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Uncertain Entailment and Modus Ponens in the Framework of Uncertain Logic

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Abstract

Uncertain entailment is a methodology for calculating the truth value of an uncertain formula via the maximum uncertainty principle when the truth values of other uncertain formulas are given. In order to find the truth value of additional formula, this paper will introduce an entailment model. As applications of uncertain entailment, this paper will also discuss modus ponens, modus tollens, and hypothetical syllogism.

Keywords: uncertain logic, entailment model, modus ponens, modus tollens, hypothetical syllogism

1 Introduction

Most human decisions are made in the state of uncertainty. What is uncertainty? This is a debatable topic. Perhaps it is impossible to use natural language to define uncertainty clearly, and all existing definitions by natural language are specious. A very personal and ultra viewpoint is that the words like randomness, fuzziness, roughness, vagueness, greyness, and uncertainty are nothing but ambiguity of human language!

However, fortunately, some "mathematical scales" have been invented to measure the truth degree of an event, for example, probability measure, capacity [3], fuzzy measure [13], possibility measure [15], credibility measure [7] as well as uncertain measure [8]. All of those measures may be defined clearly and precisely by axiomatic methods.

Let us go back to the question "what is uncertainty". Perhaps we can answer it this way. If it happened that some phenomena can be quantified by uncertain measure, then we call the phenomena "uncertainty". In other words, uncertainty is any concept that satisfies the axioms of uncertain measure. Thus there are various valid possibilities (*e.g.*, a personal belief degree) to interpret uncertainty.

In order to develop a theory of uncertain measure, Liu [8] founded an *uncertainty theory* that is a branch of mathematics based on normality, monotonicity, self-duality, and countable subadditivity axioms. As an application of uncertainty theory, Liu [11] proposed a spectrum of uncertain programming which is a type of mathematical programming involving uncertain variables, and applied uncertain programming to industrial engineering and management science. As a counterpart of Brownian motion, Liu [10] designed a canonical process that is a Lipschitz continuous uncertain process with stationary and independent increments. Following that, uncertain calculus was initialized by Liu [10] to deal with differentiation and integration of functions of uncertain processes. Based on uncertain calculus, Liu [9] proposed the concept of uncertain differential equations. For exploring the developments of uncertainty theory, the readers may consult Chen and Liu [1], Gao [4], Gao and Ralescu [5], You [14], and Liu [12].

Especially, based on uncertainty theory, Li and Liu [6] presented uncertain logic in which the truth value is defined as the uncertain measure that the proposition is true. Another important contribution is the truth value theorem by Chen and Ralescu [2] that provides a numerical method for calculating the truth value of uncertain formulas. Furthermore, uncertain inference was pioneered by Liu [10][12] as a process of deriving consequences from uncertain knowledge or evidence via the tool of conditional uncertainty.

Uncertain entailment is a methodology for calculating the truth value of an uncertain formula via the maximum uncertainty principle when the truth values of other uncertain formulas are given. In order to solve this problem, this paper will introduce an uncertain entailment model. As applications of uncertain entailment, this paper will also discuss modus ponens, modus tollens, and hypothetical syllogism.

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2 Uncertain Logic

Uncertain logic was designed by Li and Liu [6] in 2009 as a generalization of mathematical logic for dealing with uncertain knowledge via uncertainty theory. An uncertain proposition is a statement whose truth value is quantified by an uncertain measure, and an uncertain formula is a finite sequence of uncertain propositions and connective symbols that must make sense.

Truth value is a key concept in uncertain logic, and is defined as the uncertain measure that the uncertain formula is true. Then the truth value of uncertain formula X is the uncertain measure that X is true, i.e.,

$$T(X) = \mathcal{M}\{X = 1\}. \tag{1}$$

Assume that $\xi_1, \xi_2, \cdots, \xi_n$ are independent uncertain propositions with truth values $a_1, a_2, \cdots, a_n$, respectively. If X is an uncertain formula containing $\xi_1, \xi_2, \cdots, \xi_n$ with truth function f, then the truth value of X may be determined by the Chen-Ralescu's theorem [2] as follows,

$$T(X) = \begin{cases} \sup\limits_{f(x_1,x_2,\cdots,x_n)=1} \min\limits_{1\le i\le n} \nu_i(x_i), & \text{if } \sup\limits_{f(x_1,x_2,\cdots,x_n)=1} \min\limits_{1\le i\le n} \nu_i(x_i) < 0.5 \\ 1 - \sup\limits_{f(x_1,x_2,\cdots,x_n)=0} \min\limits_{1\le i\le n} \nu_i(x_i), & \text{if } \sup\limits_{f(x_1,x_2,\cdots,x_n)=1} \min\limits_{1\le i\le n} \nu_i(x_i) \ge 0.5 \end{cases} \tag{2}$$

where x_i take values either 0 or 1, and v_i are defined by

$$\nu_i(x_i) = \begin{cases} a_i, & \text{if } x_i = 1 \\ 1 - a_i, & \text{if } x_i = 0 \end{cases} \tag{3}$$

for $i = 1, 2, \cdots, n$, respectively.

3 Entailment Model

Assume $\xi_1, \xi_2, \cdots, \xi_n$ are independent uncertain propositions with unknown truth values $\alpha_1, \alpha_2, \cdots, \alpha_n$, respectively. We also assume $X_1, X_2, \cdots, X_m$ are uncertain formulas containing $\xi_1, \xi_2, \cdots, \xi_n$ with known truth values $\beta_1, \beta_2, \cdots, \beta_m$, respectively. Now let X be an additional uncertain formula containing $\xi_1, \xi_2, \cdots, \xi_n$. What is the truth value of X?

This is just the uncertain entailment problem. In order to solve it, let us consider what values $\alpha_1, \alpha_2, \cdots, \alpha_n$ may take. The first constraint is

$$0 \le \alpha_j \le 1, \quad j = 1, 2, \cdots, n. \tag{4}$$

We also hope

$$T(X_i) = \beta_i, \quad i = 1, 2, \cdots, m \tag{5}$$

where each $T(X_i)$ $(1 \le i \le m)$ is determined by the truth function f_i of X_i and Chen-Ralescu's theorem as follows,

$$T(X_i) = \begin{cases} \sup\limits_{f_i(x_1,x_2,\cdots,x_n)=1} \min\limits_{1\le j\le n} \nu_j(x_j), & \text{if } \sup\limits_{f_i(x_1,x_2,\cdots,x_n)=1} \min\limits_{1\le j\le n} \nu_j(x_j) < 0.5 \\ 1 - \sup\limits_{f_i(x_1,x_2,\cdots,x_n)=0} \min\limits_{1\le j\le n} \nu_j(x_j), & \text{if } \sup\limits_{f_i(x_1,x_2,\cdots,x_n)=1} \min\limits_{1\le j\le n} \nu_j(x_j) \ge 0.5 \end{cases} \tag{6}$$

and

$$\nu_j(x_j) = \begin{cases} \alpha_j, & \text{if } x_j = 1 \\ 1 - \alpha_j, & \text{if } x_j = 0 \end{cases} \tag{7}$$

for $j = 1, 2, \cdots, n$.

Furthermore, based on the truth values $\alpha_1, \alpha_2, \cdots, \alpha_n$ and truth function f of X, the truth value of X is determined by Chen-Ralescu's theorem as follows,

$$T(X) = \begin{cases} \sup\limits_{f(x_1,x_2,\cdots,x_n)=1} \min\limits_{1\le j\le n} \nu_j(x_j), & \text{if } \sup\limits_{f(x_1,x_2,\cdots,x_n)=1} \min\limits_{1\le j\le n} \nu_j(x_j) < 0.5 \\ 1 - \sup\limits_{f(x_1,x_2,\cdots,x_n)=0} \min\limits_{1\le j\le n} \nu_j(x_j), & \text{if } \sup\limits_{f(x_1,x_2,\cdots,x_n)=1} \min\limits_{1\le j\le n} \nu_j(x_j) \ge 0.5. \end{cases} \tag{8}$$

Since the truth values $\alpha_1, \alpha_2, \cdots, \alpha_n$ are not uniquely determined, the truth value $T(X)$ is not unique too. For this case, we have to use the maximum uncertainty principle of Liu [8] to determine the truth value $T(X)$. That is, $T(X)$ should be assigned the value as close to 0.5 as possible. In other words, we should minimize the value $|T(X) - 0.5|$ via choosing appreciate values of $\alpha_1, \alpha_2, \cdots, \alpha_n$.

The problem is to find the optimal solution $(\alpha_1, \alpha_2, \cdots, \alpha_n)$ that solves the following entailment model,

$$\begin{cases} \min |T(X) - 0.5| \\ \text{subject to:} \\ \qquad T(X_i) = \beta_i, \quad i = 1, 2, \cdots, m \\ \qquad 0 \le \alpha_j \le 1, \quad j = 1, 2, \cdots, n \end{cases} \tag{9}$$

where $T(X_1), T(X_2), \cdots, T(X_m), T(X)$ are functions of $\alpha_1, \alpha_2, \cdots, \alpha_n$ via (6) and (8).

If the entailment model (9) has no feasible solution, then the truth values $\beta_1, \beta_2, \cdots, \beta_m$ are inconsistent with each other. For this case, we cannot entail anything on the uncertain formula X.

If the entailment model (9) has an optimal solution $(\alpha_1^*, \alpha_2^*, \cdots, \alpha_n^*)$, then the truth value of X is just (8) except for

$$\nu_j(x_j) = \begin{cases} \alpha_j^*, & \text{if } x_j = 1 \\ 1 - \alpha_j^*, & \text{if } x_j = 0 \end{cases} \tag{10}$$

for $j = 1, 2, \cdots, n$.

Example 1: Let ξ_1 and ξ_2 be independent uncertain propositions with unknown truth values α_1 and α_2, respectively. It is known that

$$T(\xi_1 \vee \xi_2) = \beta_1, \quad T(\xi_1 \wedge \xi_2) = \beta_2. \tag{11}$$

What is the truth value of $\xi_1 \to \xi_2$? In order to answer this question, we write

$$X_1 = \xi_1 \vee \xi_2, \quad X_2 = \xi_1 \wedge \xi_2, \quad X = \xi_1 \to \xi_2.$$

Then we have

$$T(X_1) = \alpha_1 \vee \alpha_2 = \beta_1,$$
$$T(X_2) = \alpha_1 \wedge \alpha_2 = \beta_2,$$
$$T(X) = (1 - \alpha_1) \vee \alpha_2.$$

For this case, the entailment model (9) becomes

$$\begin{cases} \min |(1 - \alpha_1) \vee \alpha_2 - 0.5| \\ \text{subject to:} \\ \qquad \alpha_1 \vee \alpha_2 = \beta_1 \\ \qquad \alpha_1 \wedge \alpha_2 = \beta_2 \\ \qquad 0 \le \alpha_1 \le 1 \\ \qquad 0 \le \alpha_2 \le 1. \end{cases} \tag{12}$$

When $\beta_1 \ge \beta_2$, there are only two feasible solutions $(\alpha_1, \alpha_2) = (\beta_1, \beta_2)$ and $(\alpha_1, \alpha_2) = (\beta_2, \beta_1)$. If $\beta_1 + \beta_2 < 1$, the optimal solution produces

$$T(X) = (1 - \alpha_1^*) \vee \alpha_2^* = 1 - \beta_1;$$

if $\beta_1 + \beta_2 = 1$, the optimal solution produces

$$T(X) = (1 - \alpha_1^*) \vee \alpha_2^* = \beta_1 \text{ or } \beta_2;$$

if $\beta_1 + \beta_2 > 1$, the optimal solution produces

$$T(X) = (1 - \alpha_1^*) \vee \alpha_2^* = \beta_2.$$

When $\beta_1 < \beta_2$, there is no feasible solution and the truth values are ill-assigned. As a summary, we have

$$T(\xi_1 \to \xi_2) = \begin{cases} 1-\beta_1, & \text{if } \beta_1 \ge \beta_2 \text{ and } \beta_1+\beta_2 < 1 \\ \beta_1 \text{ or } \beta_2, & \text{if } \beta_1 \ge \beta_2 \text{ and } \beta_1+\beta_2 = 1 \\ \beta_2, & \text{if } \beta_1 \ge \beta_2 \text{ and } \beta_1+\beta_2 > 1 \\ \text{illness}, & \text{if } \beta_1 < \beta_2. \end{cases} \tag{13}$$

Example 2: Let ξ_1, ξ_2, ξ_3 be independent uncertain propositions with unknown truth values $\alpha_1, \alpha_2, \alpha_3$, respectively. It is known that

$$T(\xi_1 \to \xi_2) = \beta_1, \quad T(\xi_2 \to \xi_3) = \beta_2. \tag{14}$$

What is the truth value of ξ_2? In order to answer this question, we write

$$X_1 = \xi_1 \to \xi_2, \quad X_2 = \xi_2 \to \xi_3, \quad X = \xi_2.$$

Then we have

$$T(X_1) = (1-\alpha_1) \vee \alpha_2 = \beta_1,$$

$$T(X_2) = (1-\alpha_2) \vee \alpha_3 = \beta_2,$$

$$T(X) = \alpha_2.$$

For this case, the entailment model (9) becomes

$$\begin{cases} \min |\alpha_2 - 0.5| \\ \text{subject to:} \\ \qquad (1-\alpha_1) \vee \alpha_2 = \beta_1 \\ \qquad (1-\alpha_2) \vee \alpha_3 = \beta_2 \\ \qquad 0 \le \alpha_1 \le 1 \\ \qquad 0 \le \alpha_2 \le 1 \\ \qquad 0 \le \alpha_3 \le 1. \end{cases} \tag{15}$$

The optimal solution $(\alpha_1^*, \alpha_2^*, \alpha_3^*)$ produces

$$T(\xi_2) = \begin{cases} \beta_1, & \text{if } \beta_1+\beta_2 \ge 1 \text{ and } \beta_1 < 0.5 \\ 1-\beta_2, & \text{if } \beta_1+\beta_2 \ge 1 \text{ and } \beta_2 < 0.5 \\ 0.5, & \text{if } \beta_1 \ge 0.5 \text{ and } \beta_2 \ge 0.5 \\ \text{illness}, & \text{if } \beta_1+\beta_2 < 1. \end{cases} \tag{16}$$

Example 3: Let ξ_1, ξ_2, ξ_3 be independent uncertain propositions with unknown truth values $\alpha_1, \alpha_2, \alpha_3$, respectively. It is known that

$$T(\xi_1 \to \xi_2) = \beta_1, \quad T(\xi_1 \to \xi_3) = \beta_2. \tag{17}$$

What is the truth value of $\xi_1 \to \xi_2 \wedge \xi_3$? In order to answer this question, we write

$$X_1 = \xi_1 \to \xi_2, \quad X_2 = \xi_1 \to \xi_3, \quad X = \xi_1 \to \xi_2 \wedge \xi_3.$$

Then we have

$$T(X_1) = (1-\alpha_1) \vee \alpha_2 = \beta_1,$$

$$T(X_2) = (1-\alpha_1) \vee \alpha_3 = \beta_2,$$

$$T(X) = (1-\alpha_1) \vee (\alpha_2 \wedge \alpha_3).$$

For this case, the entailment model (9) becomes

$$\begin{cases} \min |(1-\alpha_1) \vee (\alpha_2 \wedge \alpha_3) - 0.5| \\ \text{subject to:} \\ \qquad (1-\alpha_1) \vee \alpha_2 = \beta_1 \\ \qquad (1-\alpha_1) \vee \alpha_3 = \beta_2 \\ \qquad 0 \le \alpha_1 \le 1 \\ \qquad 0 \le \alpha_2 \le 1 \\ \qquad 0 \le \alpha_3 \le 1. \end{cases} \tag{18}$$

The optimal solution $(\alpha_1^*, \alpha_2^*, \alpha_3^*)$ produces $T(\xi_1 \to \xi_2 \wedge \xi_3) = \beta_1 \wedge \beta_2$.

Example 4: Let ξ_1, ξ_2, ξ_3 be independent uncertain propositions with unknown truth values $\alpha_1, \alpha_2, \alpha_3$, respectively. It is known that

$$T(\xi_1 \to \xi_2) = \beta_1, \quad T(\xi_1 \to \xi_3) = \beta_2. \tag{19}$$

What is the truth value of $\xi_1 \to \xi_2 \vee \xi_3$? In order to answer this question, we write

$$X_1 = \xi_1 \to \xi_2, \quad X_2 = \xi_1 \to \xi_3, \quad X = \xi_1 \to \xi_2 \vee \xi_3.$$

Then we have

$$T(X_1) = (1-\alpha_1) \vee \alpha_2 = \beta_1,$$

$$T(X_2) = (1-\alpha_1) \vee \alpha_3 = \beta_2,$$

$$T(X) = (1-\alpha_1) \vee (\alpha_2 \vee \alpha_3).$$

For this case, the entailment model (9) becomes

$$\begin{cases} \min |(1-\alpha_1) \vee (\alpha_2 \vee \alpha_3) - 0.5| \\ \text{subject to:} \\ \qquad (1-\alpha_1) \vee \alpha_2 = \beta_1 \\ \qquad (1-\alpha_1) \vee \alpha_3 = \beta_2 \\ \qquad 0 \le \alpha_1 \le 1 \\ \qquad 0 \le \alpha_2 \le 1 \\ \qquad 0 \le \alpha_3 \le 1. \end{cases} \tag{20}$$

The optimal solution $(\alpha_1^*, \alpha_2^*, \alpha_3^*)$ produces $T(\xi_1 \to \xi_2 \vee \xi_3) = \beta_1 \vee \beta_2$.

Example 5: Let ξ_1, ξ_2, ξ_3 be independent uncertain propositions with unknown truth values $\alpha_1, \alpha_2, \alpha_3$, respectively. It is known that

$$T(\xi_1 \to \xi_3) = \beta_1, \quad T(\xi_2 \to \xi_3) = \beta_2. \tag{21}$$

What is the truth value of $\xi_1 \vee \xi_2 \to \xi_3$? In order to answer this question, we write

$$X_1 = \xi_1 \to \xi_3, \quad X_2 = \xi_2 \to \xi_3, \quad X = \xi_1 \vee \xi_2 \to \xi_3.$$

Then we have

$$T(X_1) = (1-\alpha_1) \vee \alpha_3 = \beta_1,$$

$$T(X_2) = (1-\alpha_2) \vee \alpha_3 = \beta_2,$$

$$T(X) = (1-\alpha_1 \vee \alpha_2) \vee \alpha_3.$$

For this case, the entailment model (9) becomes

$$\begin{cases} \min |(1-\alpha_1 \vee \alpha_2) \vee \alpha_3 - 0.5| \\ \text{subject to:} \\ \qquad (1-\alpha_1) \vee \alpha_3 = \beta_1 \\ \qquad (1-\alpha_2) \vee \alpha_3 = \beta_2 \\ \qquad 0 \le \alpha_1 \le 1 \\ \qquad 0 \le \alpha_2 \le 1 \\ \qquad 0 \le \alpha_3 \le 1. \end{cases} \tag{22}$$

The optimal solution $(\alpha_1^*, \alpha_2^*, \alpha_3^*)$ produces $T(\xi_1 \vee \xi_2 \to \xi_3) = \beta_1 \wedge \beta_2$.

Example 6: Let ξ_1, ξ_2, ξ_3 be independent uncertain propositions with unknown truth values $\alpha_1, \alpha_2, \alpha_3$, respectively. It is known that

$$T(\xi_1 \to \xi_3) = \beta_1, \quad T(\xi_2 \to \xi_3) = \beta_2. \tag{23}$$

What is the truth value of $\xi_1 \wedge \xi_2 \to \xi_3$? In order to answer this question, we write

$$X_1 = \xi_1 \to \xi_3, \quad X_2 = \xi_2 \to \xi_3, \quad X = \xi_1 \wedge \xi_2 \to \xi_3.$$

Then we have

$$T(X_1) = (1-\alpha_1) \vee \alpha_3 = \beta_1,$$
$$T(X_2) = (1-\alpha_2) \vee \alpha_3 = \beta_2,$$
$$T(X) = (1-\alpha_1 \wedge \alpha_2) \vee \alpha_3.$$

For this case, the entailment model (9) becomes

$$\begin{cases} \min |(1-\alpha_1 \wedge \alpha_2) \vee \alpha_3 - 0.5| \\ \text{subject to:} \\ \qquad (1-\alpha_1) \vee \alpha_3 = \beta_1 \\ \qquad (1-\alpha_2) \vee \alpha_3 = \beta_2 \\ \qquad 0 \le \alpha_1 \le 1 \\ \qquad 0 \le \alpha_2 \le 1 \\ \qquad 0 \le \alpha_3 \le 1. \end{cases} \tag{24}$$

The optimal solution $(\alpha_1^*, \alpha_2^*, \alpha_3^*)$ produces $T(\xi_1 \wedge \xi_2 \to \xi_3) = \beta_1 \vee \beta_2$.

4 Modus Ponens

Classical modus ponens tells us that if both ξ and $\xi \to \eta$ are true, then η is true. This section provides a version of modus ponens in the framework of uncertain logic.

Theorem 1 *Let ξ and η be independent uncertain propositions. Suppose ξ and $\xi \to \eta$ are two uncertain formulas with truth values β_1 and β_2, respectively. Then the truth value of η is*

$$T(\eta) = \begin{cases} \beta_2, & \text{if } \beta_1 + \beta_2 > 1 \\ 0.5 \wedge \beta_2, & \text{if } \beta_1 + \beta_2 = 1 \\ \text{illness}, & \text{if } \beta_1 + \beta_2 < 1. \end{cases} \tag{25}$$

Proof: Denote the truth values of ξ and η by α_1 and α_2, respectively, and write

$$X_1 = \xi, \quad X_2 = \xi \to \eta, \quad X = \eta.$$

It is clear that

$$T(X_1) = \alpha_1 = \beta_1,$$

$$T(X_2) = (1 - \alpha_1) \vee \alpha_2 = \beta_2,$$

$$T(X) = \alpha_2.$$

For this case, the entailment model (9) becomes

$$\begin{cases} \min |\alpha_2 - 0.5| \\ \text{subject to:} \\ \qquad \alpha_1 = \beta_1 \\ \qquad (1-\alpha_1) \vee \alpha_2 = \beta_2 \\ \qquad 0 \le \alpha_1 \le 1 \\ \qquad 0 \le \alpha_2 \le 1. \end{cases} \tag{26}$$

When $\beta_1 + \beta_2 > 1$, there is only one feasible solution and then the optimal solution is

$$\alpha_1^* = \beta_1, \quad \alpha_2^* = \beta_2.$$

Thus $T(\eta) = \alpha_2^* = \beta_2$. When $\beta_1 + \beta_2 = 1$, the feasible set is $\{\beta_1\} \times [0, \beta_2]$ and the optimal solution is

$$\alpha_1^* = \beta_1, \quad \alpha_2^* = 0.5 \wedge \beta_2.$$

Thus $T(\eta) = \alpha_2^* = 0.5 \wedge \beta_2$. When $\beta_1 + \beta_2 < 1$, there is no feasible solution and the truth values are ill-assigned. The theorem is proved.

Remark 1: Different from the classical logic, the uncertain propositions ξ and η in $\xi \to \eta$ are statements with some truth values rather than pure statements. Thus the truth value of $\xi \to \eta$ is understood as

$$T(\xi \to \eta) = (1 - T(\xi)) \vee T(\eta). \tag{27}$$

Remark 2: Note that $T(\eta)$ in (25) does not necessarily represent the objective truth of η. For example, if $T(\xi)$ is small, then $T(\eta)$ is the truth value that η might (not must) be true.

5 Modus Tollens

Classical modus tollens tells us that if $\xi \to \eta$ is true and η is false, then ξ is false. This section provides a version of modus tollens in the framework of uncertain logic.

Theorem 2 *Let ξ and η be independent uncertain propositions. Suppose $\xi \to \eta$ and η are two uncertain formulas with truth values β_1 and β_2, respectively. Then the truth value of ξ is*

$$T(\xi) = \begin{cases} 1 - \beta_1, & \text{if } \beta_1 > \beta_2 \\ (1-\beta_1) \vee 0.5, & \text{if } \beta_1 = \beta_2 \\ \text{illness}, & \text{if } \beta_1 < \beta_2. \end{cases} \tag{28}$$

Proof: Denote the truth values of ξ and η by α_1 and α_2, respectively, and write

$$X_1 = \xi \to \eta, \quad X_2 = \eta, \quad X = \xi.$$

It is clear that

$$T(X_1) = (1 - \alpha_1) \vee \alpha_2 = \beta_1,$$

$$T(X_2) = \alpha_2 = \beta_2,$$

$$T(X) = \alpha_1.$$

For this case, the entailment model (9) becomes

$$\begin{cases} \min |\alpha_1 - 0.5| \\ \text{subject to:} \\ \qquad (1-\alpha_1) \vee \alpha_2 = \beta_1 \\ \qquad \alpha_2 = \beta_2 \\ \qquad 0 \le \alpha_1 \le 1 \\ \qquad 0 \le \alpha_2 \le 1. \end{cases} \tag{29}$$

When $\beta_1 > \beta_2$, there is only one feasible solution and then the optimal solution is

$$\alpha_1^* = 1 - \beta_1, \quad \alpha_2^* = \beta_2.$$

Thus $T(\xi) = \alpha_1^* = 1 - \beta_1$. When $\beta_1 = \beta_2$, the feasible set is $[1-\beta_1, 1] \times \{\beta_2\}$ and the optimal solution is

$$\alpha_1^* = (1-\beta_1) \vee 0.5, \quad \alpha_2^* = \beta_2.$$

Thus $T(\xi) = \alpha_1^* = (1-\beta_1) \vee 0.5$. When $\beta_1 < \beta_2$, there is no feasible solution and the truth values are ill-assigned. The theorem is proved.

6 Hypothetical Syllogism

Classical hypothetical syllogism tells us that if both $\xi \to \eta$ and $\eta \to \tau$ are true, then $\xi \to \tau$ is true. This section provides a version of hypothetical syllogism in the framework of uncertain logic.

Theorem 3 *Let ξ, η, τ be independent uncertain propositions. Suppose $\xi \to \eta$ and $\eta \to \tau$ are two uncertain formulas with truth values β_1 and β_2, respectively. Then the truth value of $\xi \to \tau$ is*

$$T(\xi \to \tau) = \begin{cases} \beta_1 \wedge \beta_2, & \text{if } \beta_1 \wedge \beta_2 \ge 0.5 \\ 0.5, & \text{if } \beta_1 + \beta_2 \ge 1 \text{ and } \beta_1 \wedge \beta_2 < 0.5 \\ \text{illness}, & \text{if } \beta_1 + \beta_2 < 1. \end{cases} \tag{30}$$

Proof: Denote the truth values of ξ, η, τ by $\alpha_1, \alpha_2, \alpha_3$, respectively, and write

$$X_1 = \xi \to \eta, \quad X_2 = \eta \to \tau, \quad X = \xi \to \tau.$$

It is clear that

$$\begin{aligned} T(X_1) &= (1-\alpha_1) \vee \alpha_2 = \beta_1, \\ T(X_2) &= (1-\alpha_2) \vee \alpha_3 = \beta_2, \\ T(X) &= (1-\alpha_1) \vee \alpha_3. \end{aligned}$$

For this case, the entailment model (9) becomes

$$\begin{cases} \min |(1-\alpha_1) \vee \alpha_3 - 0.5| \\ \text{subject to:} \\ \qquad (1-\alpha_1) \vee \alpha_2 = \beta_1 \\ \qquad (1-\alpha_2) \vee \alpha_3 = \beta_2 \\ \qquad 0 \le \alpha_1 \le 1 \\ \qquad 0 \le \alpha_2 \le 1 \\ \qquad 0 \le \alpha_3 \le 1. \end{cases} \tag{31}$$

When $\beta_1 \wedge \beta_2 \ge 0.5$, we have

$$T(\xi \to \tau) = (1-\alpha_1^*) \vee \alpha_3^* = \beta_1 \wedge \beta_2.$$

When $\beta_1 + \beta_2 \ge 1$ and $\beta_1 \wedge \beta_2 < 0.5$, we have

$$T(\xi \to \tau) = (1-\alpha_1^*) \vee \alpha_3^* = 0.5.$$

When $\beta_1 + \beta_2 < 1$, there is no feasible solution and the truth values are ill-assigned. The theorem is proved.

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References

[1] Chen, X,W., and B. Liu, Existence of uniqueness theorem for uncertain differential equations, *Fuzzy Optimization and Decision Making*, to be published.

[2] Chen, X.W., and D.A. Ralescu, A note on truth value in uncertain logic, *Proceedings of the Eighth International Conference on Information and Management Sciences*, Kunming, China, July 20-28, 2009, pp.739–741. (Also available at http://orsc.edu.cn/online/090211.pdf)

[3] Choquet, G., Theory of capacities, *Annals de l'Institute Fourier*, vol.5, pp.131–295, 1954.

[4] Gao, X., Some properties of continuous uncertain measure, *International Journal of Uncertainty, Fuzziness & Knowledge-Based Systems*, vol.17, no.3, pp.419–426, 2009.

[5] Gao, X., and D.A. Ralescu, On Liu's inference rule for uncertain systems, *Proceedings of the Eighth International Conference on Information and Management Sciences*, Kunming, China, July 20-28, 2009, pp.958–964. (Also available at http://orsc.edu.cn/online/090121.pdf)

[6] Li, X., and B. Liu, Hybrid logic and uncertain logic, *Journal of Uncertain Systems*, vol.3, no.2, pp.83–94, 2009.

[7] Liu, B., and Y.K. Liu, Expected value of fuzzy variable and fuzzy expected value models, *IEEE Transactions on Fuzzy Systems*, vol.10, no.4, pp.445–450, 2002.

[8] Liu, B., *Uncertainty Theory*, 2nd ed., Springer-Verlag, Berlin, 2007.

[9] Liu, B., Fuzzy process, hybrid process and uncertain process, *Journal of Uncertain Systems*, vol.2, no.1, pp.3–16, 2008.

[10] Liu, B., Some research problems in uncertainty theory, *Journal of Uncertain Systems*, vol.3, no.1, pp.3–10, 2009.

[11] Liu, B., *Theory and Practice of Uncertain Programming*, 2nd ed., Springer-Verlag, Berlin, 2009.

[12] Liu, B., *Uncertainty Theory*, 3rd ed., http://orsc. edu.cn/liu/ut.pdf.

[13] Sugeno, M., *Theory of Fuzzy Integrals and its Applications*, Ph.D. Dissertation, Tokyo Institute of Technology, 1974.

[14] You, C., Some convergence theorems of uncertain sequences, *Mathematical and Computer Modelling*, vol.49, nos.3-4, pp.482–487, 2009.

[15] Zadeh, L.A., Fuzzy sets as a basis for a theory of possibility, *Fuzzy Sets and Systems*, vol.1, pp.3–28, 1978.