Anti-synchronization Between Two Different Hyperchaotic Systems

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Received 30 June 2008; Revised 22 October 2008

Abstract

The purpose of this paper is to study hyperchaos anti-synchronization of two identical and different hyperchaotic systems using active control. The sufficient conditions for achieving anti-synchronization of two high dimensional chaotic systems are derived based on Lyapunov stability theory, where the controllers are designed by using the sum of the relevant variables in hyperchaotic systems such that the hyperchaotic Lü system is controlled to be hyperchaotic Lorenz system. Theoretical analysis and numerical simulations are shown to verify the results.

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Keywords: hyperchaotic, anti-synchronization, active control, Lyapunov direct method

1 Introduction

Since the pioneering work of Pecora and Carroll [1], chaos synchronization, as a very active topic in nonlinear science, has been intensively studied in the last few years and widely explored in a variety of fields including physical, chemical and ecological system [2]. Hence various synchronization schemes such as adaptive control [3, 4, 5], linear and nonlinear feedback synchronization methods [6, 7], active control [8], and backstepping design technique [9] have been successfully applied to chaos synchronization. The concept of synchronization has been extended to the scope such as generalized synchronization [10, 11], phase synchronization [10], lag synchronization [12], and even anti phase synchronization (APS) [13, 14, 15]. APS can also be interpreted as anti-synchronization (AS), which is a phenomenon that the state vectors of the synchronized systems have the same amplitude but opposite signs as those of the driving system. Therefore, the sum of two signals are expected to converge to zero when either AS or APS appears. Recently, active control has been applied to anti-synchronize two identical chaotic systems [16, 17, 18]. Moreover, it is examined in different types of chaotic systems [19]. In fact, in engineering, it is hardly the case that every component can be assumed to be identical. Thus, it is much more attractive and challengeable to realize anti-synchronization of two different chaotic systems. The aim of this work is to further develop the state observer method for constructing anti-synchronized of the high dimensional system, since the aforementioned method is mainly concern with the synchronization of chaotic systems with low dimensional attractor which is characterized by one positive Lyapunov exponent. This feature limits the complexity of the chaotic dynamics. It is believed that the chaotic systems with higher dimensional attractor have much wider application. In fact, the adoption of higher dimensional chaotic systems has been proposed for secure communication and the presence of more than one Lyapunov exponent clearly improves security of the communication scheme by generating more complex dynamics. Recently, hyperchaotic systems were also considered with quickly increasing interest. Hyperchaotic system is usually classified as a chaotic system with more than one positive Lyapunov exponent.

The rest of the paper is organized as follows. Section 2 presents a brief description of the active control method. Section 3 gives a brief description of the two systems. In Sections 4 and 5, we present chaos anti-synchronization of two identical hyperchaotic systems, and in Section 6 we present chaos anti-synchronization of two different hyperchaotic systems. Finally, concluding remark is given in Section 7.

2 Anti-synchronization

Consider a chaotic continuous system described by

\[ \dot{x} = f(x(t), t), \]  

(1)

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where \( x \in \mathbb{R}^n \) is a \( n \)-dimensional state vector of the system, and \( f : \mathbb{R}^n \rightarrow \mathbb{R}^n \) defines a vector field in \( n \)-dimensional space. We decompose the function \( f(x(t), t) \) as

\[
f(x(t), t) = g(x(t), t) + h(x(t), t),
\]

where \( g(x(t), t) \) is the linear part of \( f(x(t), t) \), and described by

\[
g(x(t), t) = Ax(t),
\]

where \( A \) is a full rank constant matrix and all eigenvalues of \( A \) have negative real parts, \( h(x(t), t) = f(x(t), t) - g(x(t), t) \) is the nonlinear part of \( f(x(t), t) \). Then the system (1) can be written as

\[
\dot{x} = g(x(t), t) + h(x(t), t).
\]

The chaotic anti-synchronization discussed in this paper is defined as the complete Anti-synchronization, which means that the state vectors of synchronized systems have the same absolute values but opposite signs. Anti-synchronization of two system can be used for the adjustment of parameters to satisfy the stability criterion of linear system that all eigenvalues have negative real.

\section{3 Systems Description}

The hyperchaotic Lorenz system \cite{20, 21} is given by

\[
\begin{align*}
\dot{x} &= a(y - x) + w \\
\dot{y} &= -xz + rx - y \\
\dot{z} &= xy - bz \\
\dot{w} &= -xz + dw,
\end{align*}
\]

where \( x, y, z \) and \( w \) are state variables, and \( a, b, c \) and \( d \) are real constants. When parameters \( a = 10, r = 28, b = 8/3 \) and \( d = 1.3 \), system (8) shows hyperchaotic behavior, the projections of the hyperchaotic attractor is shown in Figure 1.

Chen et al. \cite{22} constructed hyperchaotic system by introducing an additional state into the third-order Lü chaotic system. The four-dimensional autonomous hyperchaotic system is described by

\[
\begin{align*}
\dot{x} &= a(y - x) \\
\dot{y} &= -xz + cy \\
\dot{z} &= xy - bz \\
\dot{w} &= xz + dw,
\end{align*}
\]

where \( x, y, z \), and \( w \) are state variables and \( a, b, c, \) and \( d \), are constant parameters. When parameters \( a = 36, b = 3, c = 20 \) and \( d = 1.3 \), system (9) is hyperchaotic, the projections of the hyperchaotic attractor are shown in Figure 2.
4 Anti-synchronization of Hyperchaotic Lorenz System

In order to observe the anti-synchronization behavior in the Lorenz hyperchaotic system (8), we have two Lorenz hyperchaotic systems where the drive system with four state variables denoted by the subscript 1 and the response system having identical equations denoted by the subscript 2. However, the initial condition on the drive system is different from that of the response system. The two Lorenz systems are described, respectively, by the following equations:

\[
\begin{align*}
    \dot{x}_1 &= a(y_1 - x_1) + w_1 \\
    \dot{y}_1 &= -x_1z_1 + rx_1 - y_1 \\
    \dot{z}_1 &= x_1y_1 - bz_1 \\
    \dot{w}_1 &= -x_1z_1 + dw_1
\end{align*}
\]

and

\[
\begin{align*}
    \dot{x}_2 &= a(y_2 - x_2) + w_2 + u_1(t) \\
    \dot{y}_2 &= -x_2z_2 + rx_2 - y_2 + u_2(t) \\
    \dot{z}_2 &= x_2y_2 - bz_2 + u_3(t) \\
    \dot{w}_2 &= -x_2z_2 - dw_2 + u_4(t)
\end{align*}
\]

be the response system, where we have introduced four control functions \(u_1(t), u_2(t), u_3(t)\) and \(u_4(t)\) in Eq. (11). These functions are to be determined for the purpose of anti-synchronizing the two Lorenz hyperchaotic dynamical system with the same unknown parameters and different initial conditions. Let us define the state errors between the response system that is to be controlled and the controlling derive system as

\[
e_1 = x_2 + x_1, \quad e_2 = y_2 + y_1, \quad e_3 = z_2 + z_1, \quad e_4 = w_2 + w_1.
\]
Figure 2: Typical dynamical behaviors of system (9): (a) Projection in \((x, y, z)\) space; (b) Projection in \((x, y, w)\) space; (c) Projection in \((x, z, w)\) space; (d) Projection in \((y, z, w)\) space.

By adding Eq.(10) to Eq.(11) yields the error dynamical system between Eqs.(10) and (11)

\[
\begin{align*}
\dot{e}_1 &= a(e_2 - e_1) + e_4 + u_1(t) \\
\dot{e}_2 &= -x_2z_2 - x_1z_1 + re_1 - e_2 + u_2(t) \\
\dot{e}_3 &= x_2y_2 + x_1y_1 - be_3 + u_3(t) \\
\dot{e}_4 &= -x_1z_1 - x_2z_2 + de_4 + u_4(t).
\end{align*}
\]  

The anti-synchronization problem for Lorenz hyperchaotic dynamical system is to achieve the asymptotic stability of the zero solution of the error system (13). To this end we take the active control functions \(u_1(t), u_2(t), u_3(t)\) and \(u_4(t)\) as follows:

\[
\begin{align*}
u_1(t) &= V_1(t), \quad u_2(t) = V_2(t) + x_2z_2 + x_1z_1, \\
u_3(t) &= V_3(t) - x_2y_2 - x_1y_1, \quad u_4(t) = V_4(t) + x_1z_1 + x_2z_2.
\end{align*}
\]  

Hence the error system (13) becomes

\[
\begin{align*}
\dot{e}_1 &= a(e_2 - e_1) + e_4 + V_1, \quad \dot{e}_2 = re_1 - e_2 + V_2, \quad \dot{e}_3 = -be_3 + V_3, \quad \dot{e}_4 = de_4 + V_4.
\end{align*}
\]  

Eq.(15) describe the error dynamics and can be considered in terms of a control problem where the system to be controlled is a linear system with a control input \(V_1(t), V_2(t), V_3(t)\) and \(V_4(t)\) as function of \(e_1, e_2, e_3\) and \(e_4\). As long as these feedbacks stabilize the system, \(e_1, e_2, e_3\) and \(e_4\) converge to zero as time \(t\) goes to infinity. This implies that two Lorenz hyperchaotic systems are anti-synchronized with feedback control. There are many possible choices for the control \(V_1(t), V_2(t), V_3(t)\) and \(V_4(t)\). If we choose

\[
\begin{align*}
V_1(t) &= -ae_2 - e_4, \quad V_2(t) = -re_1, \quad V_3(t) = 0, \quad V_4(t) = -(1 + d)e_4,
\end{align*}
\]  

then the error dynamical system is

\[
\begin{align*}
\dot{e}_1 &= -ae_1, \quad \dot{e}_2 = -e_2, \quad \dot{e}_3 = -be_3, \quad \dot{e}_4 = -e_4.
\end{align*}
\]
Eq. (17) describes the error dynamics. Now we define the Lyapunov function for the system (17) as follows:

$$V(e_1, e_2, e_3, e_4) = \frac{1}{2}e_1^2 + \frac{1}{2}e_2^2 + \frac{1}{2}e_3^2 + \frac{1}{2}e_4^2. \quad (18)$$

This function is positive definite and equal zero at the equilibrium of the system (17). Moreover, the derivative of the Lyapunov function (18) has the form

$$\dot{V} = -[ae_1^2 + e_2^2 + be_3^2 + e_4^2], \quad (19)$$

which is negative definite. From Lyapunov direct method, we have that the zero solution of the system (17) is asymptotically stable. This implies that the two Lorenz hyperchaotic systems are anti-synchronized.

4.1 Numerical Results

Fourth order Runge-Kutta integration method is used to solve the systems of differential equations (10) and (11). In addition, a time step size 0.001 is employed. We select the parameters of Lorenz hyperchaotic system (8) as $a = 10, b = 8/3, r = 28, d = 1.3$, so that system (8) exhibits a hyperchaotic behavior. The initial values of the drive and response systems are $x_1(0) = -0.1, y_1(0) = 0.2, z_1(0) = -0.6, w_1(0) = 0.4$ and $x_2(0) = -1, y_2(0) = 0.4, z_2(0) = 0.2, w_2(0) = 1$, respectively, and the initial states of the error system are $e_1(0) = -1.1, e_2(0) = 0.6, e_3(0) = -0.8, e_4(0) = 1.4$. Figure 3(a)–(d) display the time response of states $x_1, y_1, z_1, w_1$ for the drive system (10) and the states $x_2, y_2, z_2, w_2$ for the response system (11).

![Figure 3](image)

Figure 3: The time response of states for drive system (10) and the response system (11) via active control (a) signals $x_1$ and $x_2$; (b) signals $y_1$ and $y_2$; (c) signals $z_1$ and $z_2$; (d) signals $w_1$ and $w_2$

5 Anti-synchronization of Hyperchaotic Lü System

In order to observe the anti-synchronization behavior in the hyperchaotic Lü system, we have two identical hyperchaotic Lü system where the the drive system with four state variables denoted by the subscript 1 drives the response system having identical equations denoted by the subscript 2. However, the initial condition on the drive system is different from that of the response system. The two hyperchaotic Lü system are described, respectively, by the following equations:

$$\begin{align*}
\dot{x}_1 &= a(y_1 - x_1) + w_1 \\
\dot{y}_1 &= -x_1z_1 + rx_1 - y_1 \\
\dot{z}_1 &= x_1y_1 - bz_1 \\
\dot{w}_1 &= -x_1z_1 + dw_1
\end{align*} \quad (20)$$
and
\[
\begin{align*}
\dot{x}_2 &= a(y_2 - x_2) + w_2 + \beta_1(t) \\
\dot{y}_2 &= -x_2z_2 + cy_2 + \beta_2(t) \\
\dot{z}_2 &= x_2y_2 - bz_2 + \beta_3(t) \\
\dot{w}_2 &= x_2z_2 + dw_2 + \beta_4(t).
\end{align*}
\]

We have introduced four control functions \(\beta_1(t), \beta_2(t), \beta_3(t)\) and \(\beta_4(t)\) in (21). These functions are to be determined. Let us define the error states between the response system (21) that is to be controlled and the controlling drive system (20) as
\[
e_1 = x_2 + x_1, \quad e_2 = y_2 + y_1, \quad e_3 = z_2 + z_1, \quad e_4 = w_2 + w_1.
\]

By adding Eq.(20) to Eq.(21) yields the error dynamical system between Eqs.(20) and (21)
\[
\begin{align*}
\dot{e}_1 &= a(e_2 - e_1) + e_4 + \beta_1(t) \\
\dot{e}_2 &= -x_2z_2 - x_1z_1 + e_2 + \beta_2(t) \\
\dot{e}_3 &= x_2y_2 + x_1y_1 - be_3 + \beta_3(t) \\
\dot{e}_4 &= x_2z_2 + x_1z_1 + de_4 + \beta_4(t).
\end{align*}
\]

We define the active control functions \(\beta_1(t), \beta_2(t), \beta_3(t)\) and \(\beta_4(t)\) as follows:
\[
\begin{align*}
\beta_1(t) &= V_1(t), \quad \beta_2(t) = V_2(t) - x_2z_2 - x_1z_1, \\
\beta_3(t) &= V_3(t) + x_2y_2 + x_1y_1, \quad \beta_4(t) = V_4(t) + x_2z_2 + x_1z_1.
\end{align*}
\]

Hence the error system (23) becomes
\[
\dot{e}_1 = a(e_2 - e_1) + e_4 + V_1, \quad \dot{e}_2 = ce_1 + V_2, \quad \dot{e}_3 = -be_3 + V_3, \quad \dot{e}_4 = de_4 + V_4.
\]

Eq.(25) describe the error dynamics and can be considered in terms of a control problem where the system to be controlled is a linear system with a control input \(V_1(t), V_2(t), V_3(t)\) and \(V_4(t)\) as function of \(e_1, e_2, e_3\) and \(e_4\). As long as these feedbacks stabilize the system, \(e_1, e_2, e_3\) and \(e_4\) converge to zero as time \(t\) goes to infinity. This implies that two L"{u} hyperchaotic systems are anti-synchronized with feedback control. There are many possible choices for the control \(V_1(t), V_2(t), V_3(t)\) and \(V_4(t)\). If we choose
\[
V_1(t) = -ae_2 - e_4, \quad V_2(t) = -(1 + c)e_2, \quad V_3(t) = 0, \quad V_4(t) = -(1 + d)e_4,
\]

then the error dynamical system is
\[
\dot{e}_1 = -ae_1, \quad \dot{e}_2 = -e_2, \quad \dot{e}_3 = -be_3, \quad \dot{e}_4 = -e_4.
\]

Eq.(27) describes the error dynamics. Now we define the Lyapunov function for the system (27) as follows:
\[
V(e_1, e_2, e_3, e_4) = \frac{1}{2}e_1^2 + \frac{1}{2}e_2^2 + \frac{1}{2}e_3^2 + \frac{1}{2}e_4^2.
\]

This function is positive definite and equal zero at the equilibrium of the system (27). Moreover, the derivative of the Lyapunov function (28) has the form
\[
\dot{V} = -[ae_1^2 + e_2^2 + be_3^2 + e_4^2],
\]

which is negative definite. From Lyapunov direct method, we have that the zero solution of the system (27) is asymptotically stable. This implies that the two L"{u} hyperchaotic systems are anti-synchronized.

### 5.1 Numerical Results

Fourth order Runge-Kutta integration method is used to solve the systems of differential equations (20) and (21). In addition, a time step size 0.001 is employed. We select the parameters of lorenz hyperchaotic system (9) as \(a = 36, b = 3, c = 20, d = 1.3\), so that system (8) exhibits a hyperchaotic behavior. The initial values of the drive and response systems are \(x_1(0) = 5, y_1(0) = 8, z_1(0) = -1, w_1(0) = -3\) and \(x_2(0) = 3, y_2(0) = 4, z_2(0) = 5, w_2(0) = 5\), respectively, and the initial states of the error system are \(e_1(0) = 8, e_2(0) = 12, e_3(0) = 4, e_4(0) = 2\). Figure 4(a)–(d) display the time response of states \(x_1, y_1, z_1, w_1\) for the drive system (20) and the states \(x_2, y_2, z_2, w_2\) for the response system (21).
In this section, the hyperchaotic Lü system (9) is controlled to be Lorenz hyperchaotic system (8). Therefore, we consider Lorenz hyperchaotic system as the drive system and Lü hyperchaotic system as the response system. The drive system is

\[
\begin{align*}
\dot{x}_1 &= a_1(y_1 - x_1) + w_1 \\
\dot{y}_1 &= -x_1z_1 + rx_1 - y_1 \\
\dot{z}_1 &= x_1y_1 - b_1z_1 \\
\dot{w}_1 &= -x_1z_1 + d_1w_1,
\end{align*}
\]

and the response system is

\[
\begin{align*}
\dot{x}_2 &= a_2(y_2 - x_2) + w_2 + \eta_1(t) \\
\dot{y}_2 &= -x_2z_2 + c_2y_2 + \eta_2(t) \\
\dot{z}_2 &= x_2y_2 - b_2z_2 + \eta_3(t) \\
\dot{w}_2 &= x_2z_2 + d_2w_2 + \eta_4(t),
\end{align*}
\]

where \(\eta_1(t), \eta_2(t), \eta_3(t)\) and \(\eta_4(t)\) are the active control functions introduced in system (31). These functions are to be determined. Let the error states between the response system (30) and the drive system (31) are

\[
e_1 = x_2 - x_1, \quad e_2 = y_2 - y_1, \quad e_3 = z_2 - z_1, \quad e_4 = w_2 - w_1.
\]

By adding Eq.(30) to Eq.(31) yields the error dynamical system between Eqs.(30) and (31)

\[
\begin{align*}
\dot{e}_1 &= a_2(e_2 - e_1) + e_4 + (a_1 - a_2)(y_1 - x_1) + \eta_1(t) \\
\dot{e}_2 &= c_2e_2 - (1 + c)y_1 + rx_1 - x_2z_2 - x_1z_1 + \eta_2(t) \\
\dot{e}_3 &= -b_2e_3 + (b_2 - b_1)z_1 + x_2y_2 + x_1y_1 + \eta_3(t) \\
\dot{e}_4 &= d_2e_4 + (d_1 - d_2)w_1 - x_2z_2 + x_1z_1 + \eta_4(t).
\end{align*}
\]

We define the active control functions \(\eta_1(t), \eta_2(t), \eta_3(t)\) and \(\eta_4(t)\) as follows:

\[
\begin{align*}
\eta_1(t) &= V_1(t) - (a_1 - a_2)(y_1 - x_1) \\
\eta_2(t) &= V_2(t) + (1 + c)y_1 - rx_1 + x_2z_2 + x_1z_1 \\
\eta_3(t) &= V_3(t) - (b_2 - b_1)z_1 - x_2y_2 - x_1y_1 \\
\eta_4(t) &= V_4(t) - (d_1 - d_2)w_1 + x_2z_2 - x_1z_1.
\end{align*}
\]

Hence the error system (33) becomes

\[
\begin{align*}
\dot{e}_1 &= a_2(e_2 - e_1) + e_4 + V_1, \quad \dot{e}_2 = c_2e_2 + V_2, \quad \dot{e}_3 = -b_2e_3 + V_3, \quad \dot{e}_4 = d_2e_4 + V_4.
\end{align*}
\]
The time derivative of the Lyapunov function (38) has the form $V(t)$ signals $(a)$ signals $x_1$ and $x_2$; $(b)$ signals $y_1$ and $y_2$; $(c)$ signals $z_1$ and $z_2$; $(d)$ signals $w_1$ and $w_2$.

Figure 5: The time response of states for drive system (31) and the response system (30) via active control.

Eq. (35) describes the error dynamics and can be considered in terms of a control problem where the system to be controlled is a linear system with a control input $V_1(t), V_2(t), V_3(t)$ and $V_4(t)$ as function of $e_1, e_2, e_3$ and $e_4$. As long as these feedbacks stabilize the system, $e_1, e_2, e_3$ and $e_4$ converge to zero as time $t$ goes to infinity. This implies that two Lü hyperchaotic systems are anti-synchronized with feedback control. There are many possible choices for the control $V_1(t), V_2(t), V_3(t)$ and $V_4(t)$. If we choose

$$V_1(t) = -a_2e_2 - e_4, \quad V_2(t) = -(1 + c)e_3, \quad V_3(t) = 0, \quad V_4(t) = -(1 + d)e_4,$$

then the error dynamical system is

$$\dot{e}_1 = -a_2e_1, \quad \dot{e}_2 = -e_2, \quad \dot{e}_3 = -b_2e_3, \quad \dot{e}_4 = -e_4.$$  \hfill (37)

Eq. (37) describes the error dynamics. Now we define the Lyapunov function for the system (37) as follows:

$$V(e_1, e_2, e_3, e_4) = \frac{1}{2}e_1^2 + \frac{1}{2}e_2^2 + \frac{1}{2}e_3^2 + \frac{1}{2}e_4^2.$$  \hfill (38)

This function is positive definite and equal to zero at the equilibrium of the system (37). Moreover, the derivative of the Lyapunov function (38) has the form

$$\dot{V} = -[ae_1^2 + e_2^2 + be_3^2 + e_4^2],$$  \hfill (39)

which is negative definite. From Lyapunov direct method, we have that the zero solution of the system (37) is asymptotically stable. This implies that the two Lü hyperchaotic systems are anti-synchronized.

### 6.1 Numerical Results

Fourth order Runge-Kutta integration method is used to solve the systems of differential equations (30) and (31). In addition, a time step size 0.001 is employed. We select the parameters of Lorenz hyperchaotic system (8) as $a = 10, b = 8/3, r = 28, d = 1.3$, and the parameters of Lü hyperchaotic system (9) as $a = 36, b = 3, c = 20$ and $d = 1.3$, so that each of them exhibits a hyperchaotic behavior. The initial values of the drive and response systems are $x_1(0) = -0.1, y_1(0) = 0.2, z_1(0) = -0.6, w_1(0) = 0.4$ and $x_2(0) = -1, y_2(0) = 0.4, z_2(0) = -0.2, w_2(0) = 1$, respectively, and the initial states of the error system are $e_1(0) = -1.1, e_2(0) = 0.6, e_3(0) = -0.8, e_4(0) = 1.4$. The diagram of the hyperchaotic Lü system is controlled to be hyperchaotic Lorenz is shown in Figure 5(a)–(d).

### 7 Conclusions

This study demonstrated that anti-synchronization can coexist in two different hyperchaotic systems ratchets moving in different asymmetric potentials by active control method. Several numerical simulations were provided to illustrate the anti-synchronization approach.
Acknowledgements
This work is financially supported by the Malaysian Ministry of Higher Education Grant: UKM–ST–06–FRGS0008–2008.

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