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A Fuzzy Programming for Optimizing Multi Response Surface in Robust Designs

Mahdi Bashiri[1,*], Seyed Javad Hosseininezhad[2]
[1] *Assistant Professor in Department of Industrial engineering, Shahed University, Tehran, Iran*
[2] *Department of Industrial engineering, Shahed University, Tehran, Iran*

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Abstract

Optimization of multi response surface (MRS) in robust designs is applied to determine optimum characteristics of a process in a satisfactory region and reduce variation of responses. In this paper, a methodology is proposed for optimizing multi response surface in robust designs, which optimizes mean and variance simultaneously by applying fuzzy set theory. At first, a method is proposed to constitute a regression model based on replicates of a response and aggregate regression models so that a fuzzy regression model expresses each response. The obtained regression model includes fuzzy coefficients which consider uncertainty in the collected data. Then a multi objective decision making (MODM) problem is used. After introducing deviation function based on robustness concept and using desirability function, a two objective problem is constituted. At last, a fuzzy programming method is expressed to solve the problem by applying degree of satisfaction from each objective. Then the problem is converted to a single objective model with the goals of increasing desirability and robustness simultaneously. The obtained model has the capability of assigning importance weight for each goal based on decision making preferences. Solving the model, the obtained optimum factor levels are fuzzy numbers so that a bigger satisfactory region could be provided. Finally, a numerical example is expressed to illustrate the proposed methodology.

Keywords: MRS, fuzzy regression model, desirability function, MODM, fuzzy programming

1 Introduction

Response surface methodology (RSM) is a collection of tools for fitting a surface to special set of data, and determining optimum factor levels is also part of methodology and uses a regression model for optimization of a problem [12, 14].

Kim and Lin [9] proposed a fuzzy modeling approach to optimize the dual response system. The proposed method identifies a set of process parameter conditions to simultaneously maximize the degree of satisfaction with respect to the mean and the standard deviation responses. Venter and Haftka [16] used fuzzy concepts for modeling uncertainty typical of aircraft industry. It is shown that, for the example problem considered, the fuzzy set based design is superior. Akpan *et al.* [1] proposed a practical approach for analyzing the response of structures with fuzzy parameters. RSM is used to approximate the fuzzy element response quantity at the normal point. De Munck *et al.* [5] considered a response surface based optimization techniques for the calculation of envelope frequency response function (FRFs) of imprecisely structures using the interval and fuzzy finite element method.

In practice, multiple responses and different types of scenarios are usually applied [2, 4, 17]. Derringer and Suich [6] defined a desirability function to transform several response variables into a single response. Khuri and Conlon [8] proposed an algorithm for the simultaneous optimization of several response functions that depend on the same set of controllable variables and are adequately represented by polynomial regression models. They firstly defined a distance function by considering the ideal solution, and then determined the optimal condition by minimizing this function. Tong and Hsieh [15] applied artificial neural network (ANN) to find the optimal solution to the multi response type of problem. Antony [3] utilized principle component analysis (PCA) to analyze multi response problems. Kazemzadeh *et al.* [7] proposed a general framework in MRS problems according to some

[*] Corresponding author. Email: Bashiri@shahed.ac.ir (M. Bashiri).

existing works and some types of related decision makers and attempts to aggregate all of characteristics in one approach.

Lai and Chang [10] proposed a fuzzy multi response optimization procedure to search an appropriate combination of process parameter setting. At first, fuzzy regression models are used to model the relations between process parameters and responses, and then the possibility distributions of prediction responses are obtained. They proposed a strategy of optimizing the most possible response values and minimizing the deviation from the most possible values.

Optimization of multi response surface (MRS) in robust designs is applied to determine optimum characteristics of a process in a satisfactory region and reduce variation of responses. This paper proposes a methodology for optimizing multi response surface in robust designs which optimizes mean and variance simultaneously by applying fuzzy set theory. Since in such a problem some of data are usually neglected, we cover it by applying fuzzy set theory. At first, a method is proposed to constitute a regression model based on replicates of a response and aggregate regression models so that a fuzzy regression model expresses each response. Triangular fuzzy number (TFN) is applied for considering mean and variance of data simultaneously. The obtained regression model includes fuzzy coefficients which consider uncertainty in the collected data. After introducing deviation function based on robustness concept and using desirability function, a two objective problem is constituted. At last, a fuzzy programming is expressed to solve the problem which applies degree of satisfaction from each objective. The problem is converted to a single objective model with the goals of increasing desirability and robustness simultaneously and capability of assigning importance weight for each one based on decision making preferences. After solving the model, obtained optimum factor levels are fuzzy numbers so that a bigger satisfactory region could be provided.

The paper is organized as follows. Section 2 reviews the fuzzy set theory. Section 3 introduces the proposed methodology. In Section 4, a numerical example is given to illustrate the proposed methodology. Then based on the numerical example and comparison by other works, some usability of the proposed methodology are represented. Finally, Section 5 draws the conclusions.

2 Fuzzy Set Theory

A fuzzy set $\tilde{A}$ in a universe of discourse ***X*** is characterized by a membership function $\mu_{\tilde{A}}(x)$ which associates with each element *x* in ***X*** a real number in the interval [0,1]. The function value $\mu_{\tilde{A}}(x)$ is termed the grade of membership of *x* in $\tilde{A}$ [18].

A triangular fuzzy number $\tilde{A}$ can be defined by a triplet (*l*, *m*, *u*) shown in Fig.1. The membership function $\mu_{\tilde{A}}(x)$ is defined as:

$$\mu_{\tilde{A}}(x)=\begin{cases}0, & x<l\\ \dfrac{x-l}{m-l}, & l\le x\le m\\ \dfrac{u-x}{u-m}, & m\le x\le u\\ 0, & x>u.\end{cases} \tag{1}$$

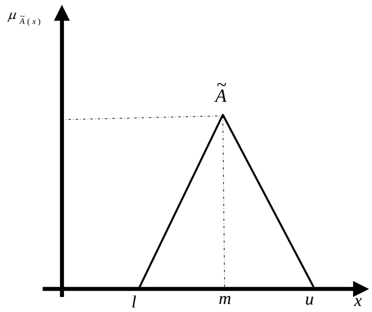

Figure1: A triangular fuzzy number $\tilde{A}$

Let $\tilde{A}=(a,b,c)$ and $\tilde{B}=(d,e,f)$ be two triangular fuzzy numbers. The operations are expressed as follows:

$$\tilde{A}\oplus\tilde{B}=(a,b,c)\oplus(d,e,f)=(a+d,b+e,c+f), \tag{2}$$

$$\tilde{A} - \tilde{B} = (a,b,c) - (d,e,f) = (a - f, b - e, c - d), \tag{3}$$

$$\tilde{A} \otimes \tilde{B} = (a,b,c).(d,e,f) = (a.d, b.e, c.f), \tag{4}$$

$$\tilde{A} / \tilde{B} = (a,b,c)/(d,e,f) = \left(\frac{a}{f}, \frac{b}{e}, \frac{c}{d}\right). \tag{5}$$

3 Proposed Methodology

In this section, steps of proposed methodology are introduced. When the experiments are done with replicates, mean and variance of collected data are considered to obtain optimum factor levels. Since in such a problem some of data are usually neglected, we cover it by applying Triangular Fuzzy Number (TFN) for considering mean and variance of data simultaneously. In robust designs optimum characteristics of a process in a satisfactory region are determined so as to reduce variation of responses, this concept is investigated for introducing deviation function. Then using desirability function, a two objective problem is constituted. Applying degree of satisfaction from each objective, the problem is converted to a single objective model. The algorithm of the proposed methodology is shown in Fig.2, steps of the approach are as following:

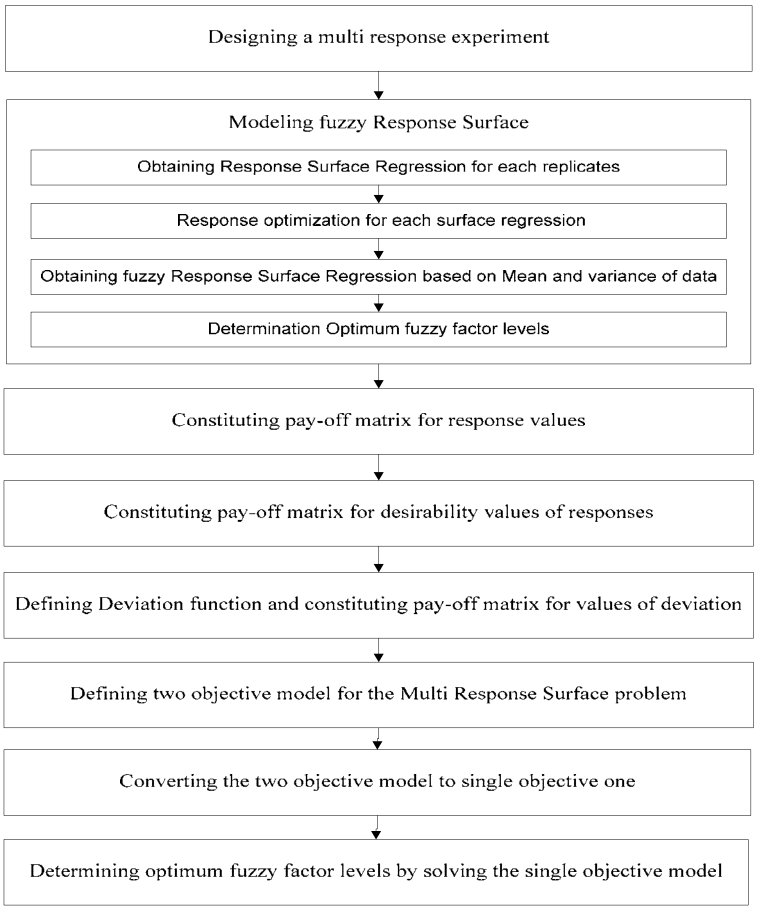

Figure 1: The algorithm of the proposed methodology

Step 1: Designing a multi response experiment.

A multi response experiment is designed which related to a process including more than one response with replicates shown in Table 1, where x_{ij} is the jth factor level value and y_{ikr} is kth response value for the rth replicates in ith experiment respectively.

Step 2: Modeling fuzzy response surface.

For kth response following steps are done,

2.1: Obtain response surface regression for *r*th replicates.

The regression model is

$$Y_r^k = \beta_0^r + \sum_{i=1}^{m} \beta_i x_i + \sum_{i=1}^{m} \beta_{ii}^r x_i^2 + \sum_{i<j}\sum \beta_{ij}^r x_i x_j + \varepsilon, r = 1,2,...,R, \tag{6}$$

where Y_r^k expresses *r*th response surface regression model for *k*th response which is obtained based on experimental data and ε represents the noise or error observed in the response value.

Table 1: Results of experiments for a multi response process

Run order	Factor levels			Responses						
	X_1	…	X_K	Y_1			…	Y_m		
				y_{i11}	…	y_{i1R}	…	Y_{im1}	…	Y_{imR}
1	x_{11}	...	x_{1K}	y_{111}	…	y_{11R}	…	y_{1m1}	…	Y_{1mR}
…	…	…	…	…	…	…	…	…	…	…
n	x_{n1}	…	x_{nK}	y_{n11}	…	y_{n1R}	…	y_{nm1}	…	y_{nmR}

2.2: Optimize response for each surface regression.

Let x_{rj}^* represent the optimum *j*th factor level for *r*th surface regression.

2.3: Obtain fuzzy response surface regression based on mean and variance of coefficient of the obtained surface regressions.

Then

$$\tilde{Y}^k = \tilde{\beta}_0 + \sum_{i=1}^{m} \tilde{\beta}_i x_i + \sum_{i=1}^{m} \tilde{\beta}_{ii} x_i^2 + \sum_{i<j}\sum \tilde{\beta}_{ij} x_i x_j + \varepsilon, \tag{7}$$

where $\tilde{Y}^k$ expresses fuzzy response surface regression model for *k*th response.

For obtaining fuzzy coefficients a procedure is applied as follows:

If $\beta_1,...,\beta_R$ are crisp values, then

2.3.1. Calculate mean of $\beta_1,...,\beta_R$, so $\beta^m = \text{Mean}(\beta_1,...\beta_R)$.

2.3.2. Calculate standard deviation of $\beta_1,...,\beta_R$.

2.3.3. $\beta^l = \text{Mean}(\beta_1,...\beta_R) - \text{Standard Deviation}(\beta_1,...\beta_R)$.

2.3.4. $\beta^u = \text{Mean}(\beta_1,...\beta_R) + \text{Standard Deviation}(\beta_1,...\beta_R)$.

2.3.5. Let $\tilde{\beta} = (\beta^l, \beta^m, \beta^u)$.

2.4: Determine optimum fuzzy factor levels.

Based on results of Step 2.2, the optimum factor levels for *k*th response are $x_{1jk}^*,...,x_{Rjk}^*$, we apply the procedure mentioned in Section 2.3 to obtain $\tilde{x}_{jk}^*$, $\tilde{Y}^{k^*}$, which are optimum fuzzy *j*th factor level and optimum fuzzy *k*th response value, respectively.

Step 3: Constituting pay-off matrix for response values.

Let $\tilde{X}^{(k)}, k = 1,...,m$ be the optimum fuzzy factor levels of the *k*th response surface and $\tilde{Y}_{ij}(X)$ be *j*th response value by replacing the optimum fuzzy factor levels of the *i*th response surface. Then the pay-off matrix for response values is constituted in Table 2.

Table 2: Pay-off matrix for response values

	$\tilde{Y}_1(X)$	$\cdots$	$\tilde{Y}_m(X)$
$\tilde{X}^{(1)}$	$\tilde{Y}_{11}(X)$	$\cdots$	$\tilde{Y}_{1m}(X)$
$\vdots$	$\vdots$	$\ddots$	$\vdots$
$\tilde{X}^{(m)}$	$\tilde{Y}_{m1}(X)$	$\cdots$	$\tilde{Y}_{mm}(X)$

Step 4: Constituting pay-off matrix for desirability values of responses.

Derringer and Suich [6] introduced desirability function for optimization of multi responses. For the Nominal-The-Best (NTB), Larger-The-Best (LTB), Smaller-The-Best (STB) responses, the desirability functions are

$$d_i = \begin{cases} \left(\dfrac{\hat{y}_i - y_{\min}}{T - y_{\min}}\right)^s, & y_{\min} \le \hat{y}_i \le T, s \ge 0 \\ \left(\dfrac{\hat{y}_i - y_{\max}}{T - y_{\max}}\right)^t, & T \le \hat{y}_i \le y_{\max}, t \ge 0 \\ 0, & \hat{y}_i < y_{\min} \quad or \quad \hat{y}_i > y_{\max}, \end{cases} \tag{8}$$

$$d_i = \begin{cases} 0, & \hat{y}_i \le y_{\min} \\ \left(\dfrac{\hat{y}_i - y_{\min}}{y_{\max} - y_{\min}}\right)^r, & y_{\min} \le y_i \le y_{\max}, r \ge 0 \\ 1, & \hat{y}_i \ge y_{\max}, \end{cases} \tag{9}$$

$$d_i = \begin{cases} 1, & \hat{y}_i \le y_{\min} \\ \left(\dfrac{\hat{y}_i - y_{\max}}{y_{\min} - y_{\max}}\right)^r, & y_{\min} \le y_i \le y_{\max}, r \ge 0 \\ 0, & \hat{y}_i \ge y_{\max}, \end{cases} \tag{10}$$

where $\hat{y}_i$, d_i, y_{min}, y_{max} and T are predicted value, desirability value, lower limit, upper limit and target of *i*th response, respectively. Meanwhile *s*, *t* and *r* are weights specified by the decision maker.

After calculating desirability values of each response using (8)-(10), if $\tilde{X}^{(k)}, k = 1,...,m$ are the optimum fuzzy factor levels of *k*th response surface and $\tilde{d}_{ij}(X)$ is desirability value of *j*th response value by replacing the optimum fuzzy factor levels of *i*th response surface, then pay-off matrix for desirability values is constituted in Table 3. Thus $\tilde{U}_k = (U_k^l, U_k^m, U_k^u) = \tilde{d}_{kk}$, $\tilde{L}_k = (L_k^l, L_k^m, L_k^u) = Min(\tilde{d}_{1k},..., \tilde{d}_{mk})$.

Table 3: Pay-off matrix for desirability values

	$\tilde{d}_1(X)$	$\cdots$	$\tilde{d}_m(X)$
$\tilde{X}^{(1)}$	$\tilde{d}_{11}(X)$	$\cdots$	$\tilde{d}_{1m}(X)$
$\vdots$	$\vdots$	$\ddots$	$\vdots$
$\tilde{X}^{(m)}$	$\tilde{d}_{m1}(X)$	$\cdots$	$\tilde{d}_{mm}(X)$

Step 5: Defining Deviation function and constituting pay-off matrix for values of deviation.

In this step, we introduce deviation function. If $\tilde{Y}_k = (Y_k^l, Y_k^m, Y_k^u)$, then let

$$D_k = Y_k^u - Y_k^m \quad k = 1,...,m. \tag{11}$$

For the deviation function of the *k*th response, it is desirable to decrease it so as to robust the experiments. So pay-off matrix for deviation values is constituted in Table 4. Thus $\tilde{P}_k = (P_k^l, P_k^m, P_k^u) = \tilde{D}_{kk}$, $\tilde{Q}_k = (Q_k^l, Q_k^m, Q_k^u) = Max(\tilde{D}_{1k},..., \tilde{D}_{mk})$.

Table 4: Pay-off matrix for deviation values

	$D_1(X)$	$\cdots$	$D_m(X)$
$\tilde{X}^{(1)}$	$\tilde{D}_{11}(X)$	$\cdots$	$\tilde{D}_{1m}(X)$
$\vdots$	$\vdots$	$\ddots$	$\vdots$
$\tilde{X}^{(m)}$	$\tilde{D}_{m1}(X)$	$\cdots$	$\tilde{D}_{mm}(X)$

Step 6: Defining two objective model for the multi response surface problem.

The MRS problem deals with a multi objective decision making (MODM), so we consider a fuzzy MODM problem in this paper. Solving these problems is attended by Lai and Hwang [11]. Final model is a two objective model expressed as

$$\begin{aligned} &Max \quad \{Fuzzy \quad DesirabilityFunction\} \\ &Min \quad \{DeviationFunction\} \\ &s.t. \qquad X \in R_{(Factor \quad Levels)}. \end{aligned} \tag{12}$$

The first objective is introduced from Step 4 and the last one is from Step 5, and $R_{(Factor \quad Levels)}$ denotes acceptable region of experiments. For example, it could be [-1,1].

Step 7: Converting the two objective models to single objective one.

For converting the model to single objective, we introduce two functions $\tilde{S}(X) = (S^l(X), S^m(X), S^u(X))$ and $\tilde{T}(X) = (T^l(X), T^m(X), T^u(X))$, which indicate the degrees of satisfaction from desirability and robustness, respectively.

If $\tilde{d}_k(X) = (d_k^l, d_k^m, d_k^u)$, then for the *k*th response, we have

$$S_k^{l,m,u}(X) = \begin{cases} 0, & d_k^{l,m,u}(X) \le L_k^{l,m,u} \\ \dfrac{d_k^{l,m,u}(X) - L_k^{l,m,u}}{U_k^{l,m,u} - L_k^{l,m,u}}, & L_k^{l,m,u} \le d_k^{l,m,u}(X) \le U_k^{l,m,u} \\ 1, & d_k^{l,m,u}(X) \ge U_k^{l,m,u}, \end{cases} \tag{13}$$

$$T_k^{l,m,u}(X) = \begin{cases} 1, & D_k^{l,m,u}(X) \le P_k^{l,m,u} \\ \dfrac{Q_k^{l,m,u} - D_k^{l,m,u}(X)}{Q_k^{l,m,u} - P_k^{l,m,u}}, & P_k^{l,m,u} \le D_k^{l,m,u}(X) \le Q_k^{l,m,u} \\ 0, & D_k^{l,m,u}(X) \ge Q_k^{l,m,u}. \end{cases} \tag{14}$$

So, it is desirable to maximize these two functions S_k^l, T_k^l, then S_k^m, T_k^m, and finally S_k^u, T_k^u separately to obtain $\tilde{S}_k = (S_k^l, S_k^m, S_k^u), \tilde{T}_k = (T_k^l, T_k^m, T_k^u)$. For this purpose, we solve

$$\begin{aligned} &Max \quad \tilde{S}_k(X) \quad k = 1,...,m \\ &Max \quad \tilde{T}_k(X) \quad k = 1,...,m \\ &s.t. \quad X \in R_{(Factor \quad Levels)}. \end{aligned} \tag{15}$$

Finally we apply Zimmerman Max-Min operator to convert *m* objectives to one, which maximizes the minimum degree of satisfaction from m objectives.

Let $Min \quad \tilde{S}_k(X) = \tilde{V}_1 = (V_1^l, V_1^m, V_1^u)$, $Min \quad \tilde{T}_k(X) = \tilde{V}_2 = (V_2^l, V_2^m, V_2^u)$. Then our problem is formulated as follows:

$$\begin{aligned} &Max \quad \tilde{V}_1 \\ &Max \quad \tilde{V}_2 \\ &s.t. \quad X \in R_{(Factor \quad Levels)}. \end{aligned} \tag{16}$$

So,

$$V_1^{l,m,u} \le \frac{d_k^{l,m,u}(X) - L_k^{l,m,u}}{U_k^{l,m,u} - L_k^{l,m,u}} \Rightarrow d_k^{l,m,u}(X) - L_k^{l,m,u} \ge V_1^{l,m,u}(U_k^{l,m,u} - L_k^{l,m,u}) \Rightarrow d_k^{l,m,u}(X) - V_1^{l,m,u}(U_k^{l,m,u} - L_k^{l,m,u}) \ge L_k^{l,m,u},$$

which guarantees $V_1^{l,m,u}$ is the lower bound for the degree of satisfaction from *k*th desirability function. By doing the same calculation, $V_2^{l,m,u}$ is the lower bound for the degree of satisfaction from *k*th deviation function.

Finally, if w_1 and w_2 indicate importance weights for desirability and robustness expressed by user, the final model with one objective is introduced as follows:

$$\begin{aligned} &Max \quad w_1 V_1^{l,m,u} + w_2 V_2^{l,m,u} \\ &s.t. \quad d_k^{l,m,u}(X) - V_1^{l,m,u}(U_k^{l,m,u} - L_k^{l,m,u}) \ge L_k^{l,m,u} \quad, k = 1,...,m \\ &\qquad D_k^{l,m,u}(X) + V_2^{l,m,u}(Q_k^{l,m,u} - P_k^{l,m,u}) \le Q_k^{l,m,u} \quad, k = 1,...,m \\ &\qquad w_1 + w_2 = 1 \\ &\qquad 0 \le V_1^{l,m,u}, V_2^{l,m,u} \le 1, X \in R_{(Factor \quad Levels)}. \end{aligned} \tag{17}$$

The final model is categorized to 3 types of models *l*, *m* and *u* and these models are solved separately.

Step 8: Determining optimum fuzzy factor levels by solving the single objective model.

After solving the model for *l, m, and u* separately, the optimum factor levels are obtained as $X_l^* = (x_1^{l*},...,x_K^{l*}),\ X_m^* = (x_1^{m*},...,x_K^{m*}), X_u^* = (x_1^{u*},...,x_K^{u*})$, where *K* is the number of factor levels.

So the optimum fuzzy factor levels are $\tilde{X}^* = (\tilde{x}_1,....,\tilde{x}_K) = \left((x_1^{l*}, x_1^{m*}, x_1^{u*}),...,(x_K^{l*}, x_K^{m*}, x_K^{u*})\right)$.

4 Numerical Example

Step1: Pignatiello [13] investigated a process, in which there are two response variables $\boldsymbol{Y_1}$, $\boldsymbol{Y_2}$ and three control factors $\boldsymbol{x_1}$, $\boldsymbol{x_2}$ and $\boldsymbol{x_3}$. It is assumed that the targets of the responses are 103 and 73 and that the specification regions are (97,109), (70, 76) for $\boldsymbol{Y_1}$, $\boldsymbol{Y_2}$, respectively. It was agreed that a simultaneous maximization of $\boldsymbol{Y_1}$ and $\boldsymbol{Y_2}$ would be desirable. The experimental data are given in Table 5.

Step2: For modeling the fuzzy response surface, we apply the following steps for two responses as mentioned before.

2.1: For the first response, 4 response surface regressions based on 4 replicates are obtained by using **MINITAB14** as follows:

$$Y_1^1 = 105.016 - 3.142x_1 - 0.567x_2 - 0.055x_3 + 2.980x_1x_2 - 0.227x_1x_3,$$
$$Y_2^1 = 105.288 - 3.082x_1 + 0.320x_2 + 0.329x_3 + 2.390x_1x_2 - 0.341x_1x_3,$$
$$Y_3^1 = 105.494 - 3.719x_1 - 1.174x_2 + 0.282x_3 + 1.393x_1x_2 - 0.743x_1x_3,$$
$$Y_4^1 = 103.671 - 2.647x_1 + 0.852x_2 - 1.355x_3 + 2.754x_1x_2 - 0.090x_1x_3.$$

The R-Sq of these regression models are 93.0%, 99.9%, 92.1% and 91.7%, respectively.

Table 5: The experimental data of example

Run order	Factor levels			Responses							
	x_1	x_2	x_3	Y_1				Y_2			
				Rep. 1	*Rep. 2*	*Rep. 3*	*Rep. 4*	*Rep. 1*	*Rep. 2*	*Rep. 3*	*Rep. 4*
1	-1	-1	-1	109.895	109.759	110.704	109.773	67.6974	67.2374	67.9620	66.9268
2	1	-1	-1	100.192	99.634	100.269	100.600	67.0264	66.1779	66.5758	67.9431
3	-1	1	-1	106.078	105.642	105.670	105.393	72.9353	72.8508	72.5768	72.3754
4	1	1	-1	104.120	104.802	104.203	104.335	72.9878	74.2487	73.9371	73.2824
5	-1	-1	1	113.515	111.121	112.854	106.666	68.2934	68.4693	68.9576	64.7051
6	1	-1	1	98.732	99.357	102.842	94.235	67.0955	63.6112	68.6470	62.4188
7	-1	1	1	103.145	106.959	107.620	103.440	71.6818	76.2657	77.4958	76.3739
8	1	1	1	104.454	105.029	99.786	104.923	76.9003	77.0322	67.9890	75.7691

2.2: For each response surface regression, we obtain optimum factor levels:

$$x_{111}^* = 0.914, x_{121}^* = 0.275, x_{131}^* = -1, x_{211}^* = 0.319, x_{221}^* = -1, x_{231}^* = -1,$$
$$x_{311}^* = 0.656, x_{321}^* = 1, x_{331}^* = -1, x_{411}^* = 0, x_{421}^* = 0, x_{431}^* = 0.495.$$

2.3: Applying the proposed procedure, we obtained fuzzy response surface:

$$\tilde{Y}^1 = (104.046, 104.867, 105.688) + (-3.587, -3.148, -2.708)x_1 + (-1.045, -0.142, 0.761)x_2$$
$$+ (-0.988, -0.2, 0.589)x_3 + (1.687, 2.379, 3.080)x_1x_2 + (-0.631, -0.350, -0.069)x_1x_3.$$

For example, to obtain $\tilde{\beta}_0$,

2.3.1. $\beta_0^m = \text{Mean}(105.016, 105.288, 105.494, 103.671)$.

2.3.2. $\text{Standard Deviation}(105.016, 105.288, 105.494, 103.671) = 0.821$.

2.3.3 $\beta^l = \text{Mean}(105.016, 105.288, 105.494, 103.671) - \text{Standard Deviation}(105.016, 105.288, 105.494, 103.671) = 104.046.$

2.3.4. $\beta^u = \text{Mean}(105.016, 105.288, 105.494, 103.671) + \text{Standard Deviation}(105.016, 105.288, 105.494, 103.671) = 105.688$.

2.3.5. $\tilde{\beta}_0 = (104.046, 104.867, 105.688)$.

2.4: Applying the proposed procedure, optimum fuzzy factor levels are as follows

$$\tilde{x}_{11}^* = (0.0743, 0.472, 0.871), \tilde{x}_{21}^* = (-0.7592, 0.0689, 0.897), \tilde{x}_{31}^* = (-1, -0.063, 0.121).$$

By substituting the fuzzy factor levels in the fuzzy response surface regression, the value of optimum fuzzy response is $\tilde{Y}^{1^*} = (103.676, 105.513, 106.482)$.

After similar computations of Step 2 for the second response, we have

$$\tilde{Y}^2 = (70.120, 70.452, 70.783) + (-1.042, -0.349, 0.345)x_1 + (2.609, 3.592, 4.576)x_2$$
$$+ (-0.046, 0.280, 0.606)x_3 + (-0.511, 0.323, 1.158)x_1x_2 + (-1.2012, -0.450, 0.301)x_1x_3,$$

$$\tilde{x}_{12}^* = (-0.0001, -0.0005, 0.0005), \tilde{x}_{22}^* = (0.432, 0.644, 0.856), \tilde{x}_{32}^* = (-0.5, 0.5, 1), \tilde{Y}^{2^*} = (71.272, 72.906, 75.305).$$

Step 3: Pay-off matrix for response values is shown in Table 6.

Table 6: Pay-off matrix for response values

	$\tilde{Y}_1(X)$	$\tilde{Y}_2(X)$
$\tilde{X}^{(1)}$	(103.676,105.513,106.482)	(68.226,70.502,76.196)
$\tilde{X}^{(2)}$	(104.092,104.676,106.929)	(71.272,72.906,75.305)

Step 4: Since the responses are Nominal-The-Best (NTB), the desirability functions are calculated by (8). So

$$d_1 = \begin{cases} \left(\dfrac{\hat{y}_1 - 97}{103-97}\right)^s, & 97 \le \hat{y}_1 \le 103, s \ge 0 \\ \left(\dfrac{\hat{y}_1 - 109}{103-109}\right)^t, & 103 \le \hat{y}_1 \le 109, t \ge 0 \\ 0, & \hat{y}_1 < 103 \quad or \quad \hat{y}_1 > 109, \end{cases} \quad d_2 = \begin{cases} \left(\dfrac{\hat{y}_2 - 70}{73-70}\right)^s, & 70 \le \hat{y}_2 \le 73, s \ge 0 \\ \left(\dfrac{\hat{y}_2 - 76}{73-76}\right)^t, & 73 \le \hat{y}_2 \le 76, t \ge 0 \\ 0, & \hat{y}_2 < 73 \quad or \quad \hat{y}_2 > 76. \end{cases}$$

After calculating desirability values of each response, pay-off matrix for desirability values is given in Table 7.

Table 7: Pay-off matrix for desirability values

	$\tilde{d}_1(X)$	$\tilde{d}_2(X)$
$\tilde{X}^{(1)}$	(0.420,0.581,0.887)	(0,0,0.167)
$\tilde{X}^{(2)}$	(0.345,0.721,0.818)	(0.232,0.424,0.969)

Table 8: Pay-off matrix for deviation values

	$D_1(X)$	$D_2(X)$
$\tilde{X}^{(1)}$	(-0.681,0.537,2.687)	(0.450,0.523,1.701)
$\tilde{X}^{(2)}$	(0.817,1.797,2.383)	(0.593.1.127,1.500)

Then,

$$\tilde{U}_1 = (U_1^l, U_1^m, U_1^u) = \tilde{d}_{11} = (0.420,0.581,0.887), \quad \tilde{L}_1 = (0.0.345,0.581,0.818),$$
$$\tilde{U}_2 = (0.232,0.424,0.969), \quad \tilde{L}_2 = (0,0,0.167).$$

Step 5:

$$D_1 = 0.821 + 0.440x_1 + 0.903x_2 + 0.789x_3 + 0.701x_1x_2 + 0.281x_1x_3,$$
$$D_2 = 0.331 + 0.694x_1 + 0.984x_2 + 0.326x_3 + 0.834x_1x_2 + 0.751x_1x_3.$$

So pay-off matrix for deviation values is shown in Table 8. Then, we have

$\tilde{P}_1 = (-0.681,0.537,2.687)$, $\tilde{Q}_1 = (0.817,1.797,2.383)$, $\tilde{P}_2 = (0.450,0.523,1.500)$, $\tilde{Q}_2 = (0.593,1.127,1.701)$.

Step 6: The final model is a two objective model expressed as

$$\begin{aligned} &Max \quad \{\tilde{d}_1(X), \tilde{d}_2(X)\} \\ &Min \quad \{D_1(X), D_2(X)\} \\ &s.t. \quad X = \{\tilde{x}_1, \tilde{x}_2, \tilde{x}_3\} \in [-1,1]. \end{aligned}$$

Step 7: Based on results of Steps 4 and 5, we have

$\tilde{U}_1 = (0.420,0.581,0.887)$, $\tilde{L}_1 = (0.0.345,0.581,0.818)$, $\tilde{U}_2 = (0.232,0.424,0.969)$, $\tilde{L}_2 = (0,0,0.167)$.

Applying (13),

$$S_1^{l,m,u}(X) = \begin{cases} 0, & d_1^{l,m,u}(X) \le (0.345,0.581,0.818) \\ \dfrac{d_1^{l,m,u}(X) - (0.345,0.581,0.818)}{(0.420,0.721,0.887) - (0.345,0.581,0.818)}, & (0.345,0.581,0.818) \le d_1^{l,m,u}(X) \le (0.420,0.721,0.887) \\ 1, & d_1^{l,m,u}(X) \ge (0.420,0.721,0.887), \end{cases}$$

$$S_2^{l,m,u}(X)=\begin{cases}0, & d_2^{l,m,u}(X)\le(0,0,0.167)\\ \dfrac{d_2^{l,m,u}(X)-(0,0,0.167)}{(0.232,0.424,0.969)-(0,0,0.167)}, & (0,0,0.167)\le d_2^{l,m,u}(X)\le(0.232,0.424,0.969)\\ 1, & d_2^{l,m,u}(X)\ge(0.232,0.424,0.969),\end{cases}$$

and

$$\tilde{P}_1=(-0.681,0.537,2.687),\quad \tilde{Q}_1=(0.817,1.797,2.383),\tilde{P}_2=(0.450,0.523,1.500),\quad \tilde{Q}_2=(0.593,1.127,1.701).$$

Applying (14),

$$T_1^{l,m,u}(X)=\begin{cases}1, & D_1^{l,m,u}(X)\le(-0.681,0.537,2.687)\\ \dfrac{(0.817,0.1797,2.383)-D_1^{l,m,u}(X)}{(0.817,0.1797,2.383)-(-0.681,0.537,2.687)}, & (-0.681,0.537,2.687)\le D_1^{l,m,u}(X)\le(0.817,0.1797,2.383)\\ 0, & D_1^{l,m,u}(X)\ge(0.817,0.1797,2.383),\end{cases}$$

$$T_2^{l,m,u}(X)=\begin{cases}1, & D_2^{l,m,u}(X)\le(0.450,0.523,1.500)\\ \dfrac{(0.593,1.127,1.701)-D_2^{l,m,u}(X)}{(0.593,1.127,1.701)-(0.450,0.523,1.500)}, & (0.450,0.523,1.500)\le D_2^{l,m,u}(X)\le(0.593,1.127,1.701)\\ 0, & D_2^{l,m,u}(X)\ge(0.593,1.127,1.701).\end{cases}$$

So

$$\begin{aligned}&Max\quad \{\tilde{S}_1(X),\tilde{S}_1(X)\}\\ &Max\quad \{\tilde{T}_1(X),\tilde{T}_2(X)\}\\ &s.t.\qquad X=\{\tilde{x}_1,\tilde{x}_2,\tilde{x}_3\}\in[-1,1].\end{aligned}$$

Then, the problem is formulated as follows:

$$\begin{aligned}&Max\quad \tilde{V}_1\\ &Max\quad \tilde{V}_2\\ &s.t.\qquad X=\{\tilde{x}_1,\tilde{x}_2,\tilde{x}_3\}\in[-1,1].\end{aligned}$$

Finally, if $w_1=w_2=0.5$, using (17) the final model is categorized to 3 types: model *l*, model *m* and model *u*. Model *l* consists of variables and constants shown by index *l* and so on.

Model *l:*

$$\begin{aligned}Max\quad & 0.5\times V_1^l+0.5\times V_2^l\\ s.t.\quad & d_1^l(X)-V_1^l(0.420-0.345)\ge 0.345\\ & d_2^l(X)-V_2^l(0.232-0)\ge 0\\ & D_1^l(X)+V_2^l(0.817-0.681)\le 0.817\\ & D_2^l(X)+V_2^l(0.593-0.450)\le 0.593\\ & w_1+w_2=1\\ & 0\le V_1^l,V_2^l\le 1, X=\{x_1^l,x_2^l,x_3^l\}\in[-1,1].\end{aligned}$$

Model *m:*

$$\begin{aligned}Max\quad & 0.5\times V_1^m+0.5\times V_2^m\\ s.t.\quad & d_1^m(X)-V_1^m(0.581-0.721)\ge 0.721\\ & d_2^m(X)-V_2^m(0.424-0)\ge 0\\ & D_1^m(X)+V_2^m(1.797-0.534)\le 1.797\\ & D_2^m(X)+V_2^m(1.127-0.523)\le 1.127\\ & w_1+w_2=1\\ & 0\le V_1^m,V_2^m\le 1, X=\{x_1^m,x_2^m,x_3^m\}\in[-1,1].\end{aligned}$$

Model *u:*

$$\begin{aligned}Max\quad & 0.5\times V_1^u+0.5\times V_2^u\\ s.t.\quad & d_1^u(X)-V_1^u(0.887-0.818)\ge 0.818\\ & d_2^u(X)-V_2^u(0.967-0.167)\ge 0.167\\ & D_1^u(X)+V_2^u(2.383-2.687)\le 2.383\\ & D_2^u(X)+V_2^u(1.701-1.450)\le 1.450\\ & w_1+w_2=1\\ & 0\le V_1^u,V_2^u\le 1, X=\{x_1^u,x_2^u,x_3^u\}\in[-1,1].\end{aligned}$$

Step 8: Using microsoft excel solver, which is shown in Fig. 3, the optimum factor levels of model *l*, m and *u* are

$$X_l^*=(x_1^{l*},x_2^{l*},x_3^{l*})=(-0.645,0.215,-1), X_m^*=(x_1^{m*},x_2^{m*},x_3^{m*})=(-0.578,1,-0.843), X_u^*=(x_1^{u*},x_2^{u*},x_3^{u*})=(-1,1,-1).$$

Thus, the optimum fuzzy factor levels are

$$\tilde{X}^*=(\tilde{x}_1^*,\tilde{x}_2^*,\tilde{x}_3^*),\tilde{x}_1^*=(-1,-0.645,-0.578),\tilde{x}_2^*=(0.215,1,1),\tilde{x}_3^*=(-1,-1,-0.843).$$

	A	B	C	D	E	F	G	H	I	J	K	L	M	N
1														
2														
3														
4		Factor Level	Min	Max		Target 1	0.7635		Target 2	1.0000		Target 3	0.5001	
5		x1	-1	1		x1	-0.6448		x1	-0.5784		x1	-1.0000	
6		x2	-1	1		x2	0.2148		x2	1.0000		x2	1.0000	
7		x3	-1	1		x3	-1.0000		x3	-0.8433		x3	-1.0000	
8						v1	1.0000		v1	1.0000		v1	0.0001	
9		Deg. of Sat.	Min	Max		v2	0.5270		v2	1.0000		v2	1.0000	
10		v1	0	1										
11		v2	0	1		Model *l*	RHS		Model *m*	RHS		Model *u*	RHS	
12						0.3453	Constraint1		0.7781	Constraint1		0.5968	Constraint1	
13		w1	0.5			0.0000	Constraint2		0.7770	Constraint2		0.8159	Constraint2	
14		w2	0.5			0.8165	Constraint3		1.7968	Constraint3		-0.2266	Constraint3	
15						0.2139	Constraint4		1.1268	Constraint4		0.4136	Constraint4	
16														
17														

Figure 3: Output of microsoft excel solver

Based on the above numerical example and comparison the proposed methodology by other works, some usability of it is expressed as follows:

1. In dual response, as a MRS optimization method, based on type of response, set mean a distinct value or range and then minimize variation. But in the proposed methodology, we optimize mean and variance simultaneously and not set mean a distinct value or range.
2. Since each replicates could be considered as a separate response, the proposed approach constitutes a regression model for them and aggregates regression models for each response by applying fuzzy concepts as a tool for considering mean and variance.
3. The proposed model by showing factor levels as a fuzzy number has a capability to change them at an acceptable range based on experimental conditions.
4. The proposed methodology by using fuzzy response surface regression model considers uncertainty of data collected and statistical errors of prediction, so it lead to a bigger decision making region for optimization problem.
5. The proposed approach considers decision making preferences for optimizing mean (desirability function) or variance (deviation function) by assigning importance weight to them in final model.

6 Conclusion

Optimization of multi response surface (MRS) in robust designs is applied to determine optimum characteristics of a process in a satisfactory region and reduce variation of responses. In this paper, a methodology was proposed for optimizing multi response surface in robust designs as a MODM problem by introducing deviation function based on robustness concept and using desirability function. The proposed approach optimizes mean and variance simultaneously by applying fuzzy set theory. At first, we constituted a regression model based on replicates of a response and aggregate regression models so that a fuzzy regression model expresses each response. Triangular fuzzy number (TFN) was applied to consider mean and variance of data simultaneously. The obtained regression model included fuzzy coefficients which consider uncertainty in the collected data. Then a fuzzy programming was expressed to solve the problem which applying degree of satisfaction from each objective, so the problem was converted to a single objective model with the goals of increasing desirability and robustness simultaneously. The proposed methodology has some capability such as optimizing mean and variance simultaneously without setting mean in a distinct value or range, applying fuzzy concepts as a tool for considering mean and variance considering uncertainty of data collected and statistical errors of prediction, showing factor levels as a fuzzy number with a capability to change them at an acceptable range based on experimental conditions and taking into consideration decision making preferences for optimizing mean (desirability function) or variance (deviation function) by assigning importance weight to them. In future, other approaches such as loss function could be considered simultaneously with desirability function to optimize multi response surface and other fuzzy techniques such as fuzzy inference system (FIS) and fuzzy multi criteria decision making (MCDM) could be used**.**

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