0.13358345×10^{-8}
0.5	0.8842364995	0.8827262200	0.8842365009	0.13414108×10^{-8}
0.6	0.8883566830	0.8870179122	0.8883566843	0.13469871×10^{-8}
0.7	0.8924768664	0.8913096044	0.8924768677	0.13525636×10^{-8}
0.8	0.8965970498	0.8956012965	0.8965970511	0.13581400×10^{-8}
0.9	0.9007172332	0.8998929886	0.9007172346	0.13637164×10^{-8}
1	0.9048374166	0.9041846808	0.9048374180	0.13692929×10^{-8}

Table 4: Comparison of results of the presented method and the method in [4] for $t = 0.1$.

α	y_2 $(n=5)$	y_2 $(in[4])$	Y_2 *(real solution)*	*Error*
0	0.9188941283	0.920407787	0.9188941297	0.13839779×10^{-8}
0.1	0.9174884571	0.918816840	0.9174884585	0.13825094×10^{-8}
0.2	0.9160827860	0.917225894	0.9160827873	0.13810409×10^{-8}
0.3	0.9146771148	0.915634948	0.9146771162	0.13795724×10^{-8}
0.4	0.9132714436	0.914044002	0.9132714450	0.13781039×10^{-8}
0.5	0.9118657725	0.912453056	0.9118657738	0.13766354×10^{-8}
0.6	0.9104601013	0.910862110	0.9104601027	0.13751668×10^{-8}
0.7	0.9090544301	0.909271163	0.9090544315	0.13736984×10^{-8}
0.8	0.9076487590	0.907680217	0.9076487603	0.13722298×10^{-8}
0.9	0.9062430878	0.906089271	0.9062430892	0.13707615×10^{-8}
1	0.9048374166	0.9044983225	0.9048374180	0.13692929×10^{-8}

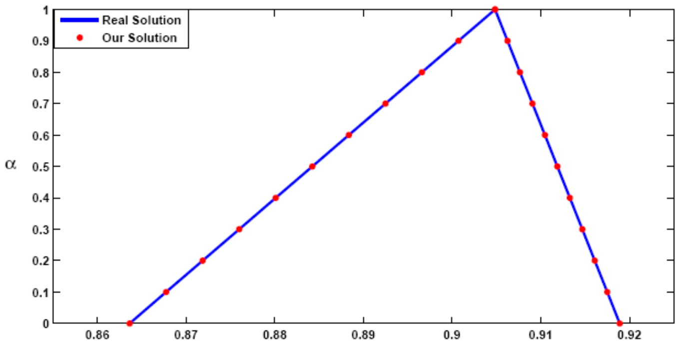

Figure 2: The results of Example 4.2 .

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