Journal of Uncertain Systems
Vol.3, No.4, pp.270-279, 2009
Online at: www.jus.org.uk

A Fuzzy Optimal Control Model

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Received 19 January 2009; Revised 28 March 2009

Abstract

Optimal control is a very important field of study not only in theory but in applications, and stochastic optimal control is also a significant branch of research in theory and applications. Based on the concept of fuzzy process, a fuzzy optimal control problem presented. Applying Bellman's Principle of Optimality, the principle of optimality for fuzzy optimal control is derived, and then a fundamental result called the equation of optimality is given in fuzzy optimal control. Finally, as an application, by using the equation of optimality, a portfolio selection model is solved.

Keywords: fuzzy optimal control, fuzzy process, principle of optimality, equation of optimality, portfolio selection

1 Introduction

Since 1950's, optimal control theory has been an important branch of modern control theory. The study of optimal control greatly attracted the attention of many mathematician because of the necessity of strict expression form in optimal control theory. With the more use of methods and results on mathematics and computer science, optimal control theory has greatly achieved development, and been applied to many fields such as production engineering, programming, economy and management.

The study of stochastic optimal control initiated in 1970's such as in Merton [6] for finance. Some investigations on optimal control of Brownian motion or stochastic differential equations and applications in finance may be found in some books such as Fleming and Rishel [3], Harrison [4] and Karatzas [5]. One of the main methods to study optimal control is based on dynamic programming. The use of dynamic programming in optimization over Ito's process was discussed in Dixit and Pindyck [1].

The complexity of the world makes the events we face uncertain in various forms. Besides randomness, fuzziness is also an important uncertainty, which plays an essential role in the real world. Fuzzy set theory has been developed very fast since it was introduced by scientist on cybernetics Zadeh [16] in 1965. A fuzzy set was characterized with its membership function by Zadeh. For the purpose of measuring fuzzy events, Zadeh [17] presented the concept of possibility measure and the term of fuzzy variable in 1978. In order to give a self-dual measure for fuzzy events, Liu and Liu [11] introduced the concept of credibility measure in 2002. Based on credibility measure, credibility theory was founded by Liu [8] in 2004 and refined by Liu [9] as a branch of mathematics for dealing with the behavior of fuzzy phenomena. A fuzzy variable may be redefined as a function from a credibility space to the set of real numbers. As fuzzy counterpart of stochastic process and Brownian motion, fuzzy process and C process were introduced by Liu [10] recently. We may call C process to be Liu process.

In order to handle an optimal control problem with fuzzy process, in the paper we will introduce and deal with a fuzzy optimal control problem by using dynamic programming. In next section, we will review some concepts such as credibility space, expected value of fuzzy variable, fuzzy process, Liu process, and fuzzy differential equation. In Section 3, we will introduce a fuzzy optimal control problem, and present the principle of optimality for fuzzy optimal control based on Bellman's principle of optimality in dynamic programming. In Section 4, we will obtain a fundamental result called the equation of optimality in fuzzy optimal control. In the last section, we will solve a portfolio selection model by using the equation of optimality.

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2 Preliminary

In convenience, we give some useful concepts at first. Let Θ be a nonempty set, and $\mathcal{P}$ the power set of Θ (i.e., all subsets of Θ).

Definition 2.1 (Liu and Liu [11]) *A set function* Cr *defined on the power set* $\mathcal{P}$ *is called a credibility measure if it satisfies the following four axioms:*

Axiom 1. $\text{Cr}\{\Theta\} = 1$*;*

Axiom 2. $\text{Cr}\{A\} \leq \text{Cr}\{B\}$ *whenever* $A \subset B$*;*

Axiom 3. $\text{Cr}\{A\} + \text{Cr}\{A^c\} = 1$ *for any* $A \in \mathcal{P}$*, where* A^c *is the complementary set of* A*;*

Axiom 4. $\text{Cr}\{\cup_i A_i\} = \sup_i\{\text{Cr}\{A_i\}$ *for any* $\{A_i\}$ *with* $\sup_i \text{Cr}\{A_i\} < 0.5$.

Definition 2.2 *Let* Θ *be a nonempty set,* $\mathcal{P}$ *the power set of* Θ *and* Cr *a credibility measure. Then the triplet* $(\Theta, \mathcal{P}, \text{Cr})$ *is said to be a credibility space.*

A fuzzy variable is defined as a function from a credibility space to the set of real numbers. If ξ is a fuzzy variable, then we may get its membership function via

$$\mu(x) = (2\text{Cr}\{\xi = x\}) \wedge 1, \quad x \in R.$$

Conversely, if a fuzzy variable ξ is given by a membership function μ, then we may get the credibility value via

$$\text{Cr}\{\xi \in B\} = \frac{1}{2}\left(\sup_{x\in B}\mu(x) + 1 - \sup_{x\in B^c}\mu(x)\right), \quad B \subset R.$$

Membership function represents the degree of possibility that the fuzzy variable ξ takes some prescribed value.

Definition 2.3 (Liu and Liu [11]) *Let* ξ *be a fuzzy variable. Then the expected value of* ξ *is defined by*

$$E[\xi] = \int_0^{+\infty} \text{Cr}\{\xi \geq r\}\text{d}r - \int_{-\infty}^0 \text{Cr}\{\xi \leq r\}\text{d}r \tag{1}$$

provided that at least one of the two integrals is finite.

Definition 2.4 (Liu and Liu [11]) *Let* ξ *be a fuzzy variable with finite expected value* e*. Then the variance of* ξ *is defined by* $V[\xi] = E[(\xi - e)^2]$.

Definition 2.5 (Liu [8]) *The fuzzy variables* ξ *and* η *are said to be identically distributed if* $\text{Cr}\{\xi \in B\} = \text{Cr}\{\eta \in B\}$ *for any set* $B \subset R$.

Definition 2.6 (Liu [8], Liu and Gao [13]) *The fuzzy variables* $\xi_1, \xi_2, \cdots, \xi_m$ *are said to be independent if*

$$\text{Cr}\left\{\bigcap_{i=1}^m \{\xi_i \in B_i\}\right\} = \min_{1\leq i\leq m} \text{Cr}\{\xi_i \in B_i\}$$

for any sets $B_1, B_2, \cdots, B_m$ *of* R.

Theorem 2.1 (Liu and Liu [12]) *Let* ξ *and* η *be independent fuzzy variables with finite expected values. Then for any numbers* a *and* b*, we have*

$$E[a\xi + b\eta] = aE[\xi] + bE[\eta].$$

Based on the credibility space, Liu introduced the concepts of fuzzy process, Liu process, fuzzy differential equation, and etc.

Definition 2.7 (Liu [10]) *Let* T *be an index set and let* $(\Theta, \mathcal{P}, \text{Cr})$ *be a credibility space. A fuzzy process is a function from* $T \times (\Theta, \mathcal{P}, \text{Cr})$ *to the set of real numbers.*

Definition 2.8 (Liu [10]) *A fuzzy process, simply denoted by X_t, is said to have independent increments if $X_{t_1} - X_{t_0}$, $X_{t_2} - X_{t_1}$, $\cdots$, $X_{t_k} - X_{t_{k-1}}$ are independent fuzzy variables for any times $t_0 < t_1 < \cdots < t_k$. A fuzzy process X_t is said to have stationary increments if, for any given $t > 0$, the increments $X_{s+t} - X_s$ are identically distributed fuzzy variables for all $s > 0$.*

Definition 2.9 (Liu [10]) *A fuzzy process C_t is said to be Liu process if*
(i) $C_0 = 0$,
(ii) C_t has stationary and independent increments,
(iii) every increment $C_{s+t} - C_s$ is a normally distributed fuzzy variable with expected value et and variance $\sigma^2 t^2$, whose membership function is

$$\mu(x) = 2\left(1 + \exp\left(\frac{\pi|x - et|}{\sqrt{6}\sigma t}\right)\right)^{-1}, \quad x \in R. \tag{2}$$

The parameters e and σ are called the drift and diffusion coefficients, respectively. The Liu process is said to be standard if $e = 0$ and $\sigma = 1$. The Liu process plays the role of Brownian motion or Wiener process.

Based on Liu process, a new kind of fuzzy differential was introduced by Liu [10]. As the inverse of fuzzy differential, a kind of integral, called Liu integral, was also introduced by Liu [10]. Liu integral is different from the fuzzy integral based on fuzzy measure given by Sugeno [15]. The following concept of fuzzy differential equation is important in theory and applications, and essential in the study of this paper.

Definition 2.10 (Liu [10]) *Suppose C_t is a standard Liu process, and g_1 and g_2 are some given functions. Then*

$$\mathrm{d}X_t = g_1(X_t, t)\mathrm{d}t + g_2(X_t, t)\mathrm{d}C_t \tag{3}$$

is called a fuzzy differential equation. A solution is a fuzzy process X_t that satisfies (3) identically in t.

Based on the concept of fuzzy differential equation, Liu [10] established a stock model for fuzzy financial market as a fuzzy counterpart of Black-Scholes stock model. Qin and Li [14] formulated an European option pricing formula for fuzzy financial market.

3 Problem of Fuzzy Optimal Control

Fuzzy optimal control problem is to choose the best decision such that some objective function related to a fuzzy process provided by a fuzzy differential equation is optimized. Because the objective function is a fuzzy variable for any decision, we can not optimize it as a real function. A question is how to compare two different fuzzy variables, or how to decide which is larger of them. In fact, there are many methods to do so but there is no single best method. These methods are established due to some criteria including, for example, expected value, optimistic value, pessimistic value, and credibility [7]. Now we make use of the expected value-based method to optimize the fuzzy objective function. That is, we assume that a fuzzy variable is larger than the other if the expected value of it is larger than the expected value of the other.

Unless stated otherwise, we assume that C_t is a standard Liu process. We consider the following fuzzy expected value optimal control problem

$$\begin{cases} J(0, x_0) \equiv \sup\limits_D E\left[\displaystyle\int_0^T f(X_s, D, s)\mathrm{d}s + G(X_T, T)\right] & (4) \\ \text{subject to} & \\ \qquad \mathrm{d}X_s = \nu(X_s, D, s)\mathrm{d}s + \sigma(X_s, D, s)\mathrm{d}C_s \quad \text{and} \quad X_0 = x_0. & (5) \end{cases}$$

In the above problem, X_s is the state variable, D the decision variable (represents the function $D(t, X_t)$ of time t and state X_t), f the objective function, and G the function of terminal reward. For a given D, $\mathrm{d}X_s$ is defined by the fuzzy differential equation (5), where ν and σ are two functions of X_s, D and time s. The function $J(0, x_0)$ is the expected optimal reward obtainable in $[0, T]$ with the initial condition that at time 0 we are in state x_0.

For any $0 < t < T$, $J(t,x)$ is the expected optimal reward obtainable in $[t,T]$ with the condition that at time t we are in state $X_t = x$. That is, we have

$$\begin{cases} J(t,x) \equiv \sup\limits_D E\left[\int_t^T f(X_s,D,s)\mathrm{d}s + G(X_T,T)\right] \\ \text{subject to} \\ \qquad \mathrm{d}X_s = \nu(X_s,D,s)\mathrm{d}s + \sigma(X_s,D,s)\mathrm{d}C_s \quad \text{and} \quad X_t = x. \end{cases} \tag{6}$$

Now we present the following *principle of optimality* for fuzzy optimal control.

Theorem 3.1 (Principle of optimality) *For any $(t,x) \in [0,T) \times R$, and $\Delta t > 0$ with $t + \Delta t < T$, we have*

$$J(t,x) = \sup_D E\left[\int_t^{t+\Delta t} f(X_s,D,s)\mathrm{d}s + J(t+\Delta t, x+\Delta X_t)\right], \tag{7}$$

where $x + \Delta X_t = X_{t+\Delta t}$.

Proof: We denote the right side of (7) by $\widetilde{J}(t,x)$. It follows from the definition of $J(t,x)$ that

$$J(t,x) \geq E\left[\int_t^{t+\Delta t} f(X_s, D|_{[t,t+\Delta t)}, s)\mathrm{d}s + \int_{t+\Delta t}^T f(X_s, D|_{[t+\Delta t,T]}, s)\mathrm{d}s + G(X_T,T)\right] \tag{8}$$

for any D, where $D|_{[t,t+\Delta t)}$ and $D|_{[t+\Delta t,T]}$ are the values of decision variable D restricted on $[t, t+\Delta t)$ and $[t+\Delta t, T]$, respectively. Since the fuzzy processes $\mathrm{d}C_s$ ($s \in [t,t+\Delta t)$) and $\mathrm{d}C_s$ ($s \in [t+\Delta t, T]$) are independent, we know that

$$\int_t^{t+\Delta t} f(X_s, D|_{[t,t+\Delta t)}, s)\mathrm{d}s \quad \text{and} \quad \int_{t+\Delta t}^T f(X_s, D|_{[t+\Delta t,T]}, s)\mathrm{d}s$$

are independent. Thus

$$J(t,x) \geq E\left[\int_t^{t+\Delta t} f(X_s, D|_{[t,t+\Delta t)}, s)\mathrm{d}s + E\left[\int_{t+\Delta t}^T f(X_s, D|_{[t+\Delta t,T]}, s)\mathrm{d}s + G(X_T,T)\right]\right] \tag{9}$$

by Theorem 2.1. Taking the supremum with respect to $D|_{[t+\Delta t,T]}$ first, and then $D|_{[t,t+\Delta t)}$ in (9), we get $J(t,x) \geq \widetilde{J}(t,x)$.

On the other hand, for all D, we have

$$\begin{aligned} & E\left[\int_t^T f(X_s,D,s)\mathrm{d}s + G(X_T,T)\right] \\ & = E\left\{\int_t^{t+\Delta t} f(X_s,D,s)\mathrm{d}s + E\left[\int_{t+\Delta t}^T f(X_s, D|_{[t+\Delta t,T]}, s)\mathrm{d}s + G(X_T,T)\right]\right\} \\ & \leq E\left[\int_t^{t+\Delta t} f(X_s,D,s)\mathrm{d}s + J(t+\Delta t, x+\Delta X_t)\right] \\ & \leq \widetilde{J}(t,x). \end{aligned}$$

Hence, $J(t,x) \leq \widetilde{J}(t,x)$, and then $J(t,x) = \widetilde{J}(t,x)$. The theorem is proved.

4 Equation of Optimality

Also, we assume C_t is a standard Liu process. Consider the fuzzy optimal control problem (6). Now let us give a fundamental result called *equation of optimality* in fuzzy optimal control.

Theorem 4.1 (Equation of optimality) *Let $J(t,x)$ be twice differentiable on $[0,T]\times R$. Then we have*

$$-J_t(t,x)=\sup_D\left\{f(x,D,t)+J_x(t,x)\nu(x,D,t)\right\}. \tag{10}$$

Proof: For any $\Delta t>0$, we have

$$\int_t^{t+\Delta t} f(X_s,D,s)\mathrm{d}s=f(x,D(t,x),t)\Delta t+o(\Delta t) \tag{11}$$

By using Taylor series expansion, we get

$$\begin{aligned}J(t+\Delta t,x+\Delta X_t)=&\ J(t,x)+J_t(t,x)\Delta t+J_x(t,x)\Delta X_t+\frac{1}{2}J_{tt}(t,x)\Delta t^2\\&+\frac{1}{2}J_{xx}(t,x)\Delta {X_t}^2+J_{tx}(t,x)\Delta t\Delta X_t+o(\Delta t)\end{aligned} \tag{12}$$

Substituting Equations (11) and (12) into Equation (7) yields

$$\begin{aligned}0=\sup_D\{&f(x,D,t)\Delta t+J_t(t,x)\Delta t+E[J_x(t,x)\Delta X_t+\frac{1}{2}J_{tt}(t,x)\Delta t^2\\&+\frac{1}{2}J_{xx}(t,x)\Delta {X_t}^2+J_{tx}(t,x)\Delta t\Delta X_t]+o(\Delta t)\}\end{aligned} \tag{13}$$

Let ξ be a fuzzy variable such that $\Delta X_t=\xi+\nu(x,D,t)\Delta t$. It follows from (13) that

$$\begin{aligned}0&=\sup_D\{f(x,D,t)\Delta t+J_t(t,x)\Delta t+J_x(t,x)\nu(x,D,t)\Delta t+E[(J_x(t,x)\\&\qquad+J_{xx}(t,x)\nu(x,D,t)\Delta t+J_{tx}(t,x)\Delta t)\xi+\frac{1}{2}J_{xx}(t,x)\xi^2]+o(\Delta t)\}\\&=\sup_D\{f(x,D,t)\Delta t+J_t(t,x)\Delta t+J_x(t,x)\nu(x,D,t)\Delta t+E[a\xi+b\xi^2]+o(\Delta t)\},\end{aligned} \tag{14}$$

where $a\equiv J_x(t,x)+J_{xx}(t,x)\nu(x,D,t)\Delta t+J_{tx}(t,x)\Delta t$, and $b\equiv\frac{1}{2}J_{xx}(t,x)$. It follows from the fuzzy differential equation, the constraint in (6), that $\xi=\Delta X_t-\nu(x,D,t)\Delta t$ is a normally distributed fuzzy variable with expected value 0 and variance $\sigma^2(x,D,t)\Delta t^2$. Simply denote $\sigma=\sigma(x,D,t)$ in sequel. Let

$$x_0=\frac{|a|}{2|b|},\quad z=\frac{\pi x}{\sqrt{6}\sigma\Delta t},\quad z_0=\frac{\pi|a|}{2\sqrt{6}|b|\sigma\Delta t}.$$

Formula (20) in the Appendix implies that

$$\begin{aligned}E[a\xi+b\xi^2]&=\int_0^{+\infty}(a+2bx)\left(1+\exp\left(\frac{\pi x}{\sqrt{6}\sigma\Delta t}\right)\right)^{-1}\mathrm{d}x-\int_0^{x_0}(a-2bx)\left(1+\exp\left(\frac{\pi x}{\sqrt{6}\sigma\Delta t}\right)\right)^{-1}\mathrm{d}x\\&=\frac{\sqrt{6}a\sigma\Delta t}{\pi}\int_{z_0}^{+\infty}\frac{1}{1+e^z}\mathrm{d}z+\frac{12b\sigma^2\Delta t^2}{\pi^2}\int_0^{z_0}\frac{z}{1+e^z}\mathrm{d}z+\frac{12b\sigma^2\Delta t^2}{\pi^2}\int_0^{+\infty}\frac{z}{1+e^z}\mathrm{d}z\\&=o(\Delta t).\end{aligned}$$

Formula (21) in the Appendix implies that

$$\begin{aligned}E[a\xi+b\xi^2]&=\int_0^{x_0}(a+2bx)\left(1+\exp\left(\frac{\pi x}{\sqrt{6}\sigma\Delta t}\right)\right)^{-1}\mathrm{d}x-\int_0^{+\infty}(a-2bx)\left(1+\exp\left(\frac{\pi x}{\sqrt{6}\sigma\Delta t}\right)\right)^{-1}\mathrm{d}x\\&=-\frac{\sqrt{6}a\sigma\Delta t}{\pi}\int_{z_0}^{+\infty}\frac{1}{1+e^z}\mathrm{d}z+\frac{12b\sigma^2\Delta t^2}{\pi^2}\int_0^{z_0}\frac{z}{1+e^z}\mathrm{d}z+\frac{12b\sigma^2\Delta t^2}{\pi^2}\int_0^{+\infty}\frac{z}{1+e^z}\mathrm{d}z\\&=o(\Delta t).\end{aligned}$$

Hence

$$E[a\xi+b\xi^2] = \pm\frac{\sqrt{6}a\sigma\Delta t}{\pi}\int_{z_0}^{+\infty}\frac{1}{1+e^z}\mathrm{d}z+\frac{12b\sigma^2\Delta t^2}{\pi^2}\int_0^{z_0}\frac{z}{1+e^z}\mathrm{d}z+\frac{12b\sigma^2\Delta t^2}{\pi^2}\int_0^{+\infty}\frac{z}{1+e^z}\mathrm{d}z$$

$$= o(\Delta t). \tag{15}$$

Substituting Equation (15) into Equation (14) yields

$$-J_t(t,x)\Delta t = \sup_D\left\{f(x,D,t)\Delta t + J_x(t,x)\nu(x,D,t)\Delta t + o(\Delta t)\right\}. \tag{16}$$

Dividing Equation (16) by Δt, and letting $\Delta t \to 0$, we can obtain the result (10).

Remark 4.1 The equation of optimality in fuzzy optimal control gives a necessary condition for an extremum. If the equation has solutions, then the optimal decision and optimal expected value of objective function are determined. If function f is convex in its arguments, then the equation will produce a minimum, and if f is concave in its arguments, then it will produce a maximum. We note that the boundary condition for the equation is $J(T,X_T) = G(X_T,T)$.

Remark 4.2 We note that in the equation of optimality for stochastic optimal control (called Hamilton-Jacobi-Bellman equation), there is an extra term $\frac{1}{2}J_{xx}(t,x)\sigma^2(x,D,t)$.

Example 4.1 Consider the following fuzzy optimization problem:

$$\begin{cases} J(t,x) \equiv \min\limits_D E\left[\int_0^T e^{-\beta s}(aX_s^2+bX_sD+cD^2)\mathrm{d}s\right] \\ \text{subject to} \\ \qquad \mathrm{d}X_s = (\nu D+\alpha X_s)\mathrm{d}s + \sigma X_s \mathrm{d}C_s, \end{cases}$$

where $a>0$, $c>0$, $\sigma>0$, $b^2-4ac\le 0$, $\nu\ne 0$, $\alpha\in R$, β is a discount factor, T is the terminal time.

We see that $f(X_s,D,s) = e^{-\beta s}(aX_s^2+bX_sD+cD^2)$. It follows from Equation (10) that

$$-J_t = \min_D\{e^{-\beta t}(ax^2+bxD+cD^2) + J_x(\nu D+\alpha x)\} = \min_D L(D), \tag{17}$$

where $L(D)$ denotes the term in the braces. The optimal decision D satisfies

$$\frac{\mathrm{d}L(D)}{\mathrm{d}D} = bxe^{-\beta t} + 2cDe^{-\beta t} + J_x\nu = 0 \quad \text{or} \quad D = -\frac{J_xe^{\beta t}\nu + bx}{2c}.$$

Equation (17) becomes

$$-J_t = e^{-\beta t}\left[ax^2 + bx\left(-\frac{J_xe^{\beta t}\nu+bx}{2c}\right) + c\left(\frac{J_xe^{\beta t}\nu+bx}{2c}\right)^2\right] + J_x\left(-\frac{J_xe^{\beta t}\nu^2+b\nu x}{2c}+\alpha x\right).$$

That is

$$-J_te^{\beta t} = \left(a-\frac{b^2}{4c}\right)x^2 - \frac{J_x^2e^{2\beta t}\nu^2}{4c} + \left(\alpha-\frac{b\nu}{2c}\right)xJ_xe^{\beta t}. \tag{18}$$

We conjecture that $J(t,x) = kx^2e^{-\beta t}$. Thus, we have

$$J_t = -\beta kx^2e^{-\beta t}, \quad J_x = 2kxe^{-\beta t}.$$

Substituting them into Equation (18) yields

$$\beta kx^2 = \left(a-\frac{b^2}{4c}\right)x^2 - \frac{\nu^2}{4c}(2kx)^2 - \frac{b\nu}{2c}(2kx^2) + \alpha(2kx^2),$$

or

$$\frac{\nu^2}{c}k^2 + \left(\beta-2\alpha+\frac{b\nu}{c}\right)k + \left(\frac{b^2}{4c}-a\right) = 0.$$

The solution of the parameter k (selecting k such that $J(t,x) \geq 0$) is

$$k = \frac{(2c\alpha - c\beta - b\nu) + \sqrt{(2c\alpha - c\beta - b\nu)^2 + \nu^2(4ac - b^2)}}{2\nu^2}.$$

Hence, the optimal decision and the optimal value of the objective function are given as follows, respectively

$$D = -\frac{(2k\nu + b)x}{2c}, \qquad J(t,x) = kx^2 e^{-\beta t}.$$

5 A Portfolio Selection Model

Portfolio selection problem is a classical problem in financial economics of allocating personal wealth among consumption, investment in a single risk asset, and investment in a risk-free security. Under the assumption that the risk asset earns a random return, Merton [6] studied a portfolio selection model by stochastic optimal control, and Kao [2] considered a generalized Merton's model. If we assume that the risk asset earns a fuzzy return, this generalized Merton's model may be solved by fuzzy optimal control.

Let X_t be the wealth of an investor at time t. The investor allocates a fraction w of the wealth in a risk asset and remainder in a sure asset. The sure asset produces a rate of return b. The risk asset is assumed to earn a fuzzy return, and yields a mean rate of return ν along with a variance of σ^2 per unit time. That is to say, the risk asset earns a return $\mathrm{d}r_t$ in time interval $(t, t+\mathrm{d}t)$, where $\mathrm{d}r_t = \nu\mathrm{d}t + \sigma\mathrm{d}C_t$, and C_t is a standard Liu process. Thus

$$\begin{aligned} X_{t+\mathrm{d}t} &= X_t + b(1-w)X_t\mathrm{d}t + \mathrm{d}r_t(wX_t) \\ &= X_t + b(1-w)X_t\mathrm{d}t + (\nu\mathrm{d}t + \sigma\mathrm{d}C_t)(wX_t) \\ &= X_t + [\nu w + b(1-w)]X_t\mathrm{d}t + \sigma w X_t\mathrm{d}C_t. \end{aligned}$$

If we consider the consumption rate by an amount p, we obtain

$$\mathrm{d}X_t = [\nu w X_t + b(1-w)X_t - p]\mathrm{d}t + \sigma w X_t \mathrm{d}C_t.$$

Assume that investor is interested in maximizing the expected utility over an infinite time horizon. Then a portfolio selection model is provided by

$$\begin{cases} J(t,x) \equiv \max\limits_{p,w} E\left[\displaystyle\int_0^\infty e^{-\beta t}\, \frac{p^\lambda + [(\nu-\sigma^2)wX_t]^\lambda}{\lambda}\, \mathrm{d}t\right] \\ \text{subject to} \\ \qquad \mathrm{d}X_t = [\nu w X_t + b(1-w)X_t - p]\mathrm{d}t + \sigma w X_t \mathrm{d}C_t, \end{cases}$$

where $\beta > 0,\ 0 < \lambda < 1$. By the equation of optimality (10), we have that

$$-J_t = \max_{p,w}\left\{e^{-\beta t}\, \frac{p^\lambda + [(\nu-\sigma^2)xw]^\lambda}{\lambda} + (\nu - b)xwJ_x + bxJ_x - pJ_x\right\} = \max_{p,w} L(p,w),$$

where $L(p,w)$ represents the term in the braces. The optimal (p,w) satisfies

$$\begin{aligned} \frac{\partial L(p,w)}{\partial p} &= e^{-\beta t}p^{\lambda-1} - J_x = 0, \\ \frac{\partial L(p,w)}{\partial w} &= e^{-\beta t}[(\nu-\sigma^2)xw]^{\lambda-1}[(\nu-\sigma^2)x] + J_x(\nu-b)x = 0, \end{aligned}$$

or

$$p = (J_x e^{\beta t})^{\frac{1}{\lambda-1}}, \qquad w = \left[\frac{(b-\nu)J_x e^{\beta t}}{\nu - \sigma^2}\right]^{\frac{1}{\lambda-1}} \frac{1}{(\nu-\sigma^2)x}.$$

Hence

$$\begin{aligned} -J_t = {} & \frac{1}{\lambda}e^{-\beta t}\left\{(J_x e^{\beta t})^{\frac{\lambda}{\lambda-1}} + \left[\frac{(b-\nu)J_x e^{\beta t}}{\nu-\sigma^2}\right]^{\frac{\lambda}{\lambda-1}}\right\} + \frac{\nu - b}{\nu-\sigma^2}J_x\left[\frac{(b-\nu)J_x e^{\beta t}}{\nu-\sigma^2}\right]^{\frac{1}{\lambda-1}} \\ & + bxJ_x - J_x(J_x e^{\beta t})^{\frac{1}{\lambda-1}}, \end{aligned}$$

or

$$-J_t e^{\beta t} = \left(\frac{1}{\lambda} - 1\right)\left[1 + \left(\frac{b-\nu}{\nu-\sigma^2}\right)^{\frac{\lambda}{\lambda-1}}\right](J_x e^{\beta t})^{\frac{\lambda}{\lambda-1}} + bxJ_x e^{\beta t}. \tag{19}$$

We conjecture that $J(t,x) = kx^\lambda e^{-\beta t}$. Then

$$J_t = -k\beta x^\lambda e^{-\beta t}, \quad J_x = k\lambda x^{\lambda-1}e^{-\beta t}.$$

Substituting them into Equation (19) yields

$$k\beta x^\lambda = \left(\frac{1}{\lambda} - 1\right)\left[1 + \left(\frac{b-\nu}{\nu-\sigma^2}\right)^{\frac{\lambda}{\lambda-1}}\right](k\lambda)^{\frac{\lambda}{\lambda-1}} x^\lambda + kb\lambda x^\lambda,$$

or

$$(k\lambda)^{\frac{1}{\lambda-1}} = \frac{\beta - b\lambda}{(1-\lambda)\left[1 + \left(\frac{b-\nu}{\nu-\sigma^2}\right)^{\frac{\lambda}{\lambda-1}}\right]}.$$

So we get

$$k\lambda = \left\{\frac{\beta - b\lambda}{(1-\lambda)\left[1 + \left(\frac{b-\nu}{\nu-\sigma^2}\right)^{\frac{\lambda}{\lambda-1}}\right]}\right\}^{\lambda-1}.$$

Therefore the optimal consumption rate and the optimal fraction of investment on risk asset is determined, respectively, by

$$p = x(k\lambda)^{\frac{1}{\lambda-1}}, \quad w = \left(\frac{b-\nu}{\nu-\sigma^2}\right)^{\frac{1}{\lambda-1}} \frac{(k\lambda)^{\frac{1}{\lambda-1}}}{\nu-\sigma^2}.$$

Remark 5.1 Note that the optimal consumption rate calls for the investor to consume a constant fraction of wealth at each moment, and optimal fraction of investment on risk asset is independent of total wealth. These conclusions are similar to that in the case of randomness [2].

Appendix

Let us give a formula for computing the expected value of $a\xi + b\xi^2$ if ξ is a fuzzy variable.

Theorem 5.1 *Let ξ be a fuzzy variable with an even and integrable membership function $\mu(x)$ which is decreasing on $[0, +\infty)$ and $\mu(0) = 1$. Then*

$$E[a\xi + b\xi^2] = \frac{1}{2}\int_0^{+\infty} \mu(x)(a+2bx)\mathrm{d}x - \frac{1}{2}\int_0^{x_0} \mu(x)(a-2bx)\mathrm{d}x \tag{20}$$

if $a \ge 0, b > 0$, or $a \le 0, b < 0$; and

$$E[a\xi + b\xi^2] = \frac{1}{2}\int_0^{x_0} \mu(x)(a+2bx)\mathrm{d}x - \frac{1}{2}\int_0^{+\infty} \mu(x)(a-2bx)\mathrm{d}x \tag{21}$$

if $a \ge 0, b < 0$, or $a \le 0, b > 0$, where $x_0 = |a|/(2|b|)$.

Proof: (1) If $b > 0$, then $a\xi + b\xi^2 \ge -\frac{a^2}{4b} \equiv y_0$. For any $y \ge y_0$, let

$$x_1 = \frac{-a + \sqrt{a^2 + 4by}}{2b}, \quad x_2 = \frac{-a - \sqrt{a^2 + 4by}}{2b}.$$

Then

$$\{a\xi + b\xi^2 = y\} = \{\xi = x_1\} \cup \{\xi = x_2\}.$$

Thus, the membership function of fuzzy variable $a\xi + b\xi^2$ is that if $y < y_0$, $\mu_{a\xi+b\xi^2}(y) = 0$, if $y \geq y_0$,

$$\begin{aligned}\mu_{a\xi+b\xi^2}(y) &= 2\mathrm{Cr}\{a\xi + b\xi^2 = y\} \wedge 1 \\ &= 2(\mathrm{Cr}\{\xi = x_1\} \vee \mathrm{Cr}\{\xi = x_2\}) \wedge 1 \\ &= (2\mathrm{Cr}\{\xi = x_1\} \wedge 1) \vee (2\mathrm{Cr}\{\xi = x_2\} \wedge 1) \\ &= \mu(x_1) \vee \mu(x_2).\end{aligned}$$

If $a \geq 0$, then $\mu(x_2) \leq \mu(x_1)$. So, $\mu_{a\xi+b\xi^2}(y) = \mu(x_1)$ for $y \geq y_0$. Thus

$$\begin{aligned}E[a\xi + b\xi^2] &= \int_0^{+\infty} \mathrm{Cr}\{a\xi + b\xi^2 \geq y\}\mathrm{d}y - \int_{y_0}^{0} \mathrm{Cr}\{a\xi + b\xi^2 \leq y\}\mathrm{d}y \\ &= \int_0^{+\infty} \frac{1}{2}\mu(x_1)\mathrm{d}y - \int_{y_0}^{0} \frac{1}{2}\mu(x_1)\mathrm{d}y \qquad (\text{let } x_1 = x, \text{ then } ax + bx^2 = y) \\ &= \frac{1}{2}\int_0^{+\infty} \mu(x)(a + 2bx)\mathrm{d}x - \frac{1}{2}\int_{-x_0}^{0} \mu(x)(a + 2bx)\mathrm{d}x \\ &= \frac{1}{2}\int_0^{+\infty} \mu(x)(a + 2bx)\mathrm{d}x - \frac{1}{2}\int_0^{x_0} \mu(x)(a - 2bx)\mathrm{d}x.\end{aligned}$$

where $x_0 = a/(2b)$. We obtain the formula (20). If $a \leq 0$, then $\mu(x_2) \geq \mu(x_1)$. So, $\mu_{a\xi+b\xi^2}(y) = \mu(x_2)$ for $y \geq y_0$. Thus

$$\begin{aligned}E[a\xi + b\xi^2] &= \int_0^{+\infty} \mathrm{Cr}\{a\xi + b\xi^2 \geq y\}\mathrm{d}y - \int_{y_0}^{0} \mathrm{Cr}\{a\xi + b\xi^2 \leq y\}\mathrm{d}y \\ &= \int_0^{+\infty} \frac{1}{2}\mu(x_2)\mathrm{d}y - \int_{y_0}^{0} \frac{1}{2}\mu(x_2)\mathrm{d}y \qquad (\text{let } x_2 = x, \text{ then } ax + bx^2 = y) \\ &= \frac{1}{2}\int_0^{-\infty} \mu(x)(a + 2bx)\mathrm{d}x - \frac{1}{2}\int_{x_0}^{0} \mu(x)(a + 2bx)\mathrm{d}x \\ &= \frac{1}{2}\int_0^{x_0} \mu(x)(a + 2bx)\mathrm{d}x - \frac{1}{2}\int_0^{+\infty} \mu(x)(a - 2bx)\mathrm{d}x\end{aligned}$$

where $x_0 = -a/(2b)$. The formula (21) is proved.

(2) If $b < 0$, then $a\xi + b\xi^2 \leq -\frac{a^2}{4b}$. The formula (20) and (21) can be proved same as in (1).

Remark 5.2 When $b = 0$, define $x_0 = +\infty$. Letting $a = 1$, the formula (20) or (21) reduces to the formula of computing the expected value $E[\xi]$ provided by Definition 2.3. If $a = 0$, the formula (20) or (21) reduces to the formula of computing the expected value $E[b\xi^2]$ as

$$E[b\xi^2] = b\int_0^{+\infty} x\mu(x)\mathrm{d}x.$$

6 Conclusion

Based on the concept of Liu process, we studied a fuzzy optimal control problem: optimizing the expected value of an objective function subject to a fuzzy differential equation. By using the Bellman's Principle of Optimality in dynamic programming, we presented the principle of optimality and a fundamental result called equation of optimality for fuzzy optimal control. As an application of the equation of optimality, we solved a portfolio selection model.

Acknowledgments

This work is supported by the National Natural Science Foundation of China (No.60874038).

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