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Variance Formulas for Trapezoidal Fuzzy Random Variables*

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Abstract

The variance of fuzzy random variables often appears in fuzzy random programming problems. Based on the definition of the variance of a fuzzy random variable, this paper attempts to deduce several formulas for the variances of trapezoidal fuzzy random variables, in which the randomness is characterized by uniform distribution. Firstly, we give the moment formulas for trapezoidal fuzzy variables. Then, according to the obtained results, we deduce the variance formulas for trapezoidal fuzzy random variables. The obtained formulas are useful in studying the properties of fuzzy random programming problems. At last, we also provide some applications of the obtained formulas through three numerical examples.

Keywords: fuzzy variable, fuzzy random variable, expected value, variance

1 Introduction

Fuzzy random theory [10, 17] is a combination of probability theory and credibility theory [10, 11], it deals with the hybrid uncertain environment where linguistic and frequent nature coexist in a decision-making process. Fuzzy random variable is a basic concept in this theory, which was introduced by Kwakernaak [5] to depict the phenomena in which fuzziness and randomness appear simultaneously. Since then, it was studied by other researchers, aiming at different purposes such as Liu and Liu [13], Feng and Liu [1], Liu and Gao [16], Wang and Watada [21], and Qin and Hao [19]. Using fuzzy random variables, fuzzy random optimization theory has been developed rapidly and many meaningful mathematical programming models have been well developed in recent years. For instance, Luhandjula [18] adopted a semi-infinite method to convert the primal fuzzy random linear programming problem to a stochastic programming one so that the techniques of stochastic optimization can be applied. Liu and Liu [14] proposed a class of fuzzy random optimization—expected value models. Liu [7, 8, 9] presented fuzzy random chance-constrained programming and dependent-chance programming via the primitive chance of a fuzzy random event, and designed hybrid intelligent algorithms to solve them. Furthermore, Liu and Liu [15] presented fuzzy random programming by introducing the mean chance of fuzzy random events. Hao and Liu [3] studied a new class of portfolio selection model with fuzzy random returns, and employed a hybrid particle swarm optimization (PSO) algorithm to solve it.

As the case of real-valued random variables, the variance of fuzzy random variable should be used to measure the spread or dispersion of the fuzzy random variable around its expected value. The variance of fuzzy random variables is of great importance in statistical analysis, economics, linear theory of fuzzy stochastic processes and other fields of fuzzy random theory and applications. And it has been studied by a number of scholars. For example, Körner [6] studied properties of the variance of a fuzzy random variable and compared it with the common variance of real-valued random variables, and considered a linear regression problem and limit theorems by using the expectation and the variance of fuzzy random variables. Feng *et al.* [2] introduced the properties of the variance of fuzzy random variables and discussed the linear theory of fuzzy stochastic processes by applying the properties of the variance of fuzzy random variables. Hao and Liu [4] formulated the mean-variance methodology for fuzzy random portfolio selection problem, in which the expected return of a portfolio was taken as the investment return and the variance of the expected return of a portfolio was taken as the investment risk.

However, due to the difficulties involved in computing variance, studies are usually focus on searching for good approximation and simulation methods to solve complex fuzzy random programming problems. Based on the definition of the variance of a fuzzy random variable [13], this paper deduces general formulas for the

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variance of trapezoidal fuzzy random variables, so that they can be used more conveniently in fuzzy random optimization problems.

This paper is organized as follows. Section 2 recalls some basic concepts on fuzzy random variables. In Section 3, we first give the moment formulas of trapezoidal fuzzy variables. After that, applying the obtained results, we deduces the variance formulas for trapezoidal fuzzy random variables. And three numerical examples are also provided to apply the variance formulas in this section. Finally, Section 4 gives our conclusions.

2 Fuzzy Random Variables

Given a universe Γ, an ample field [20] $\mathcal{A}$ on Γ is a class of subsets of Γ that is closed under arbitrary unions, intersections, and complementation in Γ. Let Pos be a possibility measure defined on the ample field $\mathcal{A}$, a self-dual set function Cr, called *credibility measure*, is defined as [12]

$$\mathrm{Cr}(A) = \frac{1}{2}(1 + \mathrm{Pos}(A) - \mathrm{Pos}(A^c)), \quad A \in \mathcal{A}, \tag{1}$$

where A^c is the complement of A.

The triplet $(\Gamma, \mathcal{A}, \mathrm{Cr})$ is called a credibility space [11].

Definition 2.1 ([12]) Let ξ be a fuzzy variable.The expected value of ξ is defined as

$$E[\xi] = \int_0^\infty \mathrm{Cr}\{\xi \geq r\}\mathrm{d}r - \int_{-\infty}^0 \mathrm{Cr}\{\xi \leq r\}\mathrm{d}r \tag{2}$$

provided that at least one of the two integrals is finite.

Definition 2.2 ([12]) Let ξ be a fuzzy variable with finite expected value e. The variance of ξ is defined as

$$V[\xi] = E\left[(\xi - e)^2\right]. \tag{3}$$

Definition 2.3 ([13]) Let $(\Omega, \Sigma, \mathrm{Pr})$ be a probability space. A fuzzy random variable is a map $\xi : \Omega \to \mathcal{F}_v$ such that for any Borel subset B of $\Re$, the following function

$$\mathrm{Pos}\{\gamma \mid \xi_\omega(\gamma) \in B\}$$

is measurable with respect to ω, where $\mathcal{F}_v$ is a collection of fuzzy variables defined on a possibility space.

The various measurability criteria for fuzzy random variables were discussed in [1], which provided us methods to check if a map from Ω to $\mathcal{F}_v$ is a fuzzy random variable.

Definition 2.4 ([13]) Let ξ be a fuzzy random variable defined on the probability space $(\Omega, \Sigma, \mathrm{Pr})$. The expected value of ξ is defined as

$$E[\xi] = \int_0^\infty \mathrm{Pr}\{\omega \in \Omega \mid E[\xi(\omega)] \geq r\}\,\mathrm{d}r - \int_{-\infty}^0 \mathrm{Pr}\{\omega \in \Omega \mid E[\xi(\omega)] \leq r\}\,\mathrm{d}r \tag{4}$$

provided that at least one of the two integrals is finite.

Definition 2.5 ([13]) Let ξ be a fuzzy random variable defined on the probability space $(\Omega, \Sigma, \mathrm{Pr})$ with finite expected value e. The variance of ξ is defined as

$$V[\xi] = E\left[(\xi - e)^2\right]. \tag{5}$$

The readers who are interested in the detailed discussion about fuzzy random variable and its properties may refer to [9, 10, 17].

3 Variance Formulas

In this section, we deduce the formulas for variance of trapezoidal fuzzy random variables, which will be useful in the fuzzy random optimization problems.

Let ξ be a trapezoidal fuzzy random variable such that for each ω, $\xi(\omega)$ =$(X(\omega)-\delta, X(\omega)-\alpha, X(\omega)+\alpha, X(\omega)+\beta)$ is a trapezoidal fuzzy variable with the following possibility distribution

$$\mu_{\xi(\omega)}(x)=\begin{cases}\frac{x-X(\omega)+\delta}{\delta-\alpha}, & \text{if } X(\omega)-\delta\le x<X(\omega)-\alpha\\ 1, & \text{if } X(\omega)-\alpha\le x<X(\omega)+\alpha\\ \frac{-x+X(\omega)+\beta}{\beta-\alpha}, & \text{if } X(\omega)+\alpha\le x<X(\omega)+\beta\\ 0, & \text{otherwise,}\end{cases}$$

where $\delta>\alpha>0$, $\beta>\alpha>0$, and X is a random variable. Then for each ω, according to (2.1), we have $E[\xi(\omega)]=(4X(\omega)-\delta+\beta)/4$. Moreover, by using (2.4), we have $E[\xi]=(4E[X]-\delta+\beta)/4$.

Proposition 3.1 *Let ξ be a trapezoidal fuzzy random variable such that for each ω, $\xi(\omega)$ =$(X(\omega)-\beta, X(\omega)-\alpha, X(\omega)+\alpha, X(\omega)+\beta)$, and denote $m=E[X]$.*
(1) If $X(\omega)\le m-\beta$, then

$$E[(\xi(\omega)-m)^2]=X^2(\omega)-2mX(\omega)+m^2+\tfrac{\alpha^2+\beta^2+\alpha\beta}{3}.$$

(2) If $m-\beta<X(\omega)\le m-\alpha$, then

$$\begin{aligned}E[(\xi(\omega)-m)^2]=&-\tfrac{1}{6(\beta-\alpha)}\left[X^3(\omega)-3(\beta-2\alpha+m)X^2(\omega)+\left(3(\beta+m)^2-12\alpha m\right)X(\omega)\right.\\&\left.-(\beta+m)^3+2\alpha^3+6\alpha m^2\right].\end{aligned}$$

(3) If $m-\alpha<X(\omega)\le m$, then

$$E[(\xi(\omega)-m)^2]=X^2(\omega)-\left(\tfrac{\beta-\alpha}{2}+2m\right)X(\omega)+m^2+\tfrac{\beta-\alpha}{2}m+\tfrac{4\alpha^2+\alpha\beta+\beta^2}{6}.$$

(4) If $m<X(\omega)\le m+\alpha$, then

$$E[(\xi(\omega)-m)^2]=X^2(\omega)+\left(\tfrac{\beta-\alpha}{2}-2m\right)X(\omega)+m^2-\tfrac{\beta-\alpha}{2}m+\tfrac{4\alpha^2+\alpha\beta+\beta^2}{6}.$$

(5) If $m+\alpha<X(\omega)\le m+\beta$, then

$$\begin{aligned}E[(\xi(\omega)-m)^2]=&\tfrac{1}{6(\beta-\alpha)}\left[X^3(\omega)+3(\beta-2\alpha-m)X^2(\omega)+(3(\beta-m)^2+12\alpha m)X(\omega)\right.\\&\left.+(\beta-m)^3-2\alpha^3-6\alpha m^2\right].\end{aligned}$$

(6) If $X(\omega)>m+\beta$, then

$$E[(\xi(\omega)-m)^2]=X^2(\omega)-2mX(\omega)+m^2+\tfrac{\alpha^2+\beta^2+\alpha\beta}{3}.$$

Proof: We only prove the assertion (1), the rest can be proved similarly.
For any $r\ge 0$, we have

$$\mathrm{Pos}\left\{(\xi(\omega)-m)^2\ge r\right\}=\mathrm{Pos}\left\{\xi(\omega)-m\ge\sqrt{r}\right\}\vee\mathrm{Pos}\left\{\xi(\omega)-m\le-\sqrt{r}\right\}. \tag{6}$$

If $X(\omega)\le m-\beta$, then by Eq. (6)

$$\mathrm{Pos}\left\{(\xi(\omega)-m)^2\ge r\right\}=\begin{cases}1, & \text{if } 0\le r\le(X(\omega)-m-\alpha)^2\\ \frac{-\sqrt{r}-X(\omega)+m+\beta}{\beta-\alpha}, & \text{if } (X(\omega)-m-\alpha)^2<r\le(X(\omega)-m-\beta)^2\\ 0, & \text{otherwise,}\end{cases}$$

and

$$\mathrm{Pos}\left\{(\xi(\omega)-m)^2<r\right\}=\begin{cases}1, & \text{if } r>(X(\omega)-m+\alpha)^2\\ \frac{\sqrt{r}+X(\omega)-m+\beta}{\beta-\alpha}, & \text{if } (X(\omega)-m+\beta)^2<r\le(X(\omega)-m+\alpha)^2\\ 0, & \text{otherwise.}\end{cases}$$

Therefore,

$$\mathrm{Cr}\{(\xi(\omega)-m)^2\ge r\}=\begin{cases}1, & \text{if } 0\le r\le(X(\omega)-m+\beta)^2\\ \frac{-\sqrt{r}-X(\omega)+m-2\alpha+\beta}{2(\beta-\alpha)}, & \text{if } (X(\omega)-m+\beta)^2<r\le(X(\omega)-m+\alpha)^2\\ \frac{1}{2}, & \text{if } (X(\omega)-m+\alpha)^2<r\le(X(\omega)-m-\alpha)^2\\ \frac{-\sqrt{r}-X(\omega)+m+\beta}{2(\beta-\alpha)}, & \text{if } (X(\omega)-m-\alpha)^2<r\le(X(\omega)-m-\beta)^2\\ 0, & \text{otherwise.}\end{cases}$$

It follows that

$$E[(\xi(\omega)-m)^2]=\int_0^\infty \mathrm{Cr}\{(\xi(\omega)-m)^2\ge r\}\mathrm{d}r=X^2(\omega)-2mX(\omega)+m^2+\frac{\alpha^2+\beta^2+\alpha\beta}{3}.$$

The proof of the proposition is complete. □

As a consequence of Proposition 3.1, when $X\sim\mathcal{U}(a,b)$, we can obtain the following results about the variance formula for symmetric trapezoidal fuzzy random variable.

Theorem 3.1 *Let ξ be a trapezoidal fuzzy random variable such that for each ω, $\xi(\omega)$ =$(X(\omega)-\beta,X(\omega)-\alpha,X(\omega)+\alpha,X(\omega)+\beta)$, and denote $m=E[X]$.*
If $X\sim\mathcal{U}(a,b)$, and $(b-a)/2\ge\beta$, then

$$V[\xi]=\frac{(b-a)^2}{12}+\frac{3\alpha^3-\beta^3-\alpha^2\beta-\alpha\beta^2}{12(b-a)}+\frac{\alpha^2+\alpha\beta+\beta^2}{3}.$$

If $X\sim\mathcal{U}(a,b)$, and $\alpha\le(b-a)/2<\beta$, then

$$V[\xi]=\frac{(b-a)^3}{192(\beta-\alpha)}+\frac{(\beta-2\alpha)(b-a)^2}{24(\beta-\alpha)}+\frac{\beta^2(b-a)}{8(\beta-\alpha)}+\frac{\beta^3-2\alpha^3}{6(\beta-\alpha)}+\frac{15\alpha^4-4\alpha^3\beta-6\alpha^2\beta^2-4\alpha\beta^3}{12(b-a)(\beta-\alpha)}+\frac{9\alpha^3+5\alpha^2\beta+2\alpha\beta^2}{6(b-a)}.$$

If $X\sim\mathcal{U}(a,b)$, and $(b-a)/2<\alpha$, then

$$V[\xi]=\frac{(b-a)^2}{12}+\frac{(\beta-\alpha)(b-a)}{8}+\frac{4\alpha^2+\alpha\beta+\beta^2}{6}.$$

Proof. Since $X\sim\mathcal{U}(a,b)$, we have $m=(a+b)/2$. By Proposition 3.1, we have the following calculation results. If $X(\omega)\le(a+b)/2-\beta$, and $(b-a)/2\ge\beta$, then

$$N_1=\int_a^{(a+b)/2-\beta}\left(x^2-(a+b)x+(a+b)^2/4+\frac{\alpha^2+\beta^2+\alpha\beta}{3}\right)\frac{1}{b-a}\mathrm{d}x=\frac{(b-a)^2}{24}+\frac{\alpha^2+\beta^2+\alpha\beta}{6}-\frac{\beta(\alpha^2+2\beta^2+\alpha\beta)}{3(b-a)}.$$

If $X(\omega)\le(a+b)/2-\beta$, and $(b-a)/2<\beta$, then $N_1=0$.
If $(a+b)/2-\beta<X(\omega)\le(a+b)/2-\alpha$, and $(b-a)/2\ge\beta$, then

$$N_2=-\frac{5\alpha^3-5\beta^3+\alpha^2\beta-\alpha\beta^2}{8(b-a)}.$$

If $(a+b)/2-\beta<X(\omega)\le(a+b)/2-\alpha$, and $\alpha\le(b-a)/2<\beta$, then

$$N_2=\frac{(b-a)^3}{384(\beta-\alpha)}+\frac{(\beta-2\alpha)(b-a)^2}{48(\beta-\alpha)}+\frac{\beta^2(b-a)}{16(\beta-\alpha)}+\frac{\beta^3-2\alpha^3}{12(\beta-\alpha)}+\frac{15\alpha^4-4\alpha^3\beta-6\alpha^2\beta^2-4\alpha\beta^3}{24(b-a)(\beta-\alpha)}.$$

If $(a+b)/2-\beta<X(\omega)\le(a+b)/2-\alpha$, and $(b-a)/2<\alpha$, then $N_2=0$.
If $(a+b)/2-\alpha<X(\omega)\le(a+b)/2$, and $(b-a)/2\ge\alpha$, then

$$N_3=\frac{9\alpha^3+5\alpha^2\beta+2\alpha\beta^2}{12(b-a)}.$$

If $(a+b)/2-\alpha < X(\omega) \le (a+b)/2$, and $(b-a)/2 < \alpha$, then

$$N_3 = \frac{(b-a)^2}{24} + \frac{(\beta-\alpha)(b-a)}{16} + \frac{4\alpha^2+\alpha\beta+\beta^2}{12}.$$

If $(a+b)/2 < X(\omega) \le (a+b)/2+\alpha$, and $(b-a)/2 \ge \alpha$, then

$$N_4 = \frac{9\alpha^3+5\alpha^2\beta+2\alpha\beta^2}{12(b-a)}.$$

If $(a+b)/2 < X(\omega) \le (a+b)/2+\alpha$, and $(b-a)/2 < \alpha$, then

$$N_4 = \frac{(b-a)^2}{24} + \frac{(\beta-\alpha)(b-a)}{16} + \frac{4\alpha^2+\alpha\beta+\beta^2}{12}.$$

If $(a+b)/2+\alpha < X(\omega) \le (a+b)/2+\beta$, and $(b-a)/2 \ge \beta$, then

$$N_5 = -\frac{5\alpha^3-5\beta^3+\alpha^2\beta-\alpha\beta^2}{8(b-a)}.$$

If $(a+b)/2+\alpha < X(\omega) \le (a+b)/2+\beta$, and $\alpha \le (b-a)/2 < \beta$, then

$$N_5 = \frac{(b-a)^3}{384(\beta-\alpha)} + \frac{(\beta-2\alpha)(b-a)^2}{48(\beta-\alpha)} + \frac{\beta^2(b-a)}{16(\beta-\alpha)} + \frac{\beta^3-2\alpha^3}{12(\beta-\alpha)} + \frac{15\alpha^4-4\alpha^3\beta-6\alpha^2\beta^2-4\alpha\beta^3}{24(b-a)(\beta-\alpha)}.$$

If $(a+b)/2+\alpha < X(\omega) \le (a+b)/2+\beta$, and $(b-a)/2 < \alpha$, then $N_5 = 0$.
If $X(\omega) > (a+b)/2+\beta$, and $(b-a)/2 \ge \beta$, then

$$N_6 = \frac{(b-a)^2}{24} + \frac{\alpha^2+\beta^2+\alpha\beta}{6} - \frac{\beta(\alpha^2+2\beta^2+\alpha\beta)}{3(b-a)}.$$

If $X(\omega) > (a+b)/2+\beta$, and $(b-a)/2 < \beta$, then $N_6 = 0$.

Combining the above, we have
If $(b-a)/2 \ge \beta$, then

$$V[\xi] = N_1+N_2+N_3+N_4+N_5+N_6 = \frac{(b-a)^2}{12} + \frac{3\alpha^3-\beta^3-\alpha^2\beta-\alpha\beta^2}{12(b-a)} + \frac{\alpha^2+\alpha\beta+\beta^2}{3}.$$

If $\alpha \le (b-a)/2 < \beta$, then

$$\begin{aligned} V[\xi] &= N_1+N_2+N_3+N_4+N_5+N_6 \\ &= \frac{(b-a)^3}{192(\beta-\alpha)} + \frac{(\beta-2\alpha)(b-a)^2}{24(\beta-\alpha)} + \frac{\beta^2(b-a)}{8(\beta-\alpha)} + \frac{\beta^3-2\alpha^3}{6(\beta-\alpha)} + \frac{15\alpha^4-4\alpha^3\beta-6\alpha^2\beta^2-4\alpha\beta^3}{12(b-a)(\beta-\alpha)} + \frac{9\alpha^3+5\alpha^2\beta+2\alpha\beta^2}{6(b-a)}. \end{aligned}$$

If $(b-a)/2 < \alpha$, then

$$V[\xi] = N_1+N_2+N_3+N_4+N_5+N_6 = \frac{(b-a)^2}{12} + \frac{(\beta-\alpha)(b-a)}{8} + \frac{4\alpha^2+\alpha\beta+\beta^2}{6}.$$

The proof of the theorem is complete. □

Example 3.1 *Let ξ be a trapezoidal fuzzy random variable. For each ω, $\xi(\omega) = (X(\omega)-4, X(\omega)-2, X(\omega)+2, X(\omega)+4)$ is a triangular fuzzy variable with $X \sim \mathcal{U}(1,2)$. Since $\alpha = 2, \beta = 4, a = 1, b = 2$, and $(b-a)/2 = (2-1)/2 < \alpha = 2$, by Theorem 3.1, we have*

$$V[\xi] = \frac{(b-a)^2}{12} + \frac{(\beta-\alpha)(b-a)}{8} + \frac{4\alpha^2+\alpha\beta+\beta^2}{6} = \frac{(2-1)^2}{12} + \frac{(4-2)\times(2-1)}{8} + \frac{4\times 2^2+2\times 4+4^2}{6} = 7.$$

Proposition 3.2 *Let ξ be a trapezoidal fuzzy random variable such that for each ω, $\xi(\omega) = (X(\omega)-\delta, X(\omega)-\alpha, X(\omega)+\alpha, X(\omega)+\beta)$, where $\delta > \beta$, and denote $m = (4E[X]-\delta+\beta)/4$.*
(1) *If $X(\omega) \le m-\beta$, then*

$$E[(\xi(\omega)-m)^2] = X^2(\omega) - 2mX(\omega) + m^2 + \frac{(\delta-\beta)m}{2} + \frac{2\alpha^2+\beta^2+\delta^2-2\alpha\delta+4\alpha\beta}{6}.$$

(2) *If $m-\beta < X(\omega) \le m-\alpha$, then*

$$\begin{aligned} E[(\xi(\omega)-m)^2] = &-\frac{1}{6(\beta-\alpha)}X^3(\omega) + \frac{m+\beta-2\alpha}{2(\beta-\alpha)}X^2(\omega) - \left(\frac{\alpha+\delta}{2} + m + \frac{(m-\alpha)^2}{2(\beta-\alpha)}\right)X(\omega) \\ &+ \frac{(m-\alpha)^3}{6(\beta-\alpha)} + \frac{\alpha m+\delta m+\alpha\delta+m^2}{2} + \frac{(\beta-\alpha)^2}{6}. \end{aligned}$$

(3) *If* $m-\alpha < X(\omega) \le m$*, then*

$$E[(\xi(\omega)-m)^2] = X^2(\omega) - \left(\tfrac{\delta-\alpha}{2}+2m\right)X(\omega) + \alpha^2 + m^2 + \tfrac{(\delta-\alpha)(\alpha+m)}{2} + \tfrac{(\delta-\alpha)^2}{6}.$$

(4) *If* $m < X(\omega) \le m + (\delta-\beta)/2$*, and* $\delta-\beta \le 2\alpha$*, or* $m < X(\omega) \le m+\alpha$*, and* $\delta-\beta > 2\alpha$*, then*

$$\begin{aligned} E[(\xi(\omega)-m)^2] = {} & -\tfrac{2m^3(\delta+\beta-2\alpha)}{3(\delta-\beta)^2} + \left(1+\tfrac{2\alpha}{\delta-\beta}\right)m^2 + \tfrac{3\alpha+\delta}{2}m + \tfrac{4\alpha^2+\alpha\delta+\delta^2}{6} + \tfrac{2(\delta+\beta-2\alpha)}{3(\delta-\beta)^2}X^3(\omega) \\ & + \left(\tfrac{\delta-\beta+2\alpha}{\delta-\beta} - \tfrac{2m(\delta+\beta-2\alpha)}{(\delta-\beta)^2}\right)X^2(\omega) + \left(\tfrac{2m^2(\delta+\beta-2\alpha)}{(\delta-\beta)^2} - 2m - \tfrac{4\alpha}{\delta-\beta}m - \tfrac{3\alpha+\delta}{2}\right)X(\omega). \end{aligned}$$

(5) *If* $m + (\delta-\beta)/2 < X(\omega) \le m+\alpha$*, and* $\delta-\beta \le 2\alpha$*, then*

$$E[(\xi(\omega)-m)^2] \;=\; X^2(\omega) + \left(\tfrac{\beta-\alpha}{2}-2m\right)X(\omega) + \alpha^2 + m^2 + \tfrac{(\beta-\alpha)(\alpha-m)}{2} + \tfrac{(\beta-\alpha)^2}{6}.$$

If $m+\alpha < X(\omega) \le m+(\delta-\beta)/2$*, and* $\delta-\beta > 2\alpha$*, then*

$$\begin{aligned} E[(\xi(\omega)-m)^2] = {} & \tfrac{(\alpha\delta-\alpha\beta-m\delta-m\beta+2\alpha m)^3}{6(\delta-\alpha)(\beta-\alpha)(\delta-\beta)^2} - \tfrac{(\alpha-m)^3}{6(\beta-\alpha)} + \tfrac{(\delta-\alpha)^2}{6} + \tfrac{(m+\alpha)(m+\delta)}{2} \\ & + \left(\tfrac{(\delta+\beta-2\alpha)^3}{6(\delta-\alpha)(\beta-\alpha)(\delta-\beta)^2} - \tfrac{1}{6(\beta-\alpha)}\right)X^3(\omega) + \left(\tfrac{(\delta+\beta-2\alpha)^2(\alpha\delta-\alpha\beta-m\delta-m\beta+2\alpha m)}{2(\delta-\alpha)(\beta-\alpha)(\delta-\beta)^2} + \tfrac{\beta-2\alpha+m}{2(\beta-\alpha)}\right)X^2(\omega) \\ & + \left(\tfrac{(\delta+\beta-2\alpha)(\alpha\delta-\alpha\beta-m\delta-m\beta+2\alpha m)^2}{2(\delta-\alpha)(\beta-\alpha)(\delta-\beta)^2} - m - \tfrac{(\alpha-m)^2}{2(\beta-\alpha)} - \tfrac{\alpha+\delta}{2}\right)X(\omega). \end{aligned}$$

(6) *If* $m+\alpha < X(\omega) \le m+\delta$*, and* $\delta-\beta \le 2\alpha$*, or* $m+(\delta-\beta)/2 < X(\omega) \le m+\delta$*, and* $\delta-\beta > 2\alpha$*, then*

$$\begin{aligned} E[(\xi(\omega)-m)^2] = {} & \tfrac{1}{6(\delta-\alpha)}X^3(\omega) + \left(\tfrac{1}{2} - \tfrac{m+\alpha}{2(\delta-\alpha)}\right)X^2(\omega) + \left(\tfrac{\alpha+\beta}{2} - m + \tfrac{(m+\alpha)^2}{2(\delta-\alpha)}\right)X(\omega) \\ & - \tfrac{(m+\alpha)^3}{6(\delta-\alpha)} + \tfrac{(\alpha-m)^2}{2} + \tfrac{(\beta-\alpha)(\alpha-m)}{6} + \tfrac{(\beta-\alpha)^2}{6}. \end{aligned}$$

(7) *If* $X(\omega) > m+\delta$*, then*

$$E[(\xi(\omega)-m)^2] = X^2(\omega) - 2mX(\omega) + m^2 + \tfrac{(\delta-\beta)m}{2} + \tfrac{2\alpha^2+\beta^2+\delta^2-2\alpha\delta+4\alpha\beta}{6}.$$

Proof: The proof of the proposition is similar to that of Proposition 3.1. $\Box$

As a consequence of Proposition 3.2, when $X \sim \mathcal{U}(a,b)$, we obtain the following variance formulas for trapezoidal fuzzy random variable with $\delta > \beta$.

Theorem 3.2 *Let* ξ *be a trapezoidal fuzzy random variable such that for each* ω, $\xi(\omega)$ =$(X(\omega)-\delta, X(\omega)-\alpha, X(\omega)+\alpha, X(\omega)+\beta)$*, where* $\delta > \beta$*, and denote* $m = (4E[X]-\delta+\beta)/4$*. When* $X \sim \mathcal{U}(a,b)$*, we have the following results.*
If $\delta-\beta \le 2\alpha$*, and* $b-a \ge (3\delta+\beta)/2$*, then*

$$\begin{aligned} V[\xi] = {} & \tfrac{(b-a)^2}{12} + \tfrac{(\delta-\beta)(a+b)}{2} - \tfrac{(\delta-\beta)(\delta+\beta)(a+b)}{4(b-a)} - \tfrac{(\delta+3\beta)(16\alpha^2+32\alpha\beta-16\alpha\delta+6\beta\delta+23\beta^2+3\delta^2)}{192(b-a)} \\ & + \tfrac{(\delta-\beta)(64\alpha^2+24\alpha\beta-8\alpha\delta-4\beta\delta+6\beta^2+14\delta^2)}{192(b-a)} - \tfrac{(3\delta+\beta)(16\alpha^2+32\alpha\beta-16\alpha\delta+6\beta\delta+3\beta^2+23\delta^2)}{192(b-a)} \\ & - \tfrac{12\alpha^3-9\beta^3-5\delta^3-5\alpha^2\beta+\alpha^2\delta+13\alpha\beta^2-7\alpha\delta^2-16\alpha\beta\delta-10\beta^2\delta-6\beta\delta^2}{24(b-a)} + \tfrac{16\alpha^2+5\beta^2+5\delta^2+32\alpha\beta-16\alpha\delta+6\beta\delta}{48}. \end{aligned}$$

If $\delta-\beta \le 2\alpha$*, and* $(\delta+3\beta)/2 \le b-a < (3\delta+\beta)/2$*, then*

$$\begin{aligned} V[\xi] = {} & \tfrac{(b-a)^3}{384(\delta-\alpha)} + \tfrac{(13\delta-\beta-16\alpha)(b-a)^2}{192(\delta-\alpha)} + \tfrac{(3\delta+\beta)^2(b-a)}{256(\delta-\alpha)} + \tfrac{(\delta-\beta)(5a+3b)}{16} + \tfrac{48\alpha^2+68\alpha\beta-20\alpha\delta+18\beta\delta+17\beta^2+13\delta^2}{192} \\ & + \tfrac{(\delta-\beta)(\delta+3\beta)(a+b)}{16(b-a)} + \tfrac{(4\alpha-\delta+\beta)^4}{256\times 8(\delta-\alpha)(b-a)} - \tfrac{(4\alpha-\delta+\beta)^3(5\delta-\beta-8\alpha)}{384\times 4(\delta-\alpha)(b-a)} - \tfrac{(4\alpha-\delta+\beta)(16\alpha^2+4\alpha\beta+12\alpha\delta+6\beta\delta+7\beta^2+3\delta^2)}{384(b-a)} \\ & - \tfrac{(4\alpha-\delta+\beta)^2(3\delta+\beta)^2}{256\times 4(\delta-\alpha)(b-a)} + \tfrac{(\beta-\delta+2\alpha)(18\alpha^2+9\alpha\beta+\alpha\delta-\beta\delta+3\beta^2+2\delta^2)}{48(b-a)} + \tfrac{(\delta-\beta)(64\alpha^2+24\alpha\beta-8\alpha\delta-4\beta\delta+6\beta^2+14\delta^2)}{192(b-a)} \\ & - \tfrac{(4\alpha-\delta+\beta)^3}{384\times 2(\delta-\alpha)} + \tfrac{3\alpha^3+9\beta^3+15\alpha^2\beta-8\alpha^2\delta-9\alpha\beta^2+4\alpha\delta^2+12\alpha\beta\delta+6\beta^2\delta}{24(b-a)} - \tfrac{(\delta+3\beta)(16\alpha^2+32\alpha\beta-16\alpha\delta+6\beta\delta+23\beta^2+3\delta^2)}{192(b-a)}. \end{aligned}$$

If $\delta-\beta\le 2\alpha$, *and* $(\delta-\beta+4\alpha)/2\le b-a<(\delta+3\beta)/2$, *then*

$$\begin{aligned}V[\xi]=&\ \tfrac{\beta+\delta-2\alpha}{384(\delta-\alpha)(\beta-\alpha)}(b-a)^3+\left(\tfrac{5\delta-\beta-8\alpha}{192(\delta-\alpha)}+\tfrac{5\beta-\delta-8\alpha}{192(\beta-\alpha)}\right)(b-a)^2+\left(\tfrac{(3\delta+\beta)^2}{256(\delta-\alpha)}+\tfrac{(\delta+3\beta)^2}{256(\beta-\alpha)}\right)(b-a)\\&+\tfrac{16\alpha^2-8\alpha\beta+24\alpha\delta+6\beta\delta+13\beta^2-3\delta^2}{96}-\tfrac{(4\alpha-\delta+\beta)^3}{384\times2(\delta-\alpha)}-\tfrac{(4\alpha+\delta-\beta)^3}{384\times2(\beta-\alpha)}+\tfrac{(4\alpha+\delta-\beta)^4}{256\times8(\beta-\alpha)(b-a)}+\tfrac{(4\alpha-\delta+\beta)^4}{256\times8(\delta-\alpha)(b-a)}\\&+\tfrac{(\beta-\delta+2\alpha)(18\alpha^2+9\alpha\beta+\alpha\delta-\beta\delta+3\beta^2+2\delta^2)}{48(b-a)}+\tfrac{(\delta-\beta)(64\alpha^2+24\alpha\beta-8\alpha\delta-4\beta\delta+6\beta^2+14\delta^2)}{192(b-a)}+\tfrac{9\alpha^3+2\alpha\delta^2+5\alpha^2\delta}{12(b-a)}\\&-\tfrac{(4\alpha+\delta-\beta)^3(5\beta-\delta-8\alpha)}{384\times4(\beta-\alpha)(b-a)}-\tfrac{(4\alpha-\delta+\beta)^3(5\delta-\beta-8\alpha)}{384\times4(\delta-\alpha)(b-a)}-\tfrac{(4\alpha-\delta+\beta)(16\alpha^2+4\alpha\beta+12\alpha\delta+6\beta\delta+7\beta^2+3\delta^2)}{384(b-a)}\\&-\tfrac{(4\alpha-\delta+\beta)^2(3\delta+\beta)^2}{256\times4(\delta-\alpha)(b-a)}-\tfrac{(4\alpha+\delta-\beta)^2(\delta+3\beta)^2}{256\times4(\beta-\alpha)(b-a)}-\tfrac{(4\alpha+\delta-\beta)(16\alpha^2-20\alpha\beta+36\alpha\delta+6\beta\delta+19\beta^2-9\delta^2)}{384(b-a)}.\end{aligned}$$

If $\delta-\beta\le 2\alpha$, *and* $(\beta-\delta+4\alpha)/2\le b-a<(\delta-\beta+4\alpha)/2$, *then*

$$\begin{aligned}V[\xi]=&\ \tfrac{(b-a)^3}{384(\delta-\alpha)}+\tfrac{(13\delta-\beta-16\alpha)(b-a)^2}{192(\delta-\alpha)}+\tfrac{(3\delta+\beta)^2(b-a)}{256(\delta-\alpha)}+\tfrac{(\beta-\alpha)(b-a)}{16}+\tfrac{80\alpha^2-8\alpha\beta+40\alpha\delta+6\beta\delta+13\beta^2+13\delta^2}{192}\\&+\tfrac{(4\alpha-\delta+\beta)^4}{256\times8(\delta-\alpha)(b-a)}-\tfrac{(4\alpha-\delta+\beta)^3(5\delta-\beta-8\alpha)}{384\times4(\delta-\alpha)(b-a)}-\tfrac{(4\alpha-\delta+\beta)^2(3\delta+\beta)^2}{256\times4(\delta-\alpha)(b-a)}+\tfrac{(\delta-\beta)(32\alpha^2+27\alpha\beta-19\alpha\delta-5\beta\delta+5\beta^2+8\delta^2)}{192(b-a)}\\&-\tfrac{(4\alpha-\delta+\beta)^3}{384\times2(\delta-\alpha)}-\tfrac{(4\alpha-\delta+\beta)(16\alpha^2+4\alpha\beta+12\alpha\delta+6\beta\delta+7\beta^2+3\delta^2)}{384(b-a)}+\tfrac{(\beta-\delta+2\alpha)(18\alpha^2+9\alpha\beta+\alpha\delta-\beta\delta+3\beta^2+2\delta^2)}{48(b-a)}.\end{aligned}$$

If $\delta-\beta\le 2\alpha$, *and* $(\delta-\beta)/2\le b-a<(\beta-\delta+4\alpha)/2$, *then*

$$V[\xi]\ =\ \tfrac{(b-a)^2}{12}+\tfrac{\beta+\delta-2\alpha}{16}(b-a)+\tfrac{8\alpha^2+\beta^2+\delta^2+\alpha\beta+\alpha\delta}{12}+\tfrac{(\delta-\beta)(10\alpha\beta-10\alpha\delta-\beta^2+\delta^2)}{192(b-a)}.$$

If $\delta-\beta\le 2\alpha$, *and* $b-a<(\delta-\beta)/2$, *then*

$$V[\xi]\ =\ \tfrac{3\delta-\beta+2\alpha}{24(\delta-\beta)}(b-a)^2+\tfrac{11\delta^2+64\alpha^2+5\beta^2+26\alpha\beta-10\alpha\delta}{96}.$$

If $2\alpha<\delta-\beta<4\alpha$, *and* $b-a\ge(3\delta+\beta)/2$, *then*

$$\begin{aligned}V[\xi]=&\ \tfrac{(b-a)^2}{12}-\tfrac{(\delta+\beta)(64\alpha^3+\beta^3-71\delta^3+8\alpha^2\beta-8\alpha^2\delta+24\alpha\beta^2+40\alpha\delta^2-21\beta^2\delta-29\beta\delta^2)}{384(\delta-\alpha)(b-a)}+\tfrac{16\alpha^2+5\beta^2+5\delta^2+32\alpha\beta-16\alpha\delta+6\beta\delta}{48}\\&-\tfrac{(\delta+3\beta)(16\alpha^2+32\alpha\beta-16\alpha\delta+6\beta\delta+23\beta^2+3\delta^2)}{192(b-a)}-\tfrac{(3\delta+\beta)(16\alpha^2+32\alpha\beta-16\alpha\delta+6\beta\delta+3\beta^2+23\delta^2)}{192(b-a)}-\tfrac{(\delta-\beta)^3(11\delta-75\beta+128\alpha)}{256\times24(\beta-\alpha)(b-a)}\\&+\tfrac{(\delta-\beta)(a+b)}{2}-\tfrac{(\delta+\beta-2\alpha)^3(\delta-\beta-4\alpha)^4}{24\times256(\delta-\alpha)(\beta-\alpha)(\delta-\beta)^2(b-a)}+\tfrac{(\delta+\beta-2\alpha)^2(\delta-\beta-4\alpha)^3(\delta+\beta+2\alpha)}{384\times4(\delta-\alpha)(\beta-\alpha)(\delta-\beta)(b-a)}+\tfrac{(\delta-\beta-2\alpha)(\delta-\beta)^2(13\beta-\delta-24\alpha)}{384\times2(\beta-\alpha)(b-a)}\\&+\tfrac{(\delta^2-2\alpha^2)(\delta-4\alpha-\beta)^2}{64(\delta-\alpha)(b-a)}-\tfrac{(\delta+\beta-2\alpha)^2(\delta-\beta-4\alpha)^2(\delta+\beta+6\alpha)}{256\times4(\delta-\alpha)(\beta-\alpha)(b-a)}+\tfrac{(\delta-\beta-4\alpha)^3(19\beta-3\delta-36\alpha)}{256\times24(\beta-\alpha)(b-a)}+\tfrac{(\delta-\beta-4\alpha)^2(\delta-\beta)(\delta-9\beta+16\alpha)}{256\times4(\beta-\alpha)(b-a)}\\&+\tfrac{(\delta^3-2\alpha^3)(\delta-2\alpha-\beta)}{12(\delta-\alpha)(b-a)}-\tfrac{(\delta^2-\beta^2)(a+b)}{4(b-a)}-\tfrac{(3\delta-\beta+2\alpha)(\delta-\beta-4\alpha)^3}{384(\delta-\beta)(b-a)}+\tfrac{3\alpha^3+9\beta^3+15\alpha^2\beta-8\alpha^2\delta-9\alpha\beta^2+4\alpha\delta^2+12\alpha\beta\delta+6\beta^2\delta}{24(b-a)}\\&+\tfrac{(\delta+\beta-2\alpha)^2(11\delta+11\beta+42\alpha)(\delta-\beta)^2}{24\times256(\delta-\alpha)(\beta-\alpha)(b-a)}+\tfrac{(\delta+\beta-2\alpha)^2(\delta-\beta-2\alpha)(\delta+\beta+10\alpha)(\delta-\beta)}{384\times2(\delta-\alpha)(\beta-\alpha)(b-a)}-\tfrac{(\delta^2-2\alpha^2)(\delta+4\alpha-\beta)(5\delta-8\alpha-5\beta)}{64(\delta-\alpha)(b-a)}\\&+\tfrac{(\delta-\beta)^2(11\delta-5\beta+10\alpha)}{6\times256(b-a)}+\tfrac{9\alpha^2-2\beta^2+4\delta^2-5\alpha\beta+7\alpha\delta+6\beta\delta}{48(b-a)}\alpha+\tfrac{(\delta+\beta-2\alpha)(\delta-\beta-4\alpha)^4}{6\times256(\delta-\beta)^2(b-a)}.\end{aligned}$$

If $2\alpha<\delta-\beta<4\alpha$, *and* $(\delta+3\beta)/2\le b-a<(3\delta+\beta)/2$, *then*

$$\begin{aligned}V[\xi]=&\ \tfrac{(b-a)^2}{24}+\tfrac{(\delta-\beta)(5a+3b)}{16}-\tfrac{(\delta-\beta)(\delta+3\beta)(a+b)}{16(b-a)}+\tfrac{16\alpha^2+5\beta^2+5\delta^2+32\alpha\beta-16\alpha\delta+6\beta\delta}{96}-\tfrac{(\delta-\beta)^2(3\delta+\beta)^2}{256\times4(\delta-\alpha)(b-a)}\\&-\tfrac{(\delta^2-2\alpha^2)(\delta+4\alpha-\beta)(5\delta-8\alpha-5\beta)}{64(\delta-\alpha)(b-a)}+\tfrac{(\delta^3-2\alpha^3)(\delta-2\alpha-\beta)}{12(\delta-\alpha)(b-a)}+\tfrac{(\delta+\beta-2\alpha)(\delta-\beta-4\alpha)^4}{6\times256(\delta-\beta)^2(b-a)}-\tfrac{(3\delta-\beta+2\alpha)(\delta-\beta-4\alpha)^3}{384(\delta-\beta)(b-a)}\\&-\tfrac{(\delta+3\beta)(16\alpha^2+32\alpha\beta-16\alpha\delta+6\beta\delta+23\beta^2+3\delta^2)}{192(b-a)}+\tfrac{(b-a)^3}{384(\delta-\alpha)}+\tfrac{(5\delta-\beta-8\alpha)(b-a)^2}{192(\delta-\alpha)}+\tfrac{(3\delta+\beta)^2(b-a)}{256(\delta-\alpha)}-\tfrac{(4\alpha-\delta+\beta)^3}{384\times2(\delta-\alpha)}\\&+\tfrac{16\alpha^2+4\alpha\beta+12\alpha\delta+6\beta\delta+17\beta^2+3\delta^2}{192}+\tfrac{(\delta-\beta)^4}{256\times24(\delta-\alpha)(b-a)}-\tfrac{(\delta-\beta)^3(5\delta-\beta-8\alpha)}{384\times4(\delta-\alpha)(b-a)}-\tfrac{(\delta+\beta-2\alpha)^3(\delta-\beta-4\alpha)^4}{24\times256(\delta-\alpha)(\beta-\alpha)(\delta-\beta)^2(b-a)}\\&+\tfrac{(\delta-\beta)^2(11\delta-5\beta+10\alpha)}{6\times256(b-a)}+\tfrac{9\alpha^2-2\beta^2+4\delta^2-5\alpha\beta+7\alpha\delta+6\beta\delta}{48(b-a)}\alpha+\tfrac{3\alpha^3+9\beta^3+15\alpha^2\beta-8\alpha^2\delta-9\alpha\beta^2+4\alpha\delta^2+12\alpha\beta\delta+6\beta^2\delta}{24(b-a)}\\&+\tfrac{(4\alpha-\delta+\beta)^3(\delta-\beta)}{384\times4(b-a)(\delta-\alpha)}+\tfrac{(\delta+\beta-2\alpha)^2(\delta-\beta-4\alpha)^3(\delta+\beta+2\alpha)}{384\times4(\delta-\alpha)(\beta-\alpha)(\delta-\beta)(b-a)}-\tfrac{(\delta-\beta)^3(11\delta-75\beta+128\alpha)}{256\times24(\beta-\alpha)(b-a)}-\tfrac{(\delta+\beta-2\alpha)^2(\delta-\beta-4\alpha)^2(\delta+\beta+6\alpha)}{256\times4(\delta-\alpha)(\beta-\alpha)(b-a)}\\&+\tfrac{(\delta+\beta-2\alpha)^2(11\delta+11\beta+42\alpha)(\delta-\beta)^2}{24\times256(\delta-\alpha)(\beta-\alpha)(b-a)}+\tfrac{(\delta+\beta-2\alpha)^2(\delta-\beta-2\alpha)(\delta+\beta+10\alpha)(\delta-\beta)}{384\times2(\delta-\alpha)(\beta-\alpha)(b-a)}-\tfrac{(16\alpha^2+4\alpha\beta+12\alpha\delta+6\beta\delta+7\beta^2+3\delta^2)(\delta-\beta)}{384(b-a)}\\&+\tfrac{(\delta^2-2\alpha^2)(\delta-4\alpha-\beta)^2}{64(\delta-\alpha)(b-a)}+\tfrac{(\delta-\beta-4\alpha)^3(19\beta-3\delta-36\alpha)}{256\times24(\beta-\alpha)(b-a)}+\tfrac{(\delta-\beta-2\alpha)(\delta-\beta)^2(13\beta-\delta-24\alpha)}{384\times2(\beta-\alpha)(b-a)}+\tfrac{(\delta-\beta-4\alpha)^2(\delta-\beta)(\delta-9\beta+16\alpha)}{256\times4(\beta-\alpha)(b-a)}.\end{aligned}$$

If $2\alpha < \delta - \beta < 4\alpha$, *and* $(\delta - \beta + 4\alpha)/2 \le b - a < (\delta + 3\beta)/2$, *then*

$$\begin{aligned}
V[\xi] = {} & \tfrac{\beta+\delta-2\alpha}{384(\delta-\alpha)(\beta-\alpha)}(b-a)^3 + \left(\tfrac{5\delta-\beta-8\alpha}{192(\delta-\alpha)} + \tfrac{5\beta-\delta-8\alpha}{192(\beta-\alpha)}\right)(b-a)^2 + \left(\tfrac{(3\delta+\beta)^2}{256(\delta-\alpha)} + \tfrac{(\delta+3\beta)^2}{256(\beta-\alpha)}\right)(b-a) \\
& + \tfrac{16\alpha^2-8\alpha\beta+24\alpha\delta+6\beta\delta+13\beta^2-3\delta^2}{96} - \tfrac{(4\alpha-\delta+\beta)^3}{384\times2(\delta-\alpha)} - \tfrac{(4\alpha+\delta-\beta)^3}{384\times2(\beta-\alpha)} + \tfrac{(4\alpha+\delta-\beta)^4}{256\times8(\beta-\alpha)(b-a)} - \tfrac{(\delta-\beta)^4}{256\times24(\delta-\alpha)(b-a)} \\
& - \tfrac{(\delta-\beta)^3(5\delta-\beta-8\alpha)}{384\times4(\delta-\alpha)(b-a)} - \tfrac{(\delta-\beta)^2(3\delta+\beta)^2}{256\times4(\delta-\alpha)(b-a)} - \tfrac{(4\alpha+\delta-\beta)^2(\delta+3\beta)^2}{256\times4(\beta-\alpha)(b-a)} - \tfrac{(\delta-\beta+4\alpha)^3(5\beta-\delta-8\alpha)}{384\times4(\beta-\alpha)(b-a)} + \tfrac{9\alpha^3+2\alpha\delta^2+5\alpha^2\delta}{12(b-a)} \\
& - \tfrac{(4\alpha+\delta-\beta)(16\alpha^2-20\alpha\beta+36\alpha\delta+6\beta\delta+19\beta^2-9\delta^2)}{384(b-a)} + \tfrac{(\delta-\beta)(4\alpha-\delta+\beta)^3}{384\times4(\delta-\alpha)(b-a)} - \tfrac{(\delta-\beta)(16\alpha^2+4\alpha\beta+12\alpha\delta+6\beta\delta+7\beta^2+3\delta^2)}{384(b-a)} \\
& - \tfrac{(\delta+\beta-2\alpha)^3(\delta-\beta-4\alpha)^4}{24\times256(\delta-\alpha)(\beta-\alpha)(\delta-\beta)^2(b-a)} + \tfrac{(\delta+\beta-2\alpha)^2(\delta-\beta-4\alpha)^3(\delta+\beta+2\alpha)}{384\times4(\delta-\alpha)(\beta-\alpha)(\delta-\beta)(b-a)} + \tfrac{(\delta+\beta-2\alpha)^2(\delta-\beta-2\alpha)(\delta+\beta+10\alpha)(\delta-\beta)}{384\times2(\delta-\alpha)(\beta-\alpha)(b-a)} \\
& - \tfrac{(\delta+\beta-2\alpha)^2(\delta-\beta-4\alpha)^2(\delta+\beta+6\alpha)}{256\times4(\delta-\alpha)(\beta-\alpha)(b-a)} - \tfrac{(\delta-\beta)^3(11\delta-75\beta+128\alpha)}{256\times24(\beta-\alpha)(b-a)} + \tfrac{(\delta+\beta-2\alpha)(\delta-\beta-4\alpha)^4}{6\times256(\delta-\beta)^2(b-a)} - \tfrac{(3\delta-\beta+2\alpha)(\delta-\beta-4\alpha)^3}{384(\delta-\beta)(b-a)} \\
& + \tfrac{(\delta-\beta-4\alpha)^3(19\beta-3\delta-36\alpha)}{256\times24(\beta-\alpha)(b-a)} + \tfrac{(\delta-\beta-2\alpha)(\delta-\beta)^2(13\beta-\delta-24\alpha)}{384\times2(\beta-\alpha)(b-a)} + \tfrac{(\delta-\beta)^2(11\delta-5\beta+10\alpha)}{6\times256(b-a)} + \tfrac{(\delta^3-2\alpha^3)(\delta-2\alpha-\beta)}{12(\delta-\alpha)(b-a)} \\
& - \tfrac{(\delta^2-2\alpha^2)(\delta+4\alpha-\beta)(5\delta-8\alpha-5\beta)}{64(\delta-\alpha)(b-a)} + \tfrac{(\delta+\beta-2\alpha)^2(11\delta+11\beta+42\alpha)(\delta-\beta)^2}{24\times256(\delta-\alpha)(\beta-\alpha)(b-a)} + \tfrac{(\delta-\beta-4\alpha)^2(\delta-\beta)(\delta-9\beta+16\alpha)}{256\times4(\beta-\alpha)(b-a)} \\
& + \tfrac{(\delta^2-2\alpha^2)(\delta-4\alpha-\beta)^2}{64(\delta-\alpha)(b-a)} + \tfrac{9\alpha^2-2\beta^2+4\delta^2-5\alpha\beta+7\alpha\delta+6\beta\delta}{48(b-a)}\alpha.
\end{aligned}$$

If $2\alpha < \delta - \beta < 4\alpha$, *and* $(\delta - \beta)/2 \le b - a < (\delta - \beta + 4\alpha)/2$, *then*

$$\begin{aligned}
V[\xi] = {} & \tfrac{(b-a)^2}{24} + \tfrac{(\beta-\delta)(b-a)}{16} + \tfrac{32\alpha^2+3\beta^2+5\delta^2-6\alpha\beta+14\alpha\delta}{96} + \tfrac{(\delta-\beta)(32\alpha^2-3\alpha\beta+11\alpha\delta+\beta^2+6\delta^2+\beta\delta)}{192(b-a)} + \tfrac{(b-a)^3}{384(\delta-\alpha)} \\
& + \tfrac{(5\delta-\beta-8\alpha)(b-a)^2}{192(\delta-\alpha)} + \tfrac{(3\delta+\beta)^2(b-a)}{256(\delta-\alpha)} + \tfrac{16\alpha^2+4\alpha\beta+12\alpha\delta+6\beta\delta+7\beta^2+3\delta^2}{192} - \tfrac{(4\alpha-\delta+\beta)^3}{384\times2(\delta-\alpha)} + \tfrac{(\delta-\beta)^4}{256\times24(\delta-\alpha)(b-a)} \\
& + \tfrac{(\delta-\beta-4\alpha)^3(19\beta-3\delta-36\alpha)}{256\times24(\beta-\alpha)(b-a)} + \tfrac{(\delta-\beta-4\alpha)^2(\delta-\beta)(\delta-9\beta+16\alpha)}{256\times4(\beta-\alpha)(b-a)} + \tfrac{(\delta^3-2\alpha^3)(\delta-2\alpha-\beta)}{12(\delta-\alpha)(b-a)} + \tfrac{(\delta+\beta-2\alpha)(\delta-\beta-4\alpha)^4}{6\times256(\delta-\beta)^2(b-a)} \\
& + \tfrac{(\delta-\beta-2\alpha)(\delta-\beta)^2(13\beta-\delta-24\alpha)}{384\times2(\beta-\alpha)(b-a)} - \tfrac{(3\delta-\beta+2\alpha)(\delta-\beta-4\alpha)^3}{384(\delta-\beta)(b-a)} + \tfrac{(\delta^2-2\alpha^2)(\delta-4\alpha-\beta)^2}{64(\delta-\alpha)(b-a)} + \tfrac{(\delta-\beta)^2(11\delta-5\beta+10\alpha)}{6\times256(b-a)} \\
& - \tfrac{(\delta-\beta)^3(5\delta-\beta-8\alpha)}{384\times4(\delta-\alpha)(b-a)} - \tfrac{(\delta-\beta)^2(3\delta+\beta)^2}{256\times4(\delta-\alpha)(b-a)} - \tfrac{(\delta-\beta)(16\alpha^2+4\alpha\beta+12\alpha\delta+6\beta\delta+7\beta^2+3\delta^2)}{384(b-a)} + \tfrac{(\beta-\delta+4\alpha)^3(\delta-\beta)}{384\times4(\delta-\alpha)(b-a)} \\
& - \tfrac{(\delta+\beta-2\alpha)^3(\delta-\beta-4\alpha)^4}{24\times256(\delta-\alpha)(\beta-\alpha)(\delta-\beta)^2(b-a)} + \tfrac{(\delta+\beta-2\alpha)^2(\delta-\beta-4\alpha)^3(\delta+\beta+2\alpha)}{384\times4(\delta-\alpha)(\beta-\alpha)(\delta-\beta)(b-a)} + \tfrac{(\delta+\beta-2\alpha)^2(11\delta+11\beta+42\alpha)(\delta-\beta)^2}{24\times256(\delta-\alpha)(\beta-\alpha)(b-a)} \\
& + \tfrac{(\delta+\beta-2\alpha)^2(\delta-\beta-2\alpha)(\delta+\beta+10\alpha)(\delta-\beta)}{384\times2(\delta-\alpha)(\beta-\alpha)(b-a)} - \tfrac{(\delta-\beta)^3(11\delta-75\beta+128\alpha)}{256\times24(\beta-\alpha)(b-a)} - \tfrac{(\delta+\beta-2\alpha)^2(\delta-\beta-4\alpha)^2(\delta+\beta+6\alpha)}{256\times4(\delta-\alpha)(\beta-\alpha)(b-a)} \\
& - \tfrac{(\delta^2-2\alpha^2)(\delta+4\alpha-\beta)(5\delta-8\alpha-5\beta)}{64(\delta-\alpha)(b-a)} + \tfrac{9\alpha^2-2\beta^2+4\delta^2-5\alpha\beta+7\alpha\delta+6\beta\delta}{48(b-a)}\alpha.
\end{aligned}$$

If $2\alpha < \delta - \beta < 4\alpha$, *and* $(\beta - \delta + 4\alpha)/2 \le b - a < (\delta - \beta)/2$, *then*

$$\begin{aligned}
V[\xi] = {} & \left(\tfrac{1}{384(\beta-\alpha)} - \tfrac{(\delta+\beta-2\alpha)^3}{384(\delta-\alpha)(\beta-\alpha)(\delta-\beta)^2}\right)\left(\tfrac{3a^4-5b^4-6a^2b^2+12a^3b-4ab^3}{b-a}\right) + \left(\tfrac{(\delta+\beta-2\alpha)^2(\delta+\beta+2\alpha)}{96(\delta-\alpha)(\beta-\alpha)(\delta-\beta)} - \tfrac{\delta-5\beta+8\alpha}{96(\beta-\alpha)}\right) \\
& \left(\tfrac{a^3-b^3-3ab^2+3a^2b}{b-a}\right) + \left(\tfrac{(\delta+\beta-2\alpha)^2(\delta+\beta+6\alpha)}{256(\beta-\alpha)(\delta-\alpha)} - \tfrac{(\delta-\beta)(\delta-9\beta+16\alpha)}{256(\beta-\alpha)} - \tfrac{\delta^2-2\alpha^2}{16(\delta-\alpha)} - \tfrac{\delta-3\beta-6\alpha}{64}\right)(b-a) + \tfrac{\delta^3-2\alpha^3}{12(\delta-\alpha)} \\
& - \left(\tfrac{(\delta+\beta-2\alpha)^3}{96(\beta-\alpha)(\delta-\alpha)(\delta-\beta)^2} + \tfrac{\delta+\beta-2\alpha}{24(\delta-\beta)^2} - \tfrac{1}{96(\beta-\alpha)}\right)(a+b)^3 + \left(\tfrac{(\delta+\beta-2\alpha)^2(\delta+\beta+2\alpha)}{64(\beta-\alpha)(\delta-\alpha)(\delta-\beta)} + \tfrac{5\beta-8\alpha-\delta}{64(\beta-\alpha)} + \tfrac{3\delta-\beta+2\alpha}{16(\delta-\beta)}\right) \\
& (a+b)^2 + \left(\tfrac{1}{24\times256(\beta-\alpha)} - \tfrac{(\delta+\beta-2\alpha)^3}{24\times256(\beta-\alpha)(\delta-\alpha)(\delta-\beta)^2}\right)\tfrac{(\delta-\beta-4\alpha)^4}{b-a} + \left(\tfrac{(\delta+\beta-2\alpha)^2(\delta+\beta+2\alpha)}{4\times384(\beta-\alpha)(\delta-\alpha)(\delta-\beta)} - \tfrac{\delta-5\beta+8\alpha}{4\times384(\beta-\alpha)}\right) \\
& \tfrac{(\delta-\beta-4\alpha)^3}{b-a} - \left(\tfrac{(\delta+\beta-2\alpha)^2(\delta+\beta+6\alpha)}{4\times256(\beta-\alpha)(\delta-\alpha)} - \tfrac{(\delta-\beta)(\delta+16\alpha-9\beta)}{4\times256(\beta-\alpha)} - \tfrac{\delta^2-2\alpha^2}{64(\delta-\alpha)}\right)\tfrac{(\delta-\beta-4\alpha)^2}{b-a} + \tfrac{(\delta+\beta-2\alpha)^2(\delta+\beta+10\alpha)(\delta-\beta)}{384\times2(\delta-\alpha)(\beta-\alpha)} \\
& + \left(\tfrac{(\delta+\beta-2\alpha)^2(\delta+\beta+4\alpha)(\delta-\beta)}{4\times384(\beta-\alpha)(\delta-\alpha)} + \tfrac{(13\beta-\delta-24\alpha)(\delta-\beta)^2}{4\times384(\beta-\alpha)} - \tfrac{(\delta^2-2\alpha^2)(\delta-\beta)}{32(\delta-\alpha)} + \tfrac{\delta^3-2\alpha^3}{24(\delta-\alpha)}\right)\tfrac{(\delta-\beta-4\alpha)}{b-a} - \tfrac{(\delta^2-2\alpha^2)(\delta-\beta)}{16(\delta-\alpha)} \\
& + \tfrac{64\alpha^2+5\beta^2+11\delta^2+26\alpha\beta-10\alpha\delta}{192} + \tfrac{(3\delta-\beta+2\alpha)(a^3-b^3+3a^2b-3ab^2)}{24(\delta-\beta)(b-a)} + \tfrac{(\delta+\beta-2\alpha)(\delta-\beta-4\alpha)^4}{256\times6(\delta-\beta)^2(b-a)} + \tfrac{(3\delta-\beta+2\alpha)(\delta-\beta-4\alpha)^3}{384(\delta-\beta)(b-a)} \\
& + \tfrac{(\delta-3\beta-6\alpha)(\delta-\beta-4\alpha)^2}{256(b-a)} - \tfrac{(64\alpha^2+5\beta^2+11\delta^2+26\alpha\beta-10\alpha\delta)(\delta-\beta-4\alpha)}{384(b-a)} - \tfrac{(\delta+\beta-2\alpha)(5a^4-3b^4+6a^2b^2+4a^3b-12ab^3)}{96(\delta-\beta)^2(b-a)} \\
& + \tfrac{(13\beta-\delta-24\alpha)(\delta-\beta)^2}{384\times2(\beta-\alpha)}.
\end{aligned}$$

If $2\alpha < \delta - \beta < 4\alpha$, *and* $b - a < (\beta - \delta + 4\alpha)/2$, *then*

$$V[\xi] = \tfrac{3\delta-\beta+2\alpha}{24(\delta-\beta)}(b-a)^2 + \tfrac{11\delta^2+64\alpha^2+5\beta^2+26\alpha\beta-10\alpha\delta}{96}.$$

If $\delta-\beta\geq 4\alpha$, *and* $b-a\geq(3\delta+\beta)/2$, *then*

$$
\begin{aligned}
V[\xi]=&\frac{16\alpha^2+5\beta^2+5\delta^2+32\alpha\beta-16\alpha\delta+6\beta\delta}{48}-\frac{(\delta+\beta)(64\alpha^3+\beta^3+71\delta^3+8\alpha^2\beta-8\alpha^2\delta+24\alpha\beta^2+40\alpha\delta^2-21\beta^2\delta-29\beta\delta^2)}{384(\delta-\alpha)(b-a)}+\frac{(\delta-\beta)(a+b)}{2}\\
&-\frac{(3\delta+\beta)(16\alpha^2+32\alpha\beta-16\alpha\delta+6\beta\delta+3\beta^2+23\delta^2)}{192(b-a)}-\frac{(\delta+3\beta)(16\alpha^2+32\alpha\beta-16\alpha\delta+6\beta\delta+23\beta^2+3\delta^2)}{192(b-a)}+\frac{(b-a)^2}{12}-\frac{(\delta-\beta)(\delta+\beta)(a+b)}{4(b-a)}\\
&+\frac{(\delta-\beta)^2(11\delta-5\beta+10\alpha)}{6\times256(b-a)}+\frac{9\alpha^2-2\beta^2+4\delta^2-5\alpha\beta+7\alpha\delta+6\beta\delta}{48(b-a)}\alpha+\frac{(\delta+\beta-2\alpha)^2(11\delta+11\beta+42\alpha)(\delta-\beta)^2}{24\times256(\delta-\alpha)(\beta-\alpha)(b-a)}-\frac{(3\delta-\beta+2\alpha)(\delta-\beta-4\alpha)^3}{384(\delta-\beta)(b-a)}\\
&-\frac{(\delta+\beta-2\alpha)^3(\delta-\beta-4\alpha)^4}{24\times256(\delta-\alpha)(\beta-\alpha)(\delta-\beta)^2(b-a)}-\frac{(\delta-\beta)^3(11\delta-75\beta+128\alpha)}{256\times24(\beta-\alpha)(b-a)}-\frac{(\delta+\beta-2\alpha)^2(\delta-\beta-4\alpha)^2(\delta+\beta+6\alpha)}{256\times4(\delta-\alpha)(\beta-\alpha)(b-a)}+\frac{(\delta^3-2\alpha^3)(\delta-2\alpha-\beta)}{12(\delta-\alpha)(b-a)}\\
&-\frac{(\delta^2-2\alpha^2)(\delta+4\alpha-\beta)(5\delta-8\alpha-5\beta)}{64(\delta-\alpha)(b-a)}+\frac{(\delta-\beta-4\alpha)^3(19\beta-3\delta-36\alpha)}{256\times24(\beta-\alpha)(b-a)}+\frac{3\alpha^3+9\beta^3+15\alpha^2\beta-8\alpha^2\delta-9\alpha\beta^2+4\alpha\delta^2+12\alpha\beta\delta+6\beta^2\delta}{24(b-a)}\\
&+\frac{(\delta+\beta-2\alpha)^2(\delta-\beta-4\alpha)^3(\delta+\beta+2\alpha)}{384\times4(\delta-\alpha)(\beta-\alpha)(\delta-\beta)(b-a)}+\frac{(\delta-\beta-2\alpha)(\delta-\beta)^2(13\beta-\delta-24\alpha)}{384\times2(\beta-\alpha)(b-a)}+\frac{(\delta^2-2\alpha^2)(\delta-4\alpha-\beta)^2}{64(\delta-\alpha)(b-a)}+\frac{(\delta+\beta-2\alpha)(\delta-\beta-4\alpha)^4}{6\times256(\delta-\beta)^2(b-a)}\\
&+\frac{(\delta+\beta-2\alpha)^2(\delta-\beta-2\alpha)(\delta+\beta+10\alpha)(\delta-\beta)}{384\times2(\delta-\alpha)(\beta-\alpha)(b-a)}+\frac{(\delta-\beta-4\alpha)^2(\delta-\beta)(\delta-9\beta+16\alpha)}{256\times4(\beta-\alpha)(b-a)}.
\end{aligned}
$$

If $\delta-\beta\geq 4\alpha$, *and* $(\delta+3\beta)/2\leq b-a<(3\delta+\beta)/2$, *then*

$$
\begin{aligned}
V[\xi]=&\frac{(b-a)^3}{384(\delta-\alpha)}-\frac{(\delta+3\beta)(16\alpha^2+32\alpha\beta-16\alpha\delta+6\beta\delta+23\beta^2+3\delta^2)}{192(b-a)}+\frac{16\alpha^2+5\beta^2+5\delta^2+32\alpha\beta-16\alpha\delta+6\beta\delta}{96}-\frac{(\delta-\beta)(\delta+3\beta)(a+b)}{16(b-a)}\\
&+\frac{(b-a)^2}{24}+\frac{(\delta-\beta)(5a+3b)}{16}+\frac{(5\delta-\beta-8\alpha)(b-a)^2}{192(\delta-\alpha)}+\frac{(3\delta+\beta)^2(b-a)}{256(\delta-\alpha)}+\frac{16\alpha^2+4\alpha\beta+12\alpha\delta+6\beta\delta+17\beta^2+3\delta^2}{192}+\frac{(4\alpha-\delta+\beta)^3(\delta-\beta)}{384\times4(b-a)(\delta-\alpha)}\\
&+\frac{(\delta-\beta)^4}{256\times24(\delta-\alpha)(b-a)}-\frac{(\delta-\beta)^3(5\delta-\beta-8\alpha)}{384\times4(\delta-\alpha)(b-a)}-\frac{(16\alpha^2+4\alpha\beta+12\alpha\delta+6\beta\delta+7\beta^2+3\delta^2)(\delta-\beta)}{384(b-a)}+\frac{(\delta+\beta-2\alpha)^2(\delta-\beta-4\alpha)^3(\delta+\beta+2\alpha)}{384\times4(\delta-\alpha)(\beta-\alpha)(\delta-\beta)(b-a)}\\
&-\frac{(\delta-\beta)^2(3\delta+\beta)^2}{256\times4(\delta-\alpha)(b-a)}-\frac{(\delta+\beta-2\alpha)^3(\delta-\beta-4\alpha)^4}{24\times256(\delta-\alpha)(\beta-\alpha)(\delta-\beta)^2(b-a)}-\frac{(4\alpha-\delta+\beta)^3}{384\times2(\delta-\alpha)}+\frac{(\delta^3-2\alpha^3)(\delta-2\alpha-\beta)}{12(\delta-\alpha)(b-a)}-\frac{(\delta-\beta)^3(11\delta-75\beta+128\alpha)}{256\times24(\beta-\alpha)(b-a)}\\
&-\frac{(\delta+\beta-2\alpha)^2(\delta-\beta-4\alpha)^2(\delta+\beta+6\alpha)}{256\times4(\delta-\alpha)(\beta-\alpha)(b-a)}+\frac{(\delta-\beta-4\alpha)^2(\delta-\beta)(\delta-9\beta+16\alpha)}{256\times4(\beta-\alpha)(b-a)}+\frac{(\delta+\beta-2\alpha)(\delta-\beta-4\alpha)^4}{6\times256(\delta-\beta)^2(b-a)}-\frac{(3\delta-\beta+2\alpha)(\delta-\beta-4\alpha)^3}{384(\delta-\beta)(b-a)}\\
&-\frac{(\delta^2-2\alpha^2)(\delta+4\alpha-\beta)(5\delta-8\alpha-5\beta)}{64(\delta-\alpha)(b-a)}+\frac{(\delta-\beta-2\alpha)(\delta-\beta)^2(13\beta-\delta-24\alpha)}{384\times2(\beta-\alpha)(b-a)}+\frac{3\alpha^3+9\beta^3+15\alpha^2\beta-8\alpha^2\delta-9\alpha\beta^2+4\alpha\delta^2+12\alpha\beta\delta+6\beta^2\delta}{24(b-a)}\\
&+\frac{(\delta-\beta-4\alpha)^3(19\beta-3\delta-36\alpha)}{256\times24(\beta-\alpha)(b-a)}+\frac{(\delta^2-2\alpha^2)(\delta-4\alpha-\beta)^2}{64(\delta-\alpha)(b-a)}+\frac{(\delta+\beta-2\alpha)^2(11\delta+11\beta+42\alpha)(\delta-\beta)^2}{24\times256(\delta-\alpha)(\beta-\alpha)(b-a)}+\frac{(\delta-\beta)^2(11\delta-5\beta+10\alpha)}{6\times256(b-a)}\\
&+\frac{9\alpha^2-2\beta^2+4\delta^2-5\alpha\beta+7\alpha\delta+6\beta\delta}{48(b-a)}\alpha+\frac{(\delta+\beta-2\alpha)^2(\delta-\beta-2\alpha)(\delta+\beta+10\alpha)(\delta-\beta)}{384\times2(\delta-\alpha)(\beta-\alpha)(b-a)}.
\end{aligned}
$$

If $\delta-\beta\geq 4\alpha$, *and* $(\delta-\beta+4\alpha)/2\leq b-a<(\delta+3\beta)/2$, *then*

$$
\begin{aligned}
V[\xi]=&\frac{\beta+\delta-2\alpha}{384(\delta-\alpha)(\beta-\alpha)}(b-a)^3+\left(\frac{5\delta-\beta-8\alpha}{192(\delta-\alpha)}+\frac{5\beta-\delta-8\alpha}{192(\beta-\alpha)}\right)(b-a)^2+\left(\frac{(3\delta+\beta)^2}{256(\delta-\alpha)}+\frac{(\delta+3\beta)^2}{256(\beta-\alpha)}\right)(b-a)\\
&+\frac{16\alpha^2-8\alpha\beta+24\alpha\delta+6\beta\delta+13\beta^2-3\delta^2}{96}-\frac{(\delta-\beta)(16\alpha^2+4\alpha\beta+12\alpha\delta+6\beta\delta+7\beta^2+3\delta^2)}{384(b-a)}-\frac{(4\alpha+\delta-\beta)^3}{384\times2(\beta-\alpha)}-\frac{(4\alpha-\delta+\beta)^3}{384\times2(\delta-\alpha)}\\
&-\frac{(\delta-\beta)^3(5\delta-\beta-8\alpha)}{384\times4(\delta-\alpha)(b-a)}+\frac{(4\alpha+\delta-\beta)^4}{256\times8(\beta-\alpha)(b-a)}-\frac{(\delta-\beta)^4}{256\times24(\delta-\alpha)(b-a)}-\frac{(\delta-\beta+4\alpha)^3(5\beta-\delta-8\alpha)}{384\times4(\beta-\alpha)(b-a)}-\frac{(\delta-\beta)^2(3\delta+\beta)^2}{256\times4(\delta-\alpha)(b-a)}\\
&+\frac{(\delta+\beta-2\alpha)^2(\delta-\beta-4\alpha)^3(\delta+\beta+2\alpha)}{384\times4(\delta-\alpha)(\beta-\alpha)(\delta-\beta)(b-a)}+\frac{(\delta-\beta)(4\alpha-\delta+\beta)^3}{384\times4(\delta-\alpha)(b-a)}+\frac{9\alpha^3+2\alpha\delta^2+5\alpha^2\delta}{12(b-a)}-\frac{(\delta+\beta-2\alpha)^3(\delta-\beta-4\alpha)^4}{24\times256(\delta-\alpha)(\beta-\alpha)(\delta-\beta)^2(b-a)}\\
&-\frac{(4\alpha+\delta-\beta)(16\alpha^2-20\alpha\beta+36\alpha\delta+6\beta\delta+19\beta^2-9\delta^2)}{384(b-a)}-\frac{(\delta-\beta)^3(11\delta-75\beta+128\alpha)}{256\times24(\beta-\alpha)(b-a)}+\frac{(\delta-\beta-2\alpha)(\delta-\beta)^2(13\beta-\delta-24\alpha)}{384\times2(\beta-\alpha)(b-a)}\\
&+\frac{(\delta+\beta-2\alpha)(\delta-\beta-4\alpha)^4}{6\times256(\delta-\beta)^2(b-a)}-\frac{(3\delta-\beta+2\alpha)(\delta-\beta-4\alpha)^3}{384(\delta-\beta)(b-a)}+\frac{(\delta-\beta)^2(11\delta-5\beta+10\alpha)}{6\times256(b-a)}+\frac{9\alpha^2-2\beta^2+4\delta^2-5\alpha\beta+7\alpha\delta+6\beta\delta}{48(b-a)}\alpha\\
&-\frac{(\delta+\beta-2\alpha)^2(\delta-\beta-4\alpha)^2(\delta+\beta+6\alpha)}{256\times4(\delta-\alpha)(\beta-\alpha)(b-a)}+\frac{(\delta+\beta-2\alpha)^2(11\delta+11\beta+42\alpha)(\delta-\beta)^2}{24\times256(\delta-\alpha)(\beta-\alpha)(b-a)}+\frac{(\delta+\beta-2\alpha)^2(\delta-\beta-2\alpha)(\delta+\beta+10\alpha)(\delta-\beta)}{384\times2(\delta-\alpha)(\beta-\alpha)(b-a)}\\
&+\frac{(\delta-\beta-4\alpha)^2(\delta-\beta)(\delta-9\beta+16\alpha)}{256\times4(\beta-\alpha)(b-a)}-\frac{(\delta^2-2\alpha^2)(\delta+4\alpha-\beta)(5\delta-8\alpha-5\beta)}{64(\delta-\alpha)(b-a)}+\frac{(\delta^2-2\alpha^2)(\delta-4\alpha-\beta)^2}{64(\delta-\alpha)(b-a)}+\frac{(\delta^3-2\alpha^3)(\delta-2\alpha-\beta)}{12(\delta-\alpha)(b-a)}\\
&+\frac{(\delta-\beta-4\alpha)^3(19\beta-3\delta-36\alpha)}{256\times24(\beta-\alpha)(b-a)}-\frac{(4\alpha+\delta-\beta)^2(\delta+3\beta)^2}{256\times4(\beta-\alpha)(b-a)}.
\end{aligned}
$$

If $\delta-\beta\geq 4\alpha$, *and* $(\delta-\beta)/2\leq b-a<(\delta-\beta+4\alpha)/2$, *then*

$$
\begin{aligned}
V[\xi]=&\frac{1}{384(\delta-\alpha)}(b-a)^3+\frac{13\delta-\beta-16\alpha}{192(\delta-\alpha)}(b-a)^2+\left(\frac{(3\delta+\beta)^2}{256(\delta-\alpha)}+\frac{\beta-\delta}{16}\right)(b-a)+\frac{32\alpha^2+3\beta^2+5\delta^2-6\alpha\beta+14\alpha\delta}{96}\\
&+\frac{16\alpha^2+4\alpha\beta+12\alpha\delta+6\beta\delta+7\beta^2+3\delta^2}{192}+\frac{(\delta-\beta)^4}{256\times24(\delta-\alpha)(b-a)}-\frac{(4\alpha-\delta+\beta)^3}{384\times2(\delta-\alpha)}+\frac{(\delta-\beta)(32\alpha^2-3\alpha\beta+11\alpha\delta+\beta^2+6\delta^2+\beta\delta)}{192(b-a)}\\
&-\frac{(\delta-\beta)^3(5\delta-\beta-8\alpha)}{384\times4(\delta-\alpha)(b-a)}-\frac{(\delta-\beta)^2(3\delta+\beta)^2}{256\times4(\delta-\alpha)(b-a)}-\frac{(\delta-\beta)(16\alpha^2+4\alpha\beta+12\alpha\delta+6\beta\delta+7\beta^2+3\delta^2)}{384(b-a)}-\frac{(\delta-\beta)^3(11\delta-75\beta+128\alpha)}{256\times24(\beta-\alpha)(b-a)}\\
&-\frac{(\delta+\beta-2\alpha)^3(\delta-\beta-4\alpha)^4}{24\times256(\delta-\alpha)(\beta-\alpha)(\delta-\beta)^2(b-a)}+\frac{(\delta+\beta-2\alpha)^2(\delta-\beta-4\alpha)^3(\delta+\beta+2\alpha)}{384\times4(\delta-\alpha)(\beta-\alpha)(\delta-\beta)(b-a)}+\frac{(\beta-\delta+4\alpha)^3(\delta-\beta)}{384\times4(\delta-\alpha)(b-a)}+\frac{(\delta-\beta)^2(11\delta-5\beta+10\alpha)}{6\times256(b-a)}\\
&+\frac{(\delta+\beta-2\alpha)^2(\delta-\beta-2\alpha)(\delta+\beta+10\alpha)(\delta-\beta)}{384\times2(\delta-\alpha)(\beta-\alpha)(b-a)}-\frac{(\delta+\beta-2\alpha)^2(\delta-\beta-4\alpha)^2(\delta+\beta+6\alpha)}{256\times4(\delta-\alpha)(\beta-\alpha)(b-a)}+\frac{(\delta+\beta-2\alpha)^2(11\delta+11\beta+42\alpha)(\delta-\beta)^2}{24\times256(\delta-\alpha)(\beta-\alpha)(b-a)}
\end{aligned}
$$

$$+\frac{(\delta^2-2\alpha^2)(\delta-4\alpha-\beta)^2}{64(\delta-\alpha)(b-a)}+\frac{(\delta-\beta-2\alpha)(\delta-\beta)^2(13\beta-\delta-24\alpha)}{384\times 2(\beta-\alpha)(b-a)}+\frac{(\delta-\beta-4\alpha)^2(\delta-\beta)(\delta-9\beta+16\alpha)}{256\times 4(\beta-\alpha)(b-a)}+\frac{(\delta^3-2\alpha^3)(\delta-2\alpha-\beta)}{12(\delta-\alpha)(b-a)}$$
$$-\frac{(\delta^2-2\alpha^2)(\delta+4\alpha-\beta)(5\delta-8\alpha-5\beta)}{64(\delta-\alpha)(b-a)}+\frac{(\delta+\beta-2\alpha)(\delta-\beta-4\alpha)^4}{6\times 256(\delta-\beta)^2(b-a)}-\frac{(3\delta-\beta+2\alpha)(\delta-\beta-4\alpha)^3}{384(\delta-\beta)(b-a)}+\frac{(\delta-\beta-4\alpha)^3(19\beta-3\delta-36\alpha)}{256\times 24(\beta-\alpha)(b-a)}$$
$$+\frac{9\alpha^2-2\beta^2+4\delta^2-5\alpha\beta+7\alpha\delta+6\beta\delta}{48(b-a)}\alpha.$$

If $\delta-\beta\geq 4\alpha$, and $(\delta-\beta-4\alpha)/2\leq b-a<(\delta-\beta)/2$, then

$$V[\xi]=\left(\frac{1}{384(\beta-\alpha)}-\frac{(\delta+\beta-2\alpha)^3}{384(\delta-\alpha)(\beta-\alpha)(\delta-\beta)^2}\right)\left(\frac{3a^4-5b^4-6a^2b^2+12a^3b-4ab^3}{b-a}\right)+\left(\frac{(\delta+\beta-2\alpha)^2(\delta+\beta+2\alpha)}{96(\delta-\alpha)(\beta-\alpha)(\delta-\beta)}-\frac{\delta-5\beta+8\alpha}{96(\beta-\alpha)}\right)$$
$$\left(\frac{a^3-b^3-3ab^2+3a^2b}{b-a}\right)+\left(\frac{(\delta+\beta-2\alpha)^2(\delta+\beta+6\alpha)}{256(\beta-\alpha)(\delta-\alpha)}-\frac{(\delta-\beta)(\delta-9\beta+16\alpha)}{256(\beta-\alpha)}-\frac{\delta^2-2\alpha^2}{16(\delta-\alpha)}-\frac{\delta-3\beta-6\alpha}{64}\right)(b-a)+\left(\frac{1}{96(\beta-\alpha)}\right.$$
$$\left.-\frac{(\delta+\beta-2\alpha)^3}{96(\beta-\alpha)(\delta-\alpha)(\delta-\beta)^2}\right)(a+b)^3+\left(\frac{(\delta+\beta-2\alpha)^2(\delta+\beta+2\alpha)}{64(\beta-\alpha)(\delta-\alpha)(\delta-\beta)}+\frac{5\beta-8\alpha-\delta}{64(\beta-\alpha)}\right)(a+b)^2+\frac{(\delta+\beta-2\alpha)^2(\delta+\beta+10\alpha)(\delta-\beta)}{384\times 2(\delta-\alpha)(\beta-\alpha)}$$
$$+\frac{\delta^3-2\alpha^3}{12(\delta-\alpha)}+\frac{(13\beta-\delta-24\alpha)(\delta-\beta)^2}{384\times 2(\beta-\alpha)}-\frac{(\delta^2-2\alpha^2)(\delta-\beta)}{16(\delta-\alpha)}+\left(\frac{1}{24\times 256(\beta-\alpha)}-\frac{(\delta+\beta-2\alpha)^3}{24\times 256(\beta-\alpha)(\delta-\alpha)(\delta-\beta)^2}\right)\frac{(\delta-\beta-4\alpha)^4}{b-a}$$
$$+\left(\frac{(\delta+\beta-2\alpha)^2(\delta+\beta+2\alpha)}{4\times 384(\beta-\alpha)(\delta-\alpha)(\delta-\beta)}-\frac{\delta-5\beta+8\alpha}{4\times 384(\beta-\alpha)}\right)\frac{(\delta-\beta-4\alpha)^3}{b-a}-\left(\frac{(\delta+\beta-2\alpha)^2(\delta+\beta+6\alpha)}{4\times 256(\beta-\alpha)(\delta-\alpha)}-\frac{(\delta-\beta)(\delta+16\alpha-9\beta)}{4\times 256(\beta-\alpha)}-\frac{\delta^2-2\alpha^2}{64(\delta-\alpha)}\right)$$
$$\frac{(\delta-\beta-4\alpha)^2}{b-a}+\left(\frac{(\delta+\beta-2\alpha)^2(\delta+\beta+4\alpha)(\delta-\beta)}{4\times 384(\beta-\alpha)(\delta-\alpha)}+\frac{(13\beta-\delta-24\alpha)(\delta-\beta)^2}{4\times 384(\beta-\alpha)}-\frac{(\delta^2-2\alpha^2)(\delta-\beta)}{32(\delta-\alpha)}+\frac{\delta^3-2\alpha^3}{24(\delta-\alpha)}\right)\frac{\delta-\beta-4\alpha}{b-a}$$
$$-\frac{(\delta+\beta-2\alpha)(5a^4-3b^4+6a^2b^2+4a^3b-12ab^3)}{96(\delta-\beta)^2(b-a)}-\frac{\delta+\beta-2\alpha}{24(\delta-\beta)^2}(a+b)^3+\frac{3\delta-\beta+2\alpha}{16(\delta-\beta)}(a+b)^2+\frac{64\alpha^2+5\beta^2+11\delta^2+26\alpha\beta-10\alpha\delta}{192}$$
$$+\frac{(\delta+\beta-2\alpha)(\delta-\beta-4\alpha)^4}{256\times 6(\delta-\beta)^2(b-a)}+\frac{(3\delta-\beta+2\alpha)(\delta-\beta-4\alpha)^3}{384(\delta-\beta)(b-a)}+\frac{(\delta-3\beta-6\alpha)(\delta-\beta-4\alpha)^2}{256(b-a)}-\frac{(64\alpha^2+5\beta^2+11\delta^2+26\alpha\beta-10\alpha\delta)(\delta-\beta-4\alpha)}{384(b-a)}$$
$$+\frac{(3\delta-\beta+2\alpha)(a^3-b^3+3a^2b-3ab^2)}{24(\delta-\beta)(b-a)}.$$

If $\delta-\beta\geq 4\alpha$, and $b-a<(\delta-\beta-4\alpha)/2$, then

$$V[\xi]=\left(\frac{(\delta+\beta-2\alpha)^2(\delta+\beta+2\alpha)}{96(\delta-\alpha)(\beta-\alpha)(\delta-\beta)}-\frac{\delta-5\beta+8\alpha}{96(\beta-\alpha)}\right)(b-a)^2+\frac{(\delta+\beta-2\alpha)^2(\delta-\beta)(\delta+\beta+10\alpha)}{384(\delta-\alpha)(\beta-\alpha)}$$
$$+\frac{(13\beta-\delta-24\alpha)(\delta-\beta)^2}{384(\beta-\alpha)}-\frac{(\delta-\beta)(\delta^2-2\alpha^2)}{8(\delta-\alpha)}+\frac{\delta^3-2\alpha^3}{6(\delta-\alpha)}.$$

Proof: The proof of the theorem is similar to that of Theorem 3.1. □

Example 3.2 *Let ξ be a trapezoidal fuzzy random variable. For each ω, $\xi(\omega)=(X(\omega)-3,X(\omega)-1,X(\omega)+1,X(\omega)+2)$ is a triangular fuzzy variable with $X\sim\mathcal{U}(0,1)$. Since $\alpha=1,\beta=2,\delta=3,a=0,b=1$, $\delta-\beta=1<2\alpha=2$, and $(\delta-\beta)/2=1/2<b-a=1<(\beta-\delta+4\alpha)/2=3/2$, by Theorem 3.2, we have*

$$V[\xi]=\frac{(b-a)^2}{12}+\frac{\beta+\delta-2\alpha}{16}(b-a)+\frac{8\alpha^2+\beta^2+\delta^2+\alpha\beta+\alpha\delta}{12}+\frac{(\delta-\beta)(10\alpha\beta-10\alpha\delta-\beta^2+\delta^2)}{192(b-a)}=\frac{463}{192}.$$

Proposition 3.3 *Let ξ be a trapezoidal fuzzy random variable such that for each ω, $\xi(\omega)=(X(\omega)-\delta,X(\omega)-\alpha,X(\omega)+\alpha,X(\omega)+\beta)$, where $\delta<\beta$, and denote $m=(4E[X]-\delta+\beta)/4$.*
(1) If $X(\omega)\leq m-\beta$, then

$$E[(\xi(\omega)-m)^2]=X^2(\omega)-2mX(\omega)+m^2+\frac{(\delta-\beta)m}{2}+\frac{2\alpha^2+\beta^2+\delta^2-2\alpha\delta+4\alpha\beta}{6}.$$

(2) If $m-\beta<X(\omega)\leq m-\alpha$, and $\beta-\delta\leq 2\alpha$, or $m-\beta<X(\omega)\leq m-(\beta-\delta)/2$, and $\beta-\delta>2\alpha$, then

$$E[(\xi(\omega)-m)^2]=-\frac{1}{6(\beta-\alpha)}X^3(\omega)+\frac{m+\beta-2\alpha}{2(\beta-\alpha)}X^2(\omega)-\left(\frac{\alpha+\delta}{2}+m+\frac{(m-\alpha)^2}{2(\beta-\alpha)}\right)X(\omega)$$
$$+\frac{(m-\alpha)^3}{6(\beta-\alpha)}+\frac{\alpha m+\delta m+\alpha\delta+m^2}{2}+\frac{(\beta-\alpha)^2}{6}.$$

(3) If $m-\alpha<X(\omega)\leq m-(\beta-\delta)/2$, and $\beta-\delta\leq 2\alpha$, then

$$E[(\xi(\omega)-m)^2]=X^2(\omega)-\left(\frac{\delta-\alpha}{2}+2m\right)X(\omega)+\alpha^2+m^2+\frac{(\delta-\alpha)(\alpha+m)}{2}+\frac{(\delta-\alpha)^2}{6}.$$

If $m-(\beta-\delta)/2<X(\omega)\leq m-\alpha$, and $\beta-\delta>2\alpha$, then

$$E[(\xi(\omega)-m)^2]=\left(\frac{1}{6(\delta-\alpha)}-\frac{(\delta+\beta-2\alpha)^3}{6(\delta-\alpha)(\beta-\alpha)(\beta-\delta)^2}\right)X^3(\omega)+\left(\frac{(\delta+\beta-2\alpha)^2(\alpha\beta-\alpha\delta+m\delta+m\beta-2m\alpha)}{2(\delta-\alpha)(\beta-\alpha)(\beta-\delta)^2}\right.$$
$$\left.+\frac{1}{2}-\frac{\alpha+m}{2(\delta-\alpha)}\right)X^2(\omega)-\left(\frac{(\delta+\beta-2\alpha)(\alpha\delta-\alpha\beta-m\delta-m\beta+2m\alpha)^2}{2(\delta-\alpha)(\beta-\alpha)(\beta-\delta)^2}+m-\frac{(\alpha+m)^2}{2(\delta-\alpha)}-\frac{\alpha+\beta}{2}\right)X(\omega)$$
$$-\frac{(\alpha\delta-\alpha\beta-m\delta-m\beta+2\alpha m)^3}{6(\delta-\alpha)(\beta-\alpha)(\beta-\delta)^2}-\frac{(\alpha+m)^3}{6(\delta-\alpha)}+\frac{(\beta-\alpha)^2}{6}+\frac{(\alpha-m)(\beta-m)}{2}.$$

(4) *If* $m-(\beta-\delta)/2 < X(\omega) \le m$, *and* $\beta-\delta \le 2\alpha$, *or* $m-\alpha < X(\omega) \le m$, *and* $\beta-\delta > 2\alpha$, *then*

$$\begin{aligned}E[(\xi(\omega)-m)^2] =& \left(\tfrac{\delta+\beta-2\alpha}{6(\delta-\alpha)(\beta-\alpha)} - \tfrac{(\delta+\beta-2\alpha)^3}{6(\delta-\alpha)(\beta-\alpha)(\beta-\delta)^2}\right) X^3(\omega) + \left(\tfrac{(\delta+\beta-2\alpha)^2(\delta m+\beta m-2\alpha m+\alpha\beta-\alpha\delta)}{2(\delta-\alpha)(\beta-\alpha)(\beta-\delta)^2}\right.\\ &\left.+\tfrac{2\beta-\alpha-m}{2(\beta-\alpha)} - \tfrac{m+\alpha}{2(\delta-\alpha)}\right) X^2(\omega) + \left(\tfrac{(m+\alpha)^2}{2(\delta-\alpha)} - \tfrac{(\delta+\beta-2\alpha)(\delta m+\beta m-2\alpha m+\alpha\beta-\alpha\delta)^2}{2(\delta-\alpha)(\beta-\alpha)(\beta-\delta)^2} + \tfrac{(\alpha-m)^2}{2(\beta-\alpha)} + \tfrac{3\alpha+\beta-4m}{2}\right)\\ & X(\omega) - \tfrac{(m+\alpha)^3}{6(\delta-\alpha)} + \tfrac{(\alpha-m)^3}{6(\beta-\alpha)} + \tfrac{(\beta-\alpha)^2}{6} + \tfrac{(\alpha-m)(\alpha+\beta-2m)}{2} + \tfrac{(\delta m+\beta m-2\alpha m+\alpha\beta-\alpha\delta)^3}{6(\delta-\alpha)(\beta-\alpha)(\beta-\delta)^2}.\end{aligned}$$

(5) *If* $m < X(\omega) \le m+\alpha$, *then*

$$E[(\xi(\omega)-m)^2] = X^2(\omega) + \left(\tfrac{\beta-\alpha}{2} - 2m\right) X(\omega) + \alpha^2 + m^2 + \tfrac{(\beta-\alpha)(\alpha-m)}{2} + \tfrac{(\beta-\alpha)^2}{6}.$$

(6) *If* $m+\alpha < X(\omega) \le m+\delta$, *then*

$$\begin{aligned}E[(\xi(\omega)-m)^2] =& \tfrac{1}{6(\delta-\alpha)} X^3(\omega) + \left(\tfrac{1}{2} - \tfrac{m+\alpha}{2(\delta-\alpha)}\right) X^2(\omega) + \left(\tfrac{\alpha+\beta}{2} - m + \tfrac{(m+\alpha)^2}{2(\delta-\alpha)}\right) X(\omega)\\ & - \tfrac{(m+\alpha)^3}{6(\delta-\alpha)} + \tfrac{(\alpha-m)^2}{2} + \tfrac{(\beta-\alpha)(\alpha-m)}{6} + \tfrac{(\beta-\alpha)^2}{6}.\end{aligned}$$

(7) *If* $X(\omega) > m+\delta$, *then*

$$E[(\xi(\omega)-m)^2] = X^2(\omega) - 2mX(\omega) + m^2 + \tfrac{(\delta-\beta)m}{2} + \tfrac{2\alpha^2+\beta^2+\delta^2-2\alpha\delta+4\alpha\beta}{6}.$$

Proof: The proof of the proposition is similar to that of Proposition 3.1. □

As a consequence of Proposition 3.3, when $X \sim \mathcal{U}(a,b)$, we obtain the following variance formula for a trapezoidal fuzzy random variable with $\delta < \beta$.

Theorem 3.3 *Let* ξ *be a trapezoidal fuzzy random variable such that for each* ω, $\xi(\omega) = (X(\omega)-\delta, X(\omega)-\alpha, X(\omega)+\alpha, X(\omega)+\beta)$, *where* $\delta < \beta$, *and denote* $m = (4E[X]-\delta+\beta)/4$. *When* $X \sim \mathcal{U}(a,b)$, *we have the following results.*
If $\beta-\delta \le 2\alpha$, *and* $b-a \ge (\delta+3\beta)/2$, *then*

$$\begin{aligned}V[\xi] =& \tfrac{(b-a)^2}{12} - \tfrac{(\delta-\beta)(\delta+\beta)(a+b)}{4(b-a)} + \tfrac{24\alpha^3+37\beta^3+23\delta^3+60\alpha^2\beta-4\alpha^2\delta-16\alpha\beta^2+48\alpha\delta^2+24\alpha\beta\delta+41\beta^2\delta+19\beta\delta^2}{96(b-a)}\\ & + \tfrac{(\delta-\beta)(a+b)}{2} + \tfrac{16\alpha^2+5\beta^2+5\delta^2+32\alpha\beta-16\alpha\delta+6\beta\delta}{48} - \tfrac{(3\delta+\beta)(16\alpha^2+32\alpha\beta-16\alpha\delta+6\beta\delta+3\beta^2+23\delta^2)}{192(b-a)}\\ & - \tfrac{(\delta+3\beta)(16\alpha^2+32\alpha\beta-16\alpha\delta+6\beta\delta+23\beta^2+3\delta^2)}{192(b-a)} + \tfrac{(\beta-\delta)(64\alpha^2+24\alpha\delta-8\alpha\beta-4\beta\delta+6\delta^2+14\beta^2)}{192(b-a)}.\end{aligned}$$

If $\beta-\delta \le 2\alpha$, *and* $(3\delta+\beta)/2 \le b-a < (\delta+3\beta)/2$, *then*

$$\begin{aligned}V[\xi] =& \tfrac{(b-a)^3}{384(\beta-\alpha)} + \tfrac{(13\beta-\delta-16\alpha)(b-a)^2}{192(\beta-\alpha)} + \tfrac{(3\beta+\delta)^2(b-a)}{256(\beta-\alpha)} + \tfrac{(\delta-\beta)(3a+5b)}{16} + \tfrac{48\alpha^2+44\alpha\beta+4\alpha\delta+18\beta\delta+29\beta^2+\delta^2}{192}\\ & - \tfrac{(4\alpha+\delta-\beta)(16\alpha^2-20\alpha\beta+36\alpha\delta+6\beta\delta+19\beta^2-9\delta^2)}{384(b-a)} + \tfrac{84\alpha^3+\beta^3+23\delta^3+68\alpha^2\delta+20\alpha\beta^2+48\alpha\delta^2-24\alpha\beta\delta+17\beta^2\delta+19\beta\delta^2}{96(b-a)}\\ & + \tfrac{(\delta-\beta)(3\delta+\beta)(a+b)}{16(b-a)} - \tfrac{(4\alpha+\delta-\beta)^3}{384\times2(\beta-\alpha)} + \tfrac{(4\alpha+\delta-\beta)^4}{256\times8(\beta-\alpha)(b-a)} - \tfrac{(4\alpha+\delta-\beta)^3(5\beta-\delta-8\alpha)}{384\times4(\beta-\alpha)(b-a)} - \tfrac{(4\alpha+\delta-\beta)^2(3\beta+\delta)^2}{256\times4(\beta-\alpha)(b-a)}\\ & + \tfrac{(\beta-\delta)(64\alpha^2-8\alpha\beta+24\alpha\delta-4\beta\delta+14\beta^2+6\delta^2)}{192(b-a)} - \tfrac{(3\delta+\beta)(16\alpha^2+32\alpha\beta-16\alpha\delta+6\beta\delta+23\delta^2+3\beta^2)}{192(b-a)}.\end{aligned}$$

If $\beta-\delta \le 2\alpha$, *and* $(\beta-\delta+4\alpha)/2 \le b-a < (3\delta+\beta)/2$, *then*

$$\begin{aligned}V[\xi] =& \tfrac{\beta+\delta-2\alpha}{384(\delta-\alpha)(\beta-\alpha)}(b-a)^3 + \left(\tfrac{5\delta-\beta-8\alpha}{192(\delta-\alpha)} + \tfrac{5\beta-\delta-8\alpha}{192(\beta-\alpha)}\right)(b-a)^2 + \left(\tfrac{(3\delta+\beta)^2}{256(\delta-\alpha)} + \tfrac{(\delta+3\beta)^2}{256(\beta-\alpha)}\right)(b-a)\\ & + \tfrac{16\alpha^2-8\alpha\beta+24\alpha\delta+6\beta\delta+13\beta^2-3\delta^2}{96} - \tfrac{(4\alpha-\delta+\beta)^3}{384\times2(\delta-\alpha)} - \tfrac{(4\alpha+\delta-\beta)^3}{384\times2(\beta-\alpha)} + \tfrac{(4\alpha+\delta-\beta)^4}{256\times8(\beta-\alpha)(b-a)} + \tfrac{(4\alpha-\delta+\beta)^4}{256\times8(\delta-\alpha)(b-a)}\\ & - \tfrac{(4\alpha+\delta-\beta)(16\alpha^2-20\alpha\beta+36\alpha\delta+6\beta\delta+19\beta^2-9\delta^2)}{384(b-a)} - \tfrac{(4\alpha+\delta-\beta)^2(\delta+3\beta)^2}{256\times4(\beta-\alpha)(b-a)} + \tfrac{(\beta-\delta)(64\alpha^2-8\alpha\beta+24\alpha\delta-4\beta\delta+6\delta^2+14\beta^2)}{192(b-a)}\\ & - \tfrac{(4\alpha-\delta+\beta)^3(5\delta-\beta-8\alpha)}{384\times4(\delta-\alpha)(b-a)} - \tfrac{(4\alpha-\delta+\beta)^2(3\delta+\beta)^2}{256\times4(\delta-\alpha)(b-a)} + \tfrac{144\alpha^3+40\alpha^2\beta+40\alpha^2\delta+36\alpha\beta^2+36\alpha\delta^2-40\alpha\beta\delta+\beta^2\delta-5\beta\delta^2+\beta^3+3\delta^3}{96(b-a)}\\ & - \tfrac{(4\alpha-\delta+\beta)(16\alpha^2+4\alpha\beta+12\alpha\delta+6\beta\delta+7\beta^2+3\delta^2)}{384(b-a)} - \tfrac{(4\alpha+\delta-\beta)^3(5\beta-\delta-8\alpha)}{384\times4(\beta-\alpha)(b-a)}.\end{aligned}$$

If $\beta-\delta\le 2\alpha$, *and* $(\delta-\beta+4\alpha)/2\le b-a<(\beta-\delta+4\alpha)/2$, *then*

$$\begin{aligned}V[\xi]=&\tfrac{(b-a)^3}{384(\beta-\alpha)}+\tfrac{(13\beta-\delta-16\alpha)(b-a)^2}{192(\beta-\alpha)}+\tfrac{(3\beta+\delta)^2(b-a)}{256(\beta-\alpha)}+\tfrac{(\delta-\alpha)(b-a)}{16}+\tfrac{(\beta-\delta)(32\alpha^2+27\alpha\beta-19\alpha\delta-5\beta\delta+8\beta^2+5\delta^2)}{192(b-a)}\\&-\tfrac{(4\alpha+\delta-\beta)^3}{384\times 2(\beta-\alpha)}+\tfrac{(4\alpha+\delta-\beta)^4}{256\times 8(\beta-\alpha)(b-a)}-\tfrac{(4\alpha+\delta-\beta)^3(5\beta-\delta-8\alpha)}{384\times 4(\beta-\alpha)(b-a)}-\tfrac{(4\alpha+\delta-\beta)(16\alpha^2-20\alpha\beta+36\alpha\delta+6\beta\delta+19\beta^2-9\delta^2)}{384(b-a)}\\&+\tfrac{(\delta-\beta+2\alpha)(18\alpha^2+9\alpha\delta+\alpha\beta-\beta\delta+2\beta^2+3\delta^2)}{48(b-a)}-\tfrac{(4\alpha+\delta-\beta)^2(\delta+3\beta)^2}{256\times 4(\beta-\alpha)(b-a)}+\tfrac{80\alpha^2+8\alpha\beta+24\alpha\delta+6\beta\delta+29\beta^2-3\delta^2}{192}.\end{aligned}$$

If $\beta-\delta\le 2\alpha$, *and* $(\beta-\delta)/2\le b-a<(\beta-\delta+4\alpha)/2$, *then*

$$V[\xi]\ =\ \tfrac{(b-a)^2}{12}+\tfrac{(\beta+\delta-2\alpha)(b-a)}{16}+\tfrac{8\alpha^2+\beta^2+\delta^2+\alpha\beta+\alpha\delta}{12}+\tfrac{(\beta-\delta)(10\alpha\delta-10\alpha\beta+\beta^2-\delta^2)}{192(b-a)}.$$

If $\beta-\delta\le 2\alpha$, *and* $b-a<(\beta-\delta)/2$, *then*

$$V[\xi]=\tfrac{3\beta-\delta+2\alpha}{24(\beta-\delta)}(b-a)^2+\tfrac{11\beta^2+64\alpha^2+5\delta^2+26\alpha\delta-10\alpha\beta}{96}.$$

If $2\alpha<\beta-\delta<4\alpha$, *and* $b-a\ge(\delta+3\beta)/2$, *then*

$$\begin{aligned}V[\xi]=&\tfrac{(b-a)^2}{12}-\tfrac{(\delta^2-\beta^2)(a+b)}{4(b-a)}-\tfrac{(\delta+3\beta)(16\alpha^2+32\alpha\beta-16\alpha\delta+6\beta\delta+23\beta^2+3\delta^2)}{192(b-a)}-\tfrac{(3\delta+\beta)(16\alpha^2+32\alpha\beta-16\alpha\delta+6\beta\delta+3\beta^2+23\delta^2)}{192(b-a)}\\&+\tfrac{(\delta-\beta)(a+b)}{2}-\tfrac{(\delta+\beta-2\alpha)^3(\beta-\delta-4\alpha)^4}{24\times 256(\delta-\alpha)(\beta-\alpha)(\beta-\delta)^2(b-a)}+\tfrac{(\delta+\beta-2\alpha)^2(\beta-\delta-4\alpha)^3(\delta+\beta+2\alpha)}{384\times 4(\delta-\alpha)(\beta-\alpha)(\beta-\delta)(b-a)}-\tfrac{(\delta+\beta-2\alpha)^2(\beta-\delta-4\alpha)^2(\delta+\beta+6\alpha)}{256\times 4(\delta-\alpha)(\beta-\alpha)(b-a)}\\&+\tfrac{(\beta-\delta-4\alpha)^2(\beta-\delta)(\beta-9\delta+16\alpha)}{256\times 4(\delta-\alpha)(b-a)}+\tfrac{(\beta^3-2\alpha^3)(\beta-\delta-2\alpha)}{12(\beta-\alpha)(b-a)}-\tfrac{(\beta^2-2\alpha^2)(\beta-\delta)(5\beta-8\alpha-5\delta)}{64(\beta-\alpha)(b-a)}+\tfrac{9\alpha^2+4\beta^2-2\delta^2-5\alpha\delta+7\alpha\beta+6\beta\delta}{48(b-a)}\alpha\\&-\tfrac{(\beta-\delta)^3(11\beta-75\delta+128\alpha)}{256\times 24(\delta-\alpha)(b-a)}+\tfrac{(\beta-\delta-4\alpha)^3(19\delta-3\beta-36\alpha)}{256\times 24(\delta-\alpha)(b-a)}+\tfrac{(\delta+\beta-2\alpha)(\beta-\delta-4\alpha)^4}{6\times 256(\beta-\delta)^2(b-a)}+\tfrac{(\delta+\beta-2\alpha)^2(\beta-\delta-2\alpha)(\delta+\beta+10\alpha)(\beta-\delta)}{384\times 2(\delta-\alpha)(\beta-\alpha)(b-a)}\\&+\tfrac{(\beta^2-2\alpha^2)(\beta-\delta-4\alpha)^2}{64(\beta-\alpha)(b-a)}-\tfrac{(3\beta-\delta+2\alpha)(\beta-\delta-4\alpha)^3}{384(\beta-\delta)(b-a)}+\tfrac{12\alpha^3+5\beta^3+32\alpha^2\beta-4\alpha^2\delta+14\alpha\beta^2+18\alpha\delta^2-4\alpha\beta\delta+11\beta^2\delta+27\beta\delta^2+17\delta^3}{96(b-a)}\\&+\tfrac{16\alpha^2+5\beta^2+5\delta^2+32\alpha\beta-16\alpha\delta+6\beta\delta}{48}-\tfrac{(\delta+\beta)(64\alpha^3-103\beta^3+\delta^3-88\alpha^2\beta+88\alpha^2\delta+136\alpha\beta^2-8\alpha\delta^2-64\alpha\beta\delta-29\beta^2\delta+11\beta\delta^2)}{384(\beta-\alpha)(b-a)}\\&+\tfrac{(\beta-\delta)^2(11\beta-5\delta+10\alpha)}{6\times 256(b-a)}+\tfrac{(\delta+\beta-2\alpha)^2(11\delta+11\beta+42\alpha)(\beta-\delta)^2}{24\times 256(\delta-\alpha)(\beta-\alpha)(b-a)}+\tfrac{(\beta-\delta-2\alpha)(\beta-\delta)^2(13\delta-\beta-24\alpha)}{384\times 2(\delta-\alpha)(b-a)}.\end{aligned}$$

If $2\alpha<\beta-\delta<4\alpha$, *and* $(3\delta+\beta)/2\le b-a<(\delta+3\beta)/2$, *then*

$$\begin{aligned}V[\xi]=&\tfrac{(b-a)^3}{384(\beta-\alpha)}+\tfrac{(b-a)^2}{24}+\tfrac{(\delta-\beta)(3a+5b)}{16}-\tfrac{(\delta-\beta)(3\delta+\beta)(a+b)}{16(b-a)}-\tfrac{(3\delta+\beta)(16\alpha^2+32\alpha\beta-16\alpha\delta+6\beta\delta+23\delta^2+3\beta^2)}{192(b-a)}\\&+\tfrac{16\alpha^2+5\beta^2+5\delta^2+32\alpha\beta-16\alpha\delta+6\beta\delta}{96}+\tfrac{(5\beta-\delta-8\alpha)(b-a)^2}{192(\beta-\alpha)}+\tfrac{(\delta+3\beta)^2(b-a)}{256(\beta-\alpha)}+\tfrac{16\alpha^2-20\alpha\beta+36\alpha\delta+6\beta\delta+19\beta^2-9\delta^2}{192}\\&-\tfrac{(\beta^2-2\alpha^2)(\beta-\delta)(5\beta-8\alpha-5\delta)}{64(\beta-\alpha)(b-a)}+\tfrac{(\beta^3-2\alpha^3)(\beta-\delta-2\alpha)}{12(\beta-\alpha)(b-a)}+\tfrac{9\alpha^2+4\beta^2-2\delta^2-5\alpha\delta+7\alpha\beta+6\beta\delta}{48(b-a)}\alpha+\tfrac{(\delta+\beta-2\alpha)(\beta-\delta-4\alpha)^4}{6\times 256(\beta-\delta)^2(b-a)}\\&+\tfrac{(4\alpha+\delta-\beta)^3(\beta-\delta)}{384\times 4(\beta-\alpha)(b-a)}-\tfrac{(\delta+\beta-2\alpha)^3(\beta-\delta-4\alpha)^4}{24\times 256(\delta-\alpha)(\beta-\alpha)(\beta-\delta)^2(b-a)}+\tfrac{(\beta-\delta-4\alpha)^3(19\delta-3\beta-36\alpha)}{256\times 24(\delta-\alpha)(b-a)}-\tfrac{(\beta-\delta)^3(11\beta-75\delta+128\alpha)}{256\times 24(\delta-\alpha)(b-a)}\\&+\tfrac{(\beta-\delta)^4}{256\times 24(\beta-\alpha)(b-a)}-\tfrac{(\beta-\delta)^3(5\beta-\delta-8\alpha)}{384\times 4(\beta-\alpha)(b-a)}-\tfrac{(\beta-\delta)^2(\delta+3\beta)^2}{256\times 4(\beta-\alpha)(b-a)}-\tfrac{(16\alpha^2-20\alpha\beta+36\alpha\delta+6\beta\delta+19\beta^2-9\delta^2)(\beta-\delta)}{384(b-a)}\\&+\tfrac{(\delta+\beta-2\alpha)^2(\beta-\delta-4\alpha)^3(\delta+\beta+2\alpha)}{384\times 4(\delta-\alpha)(\beta-\alpha)(\beta-\delta)(b-a)}+\tfrac{(\delta+\beta-2\alpha)^2(11\delta+11\beta+42\alpha)(\beta-\delta)^2}{24\times 256(\delta-\alpha)(\beta-\alpha)(b-a)}-\tfrac{(\delta+\beta-2\alpha)^2(\beta-\delta-4\alpha)^2(\delta+\beta+6\alpha)}{256\times 4(\delta-\alpha)(\beta-\alpha)(b-a)}\\&+\tfrac{(\delta+\beta-2\alpha)^2(\beta-\delta-2\alpha)(\delta+\beta+10\alpha)(\beta-\delta)}{384\times 2(\delta-\alpha)(\beta-\alpha)(b-a)}+\tfrac{(\beta-\delta-4\alpha)^2(\beta-\delta)(\beta-9\delta+16\alpha)}{256\times 4(\delta-\alpha)(b-a)}+\tfrac{(\beta-\delta-2\alpha)(\beta-\delta)^2(13\delta-\beta-24\alpha)}{384\times 2(\delta-\alpha)(b-a)}\\&+\tfrac{(\beta^2-2\alpha^2)(\beta-\delta-4\alpha)^2}{64(\beta-\alpha)(b-a)}+\tfrac{(\beta-\delta)^2(11\beta-5\delta+10\alpha)}{6\times 256(b-a)}+\tfrac{12\alpha^3+5\beta^3+32\alpha^2\beta-4\alpha^2\delta+14\alpha\beta^2+18\alpha\delta^2-4\alpha\beta\delta+11\beta^2\delta+27\beta\delta^2+17\delta^3}{96(b-a)}\\&-\tfrac{(3\beta-\delta+2\alpha)(\beta-\delta-4\alpha)^3}{384(\beta-\delta)(b-a)}.\end{aligned}$$

If $2\alpha<\beta-\delta<4\alpha$, *and* $(\beta-\delta+4\alpha)/2\le b-a<(3\delta+\beta)/2$, *then*

$$\begin{aligned}V[\xi]=&\tfrac{\beta+\delta-2\alpha}{384(\delta-\alpha)(\beta-\alpha)}(b-a)^3+\left(\tfrac{5\delta-\beta-8\alpha}{192(\delta-\alpha)}+\tfrac{5\beta-\delta-8\alpha}{192(\beta-\alpha)}\right)(b-a)^2+\left(\tfrac{(3\delta+\beta)^2}{256(\delta-\alpha)}+\tfrac{(\delta+3\beta)^2}{256(\beta-\alpha)}\right)(b-a)\\&+\tfrac{(\delta+\beta-2\alpha)^2(\beta-\delta-4\alpha)^3(\delta+\beta+2\alpha)}{384\times 4(\delta-\alpha)(\beta-\alpha)(\beta-\delta)(b-a)}+\tfrac{(\delta+\beta-2\alpha)^2(11\delta+11\beta+42\alpha)(\beta-\delta)^2}{24\times 256(\delta-\alpha)(\beta-\alpha)(b-a)}-\tfrac{(\delta+\beta-2\alpha)^2(\beta-\delta-4\alpha)^2(\delta+\beta+6\alpha)}{256\times 4(\delta-\alpha)(\beta-\alpha)(b-a)}\\&+\tfrac{72\alpha^3+5\beta^3-3\delta^3+30\alpha\beta^2+6\alpha\delta^2+72\alpha^2\beta-32\alpha^2\delta-20\alpha\beta\delta-5\beta^2\delta+3\beta\delta^2}{96(b-a)}-\tfrac{(\beta-\delta)^2(\delta+3\beta)^2}{256\times 4(\beta-\alpha)(b-a)}-\tfrac{(4\alpha-\delta+\beta)^2(3\delta+\beta)^2}{256\times 4(\delta-\alpha)(b-a)}\\&+\tfrac{(\beta-\delta-4\alpha)^2(\beta-\delta)(\beta-9\delta+16\alpha)}{256\times 4(\delta-\alpha)(b-a)}-\tfrac{(\beta^2-2\alpha^2)(\beta-\delta)(5\beta-8\alpha-5\delta)}{64(\beta-\alpha)(b-a)}+\tfrac{(\delta+\beta-2\alpha)(\beta-\delta-4\alpha)^4}{6\times 256(\beta-\delta)^2(b-a)}-\tfrac{(3\beta-\delta+2\alpha)(\beta-\delta-4\alpha)^3}{384(\beta-\delta)(b-a)}\\&-\tfrac{(\delta+\beta-2\alpha)^3(\beta-\delta-4\alpha)^4}{24\times 256(\delta-\alpha)(\beta-\alpha)(\beta-\delta)^2(b-a)}+\tfrac{(\beta-\delta-2\alpha)(\beta-\delta)^2(13\delta-\beta-24\alpha)}{384\times 2(\delta-\alpha)(b-a)}+\tfrac{(\beta^2-2\alpha^2)(\beta-\delta-4\alpha)^2}{64(\beta-\alpha)(b-a)}+\tfrac{(\beta^3-2\alpha^3)(\beta-\delta-2\alpha)}{12(\beta-\alpha)(b-a)}\\&-\tfrac{(4\alpha-\delta+\beta)(16\alpha^2+4\alpha\beta+12\alpha\delta+6\beta\delta+7\beta^2+3\delta^2)}{384(b-a)}+\tfrac{(\beta-\delta-4\alpha)^3(19\delta-3\beta-36\alpha)}{256\times 24(\delta-\alpha)(b-a)}+\tfrac{(\delta+\beta-2\alpha)^2(\beta-\delta-2\alpha)(\delta+\beta+10\alpha)(\beta-\delta)}{384\times 2(\delta-\alpha)(\beta-\alpha)(b-a)}\end{aligned}$$

$$
\begin{aligned}
&+\tfrac{16\alpha^2-8\alpha\beta+24\alpha\delta+6\beta\delta+13\beta^2-3\delta^2}{96}-\tfrac{(4\alpha-\delta+\beta)^3}{384\times 2(\delta-\alpha)}+\tfrac{(4\alpha-\delta+\beta)^4}{256\times 8(\delta-\alpha)(b-a)}-\tfrac{(\beta-\delta)^4}{256\times 24(\beta-\alpha)(b-a)}-\tfrac{(\beta-\delta)^3(5\beta-\delta-8\alpha)}{384\times 4(\beta-\alpha)(b-a)}\\
&-\tfrac{(4\alpha-\delta+\beta)^3(5\delta-\beta-8\alpha)}{384\times 4(\delta-\alpha)(b-a)}-\tfrac{(4\alpha+\delta-\beta)^3(\beta-\delta)}{384\times 4(\beta-\alpha)(b-a)}-\tfrac{(\beta-\delta)^3(11\beta-75\delta+128\alpha)}{256\times 24(\delta-\alpha)(b-a)}-\tfrac{(\beta-\delta)(16\alpha^2-20\alpha\beta+36\alpha\delta+6\beta\delta+19\beta^2-9\delta^2)}{384(b-a)}\\
&+\tfrac{(\beta-\delta)^2(11\beta-5\delta+10\alpha)}{6\times 256(b-a)}+\tfrac{9\alpha^2+4\beta^2-2\delta^2-5\alpha\delta+7\alpha\beta+6\beta\delta}{48(b-a)}\alpha.
\end{aligned}
$$

If $2\alpha<\beta-\delta<4\alpha$, *and* $(\beta-\delta)/2\le b-a<(\beta-\delta+4\alpha)/2$, *then*

$$
\begin{aligned}
V[\xi]=&-\tfrac{(\delta+\beta-2\alpha)^3(\beta-\delta-4\alpha)^4}{24\times 256(\delta-\alpha)(\beta-\alpha)(\beta-\delta)^2(b-a)}+\tfrac{(\delta-\beta)(32\alpha^2-3\alpha\delta+11\alpha\beta+\delta^2+6\beta^2+\beta\delta)}{192(b-a)}+\tfrac{(b-a)^3}{384(\beta-\alpha)}+\tfrac{(13\beta-\delta-16\alpha)(b-a)^2}{192(\beta-\alpha)}\\
&+\tfrac{(\beta-\alpha)(b-a)}{16}+\tfrac{(\delta+3\beta)^2(b-a)}{256(\beta-\alpha)}+\tfrac{16\alpha^2-20\alpha\beta+36\alpha\delta+6\beta\delta+19\beta^2-9\delta^2}{192}-\tfrac{(\beta-\delta)^4}{256\times 24(\beta-\alpha)(b-a)}-\tfrac{(\beta-\delta)^3(5\beta-\delta-8\alpha)}{384\times 4(\beta-\alpha)(b-a)}\\
&-\tfrac{(\beta-\delta)^2(\delta+3\beta)^2}{256\times 4(\beta-\alpha)(b-a)}-\tfrac{(\beta-\delta)(16\alpha^2-20\alpha\beta+36\alpha\delta+6\beta\delta+19\beta^2-9\delta^2)}{384(b-a)}+\tfrac{(\beta-\delta)(\delta-\beta+4\alpha)^4}{384\times 4(\beta-\alpha)(b-a)}+\tfrac{32\alpha^2+5\beta^2+3\delta^2-6\alpha\delta+14\alpha\beta}{96}\\
&+\tfrac{(\delta+\beta-2\alpha)^2(\beta-\delta-4\alpha)^3(\delta+\beta+2\alpha)}{384\times 4(\delta-\alpha)(\beta-\alpha)(\beta-\delta)(b-a)}+\tfrac{(\delta+\beta-2\alpha)^2(\beta-\delta-2\alpha)(\delta+\beta+10\alpha)(\beta-\delta)}{384\times 2(\delta-\alpha)(\beta-\alpha)(b-a)}+\tfrac{(\delta+\beta-2\alpha)^2(11\delta+11\beta+42\alpha)(\beta-\delta)^2}{24\times 256(\delta-\alpha)(\beta-\alpha)(b-a)}\\
&-\tfrac{(\beta-\delta)^3(11\beta-75\delta+128\alpha)}{256\times 24(\delta-\alpha)(b-a)}-\tfrac{(\delta+\beta-2\alpha)^2(\beta-\delta-4\alpha)^2(\delta+\beta+6\alpha)}{256\times 4(\delta-\alpha)(\beta-\alpha)(b-a)}+\tfrac{(\beta-\delta-2\alpha)(\beta-\delta)^2(13\delta-\beta-24\alpha)}{384\times 2(\delta-\alpha)(b-a)}+\tfrac{(\beta^2-2\alpha^2)(\beta-\delta-4\alpha)^2}{64(\beta-\alpha)(b-a)}\\
&+\tfrac{(\beta-\delta-4\alpha)^2(\beta-\delta)(\beta-9\delta+16\alpha)}{256\times 4(\delta-\alpha)(b-a)}-\tfrac{(\beta^2-2\alpha^2)(\beta-\delta)(5\beta-8\alpha-5\delta)}{64(\beta-\alpha)(b-a)}+\tfrac{(\beta-\delta-4\alpha)^3(19\delta-3\beta-36\alpha)}{256\times 24(\delta-\alpha)(b-a)}+\tfrac{(\beta^3-2\alpha^3)(\beta-\delta-2\alpha)}{12(\beta-\alpha)(b-a)}\\
&+\tfrac{(\delta+\beta-2\alpha)(\beta-\delta-4\alpha)^4}{6\times 256(\beta-\delta)^2(b-a)}-\tfrac{(3\beta-\delta+2\alpha)(\beta-\delta-4\alpha)^3}{384(\beta-\delta)(b-a)}+\tfrac{(\beta-\delta)^2(11\beta-5\delta+10\alpha)}{6\times 256(b-a)}+\tfrac{9\alpha^2+4\beta^2-2\delta^2-5\alpha\delta+7\alpha\beta+6\beta\delta}{48(b-a)}\alpha.
\end{aligned}
$$

If $2\alpha<\beta-\delta<4\alpha$, *and* $(\delta-\beta+4\alpha)/2\le b-a<(\beta-\delta)/2$, *then*

$$
\begin{aligned}
V[\xi]=&\left(\tfrac{(\delta+\beta-2\alpha)^3}{384(\delta-\alpha)(\beta-\alpha)(\beta-\delta)^2}-\tfrac{1}{384(\delta-\alpha)}\right)\left(\tfrac{5a^4-3b^4+6a^2b^2+4a^3b-12ab^3}{b-a}\right)-\left(\tfrac{(\delta+\beta-2\alpha)^2(\delta+\beta+2\alpha)}{96(\delta-\alpha)(\beta-\alpha)(\beta-\delta)}-\tfrac{\beta-5\delta+8\alpha}{96(\delta-\alpha)}\right)\\
&\left(\tfrac{b^3-a^3+3ab^2-3a^2b}{b-a}\right)+\left(\tfrac{(\delta+\beta-2\alpha)^2(\delta+\beta+6\alpha)}{256(\beta-\alpha)(\delta-\alpha)}-\tfrac{(\beta-\delta)(\beta-9\delta+16\alpha)}{256(\delta-\alpha)}-\tfrac{\beta^2-2\alpha^2}{16(\beta-\alpha)}-\tfrac{\beta-3\delta-6\alpha}{64}\right)(b-a)-\left(\tfrac{1}{96(\delta-\alpha)}\right.\\
&\left.-\tfrac{(\delta+\beta-2\alpha)^3}{96(\beta-\alpha)(\delta-\alpha)(\beta-\delta)^2}+\tfrac{\delta+\beta-2\alpha}{24(\beta-\delta)^2}\right)(a+b)^3+\left(\tfrac{(\delta+\beta-2\alpha)^2(\delta+\beta+2\alpha)}{64(\beta-\alpha)(\delta-\alpha)(\beta-\delta)}+\tfrac{5\delta-8\alpha-\beta}{64(\delta-\alpha)}+\tfrac{3\beta-\delta+2\alpha}{16(\beta-\delta)}\right)(a+b)^2\\
&+\tfrac{\beta^3-2\alpha^3}{12(\beta-\alpha)}+\tfrac{(13\delta-\beta-24\alpha)(\beta-\delta)^2}{384\times 2(\delta-\alpha)}-\tfrac{(\beta^2-2\alpha^2)(\beta-\delta)}{16(\beta-\alpha)}+\left(\tfrac{1}{24\times 256(\delta-\alpha)}-\tfrac{(\delta+\beta-2\alpha)^3}{24\times 256(\beta-\alpha)(\delta-\alpha)(\beta-\delta)^2}\right)\tfrac{(\beta-\delta-4\alpha)^4}{b-a}\\
&+\left(\tfrac{(\delta+\beta-2\alpha)^2(\delta+\beta+2\alpha)}{4\times 384(\beta-\alpha)(\delta-\alpha)(\beta-\delta)}-\tfrac{\beta-5\delta+8\alpha}{4\times 384(\delta-\alpha)}\right)\tfrac{(\beta-\delta-4\alpha)^3}{b-a}-\left(\tfrac{(\delta+\beta-2\alpha)^2(\delta+\beta+6\alpha)}{4\times 256(\beta-\alpha)(\delta-\alpha)}-\tfrac{(\beta-\delta)(\beta+16\alpha-9\delta)}{4\times 256(\delta-\alpha)}\right)\tfrac{(\beta-\delta-4\alpha)^2}{b-a}\\
&+\left(\tfrac{(\delta+\beta-2\alpha)^2(\delta+\beta+10\alpha)(\beta-\delta)}{4\times 384(\beta-\alpha)(\delta-\alpha)}+\tfrac{(13\delta-\beta-24\alpha)(\beta-\delta)^2}{4\times 384(\delta-\alpha)}+\tfrac{\beta^3-2\alpha^3}{24(\beta-\alpha)}\right)\tfrac{\beta-\delta-4\alpha}{b-a}+\tfrac{(\beta^2-2\alpha^2)(\delta-\beta-4\alpha)(\beta-\delta-4\alpha)}{64(\beta-\alpha)}\\
&+\tfrac{(\delta+\beta-2\alpha)(3a^4-5b^4-6a^2b^2+12a^3b-4ab^3)}{96(\beta-\delta)^2(b-a)}+\tfrac{(\delta+\beta-2\alpha)^2(\delta+\beta+10\alpha)(\beta-\delta)}{384\times 2(\delta-\alpha)(\beta-\alpha)}-\tfrac{(64\alpha^2+5\delta^2+11\beta^2+26\alpha\delta-10\alpha\beta)(\beta-\delta-4\alpha)}{384(b-a)}\\
&+\tfrac{64\alpha^2+11\beta^2+5\delta^2-10\alpha\beta+26\alpha\delta}{192}+\tfrac{(3\beta-\delta+2\alpha)(a^3-b^3+3a^2b-3ab^2)}{24(\beta-\delta)(b-a)}+\tfrac{(\delta+\beta-2\alpha)(\beta-\delta-4\alpha)^4}{256\times 6(\beta-\delta)^2(b-a)}+\tfrac{(\beta-3\delta-6\alpha)(\beta-\delta-4\alpha)^2}{256(b-a)}\\
&-\tfrac{(3\beta-\delta+2\alpha)(\beta-\delta-4\alpha)^3}{384(\beta-\delta)(b-a)}.
\end{aligned}
$$

If $2\alpha<\beta-\delta<4\alpha$, *and* $b-a<(\delta-\beta+4\alpha)/2$, *then*

$$
V[\xi]=\tfrac{3\beta-\delta+2\alpha}{24(\beta-\delta)}(b-a)^2+\tfrac{11\beta^2+64\alpha^2+5\delta^2+26\alpha\delta-10\alpha\beta}{96}.
$$

If $\delta-\beta\ge 4\alpha$, *and* $b-a\ge(\delta+3\beta)/2$, *then*

$$
\begin{aligned}
V[\xi]=&\tfrac{(\delta-\beta)(a+b)}{2}-\tfrac{(\delta-\beta)(\delta+\beta)(a+b)}{4(b-a)}-\tfrac{(\delta+\beta)(64\alpha^3-103\beta^3+\delta^3-88\alpha^2\beta+88\alpha^2\delta+136\alpha\beta^2-8\alpha\delta^2-64\alpha\beta\delta-29\beta^2\delta+11\beta\delta^2)}{384(\beta-\alpha)(b-a)}\\
&-\tfrac{(\beta-\delta)^3(11\beta-75\delta+128\alpha)}{256\times 24(\delta-\alpha)(b-a)}-\tfrac{(\delta+3\beta)(16\alpha^2+32\alpha\beta-16\alpha\delta+6\beta\delta+23\beta^2+3\delta^2)}{192(b-a)}-\tfrac{(3\delta+\beta)(16\alpha^2+32\alpha\beta-16\alpha\delta+6\beta\delta+3\beta^2+23\delta^2)}{192(b-a)}\\
&+\tfrac{16\alpha^2+5\beta^2+5\delta^2+32\alpha\beta-16\alpha\delta+6\beta\delta}{48}-\tfrac{(\delta+\beta-2\alpha)^3(\beta-\delta-4\alpha)^4}{24\times 256(\delta-\alpha)(\beta-\alpha)(\beta-\delta)^2(b-a)}+\tfrac{(\delta+\beta-2\alpha)^2(\beta-\delta-4\alpha)^3(\delta+\beta+2\alpha)}{384\times 4(\delta-\alpha)(\beta-\alpha)(\beta-\delta)(b-a)}+\tfrac{(b-a)^2}{12}\\
&+\tfrac{(\delta+\beta-2\alpha)^2(11\delta+11\beta+42\alpha)(\beta-\delta)^2}{24\times 256(\delta-\alpha)(\beta-\alpha)(b-a)}+\tfrac{(\delta+\beta-2\alpha)^2(\beta-\delta-2\alpha)(\delta+\beta+10\alpha)(\beta-\delta)}{384\times 2(\delta-\alpha)(\beta-\alpha)(b-a)}-\tfrac{(\delta+\beta-2\alpha)^2(\beta-\delta-4\alpha)^2(\delta+\beta+6\alpha)}{256\times 4(\delta-\alpha)(\beta-\alpha)(b-a)}\\
&+\tfrac{(\beta-\delta-4\alpha)^3(19\delta-3\beta-36\alpha)}{256\times 24(\delta-\alpha)(b-a)}+\tfrac{(\beta-\delta-2\alpha)(\beta-\delta)^2(13\delta-\beta-24\alpha)}{384\times 2(\delta-\alpha)(b-a)}+\tfrac{(\beta-\delta-4\alpha)^2(\beta-\delta)(\beta-9\delta+16\alpha)}{256\times 4(\delta-\alpha)(b-a)}+\tfrac{(\beta^3-2\alpha^3)(\beta-\delta-2\alpha)}{12(\beta-\alpha)(b-a)}\\
&-\tfrac{(\beta^2-2\alpha^2)(\beta-\delta)(5\beta-8\alpha-5\delta)}{64(\beta-\alpha)(b-a)}+\tfrac{(\beta^2-2\alpha^2)(\beta-\delta-4\alpha)^2}{64(\beta-\alpha)(b-a)}+\tfrac{(\delta+\beta-2\alpha)(\beta-\delta-4\alpha)^4}{6\times 256(\beta-\delta)^2(b-a)}+\tfrac{9\alpha^2+4\beta^2-2\delta^2-5\alpha\delta+7\alpha\beta+6\beta\delta}{48(b-a)}\alpha\\
&-\tfrac{(3\beta-\delta+2\alpha)(\beta-\delta-4\alpha)^3}{384(\beta-\delta)(b-a)}+\tfrac{(\beta-\delta)^2(11\beta-5\delta+10\alpha)}{6\times 256(b-a)}+\tfrac{12\alpha^3+5\beta^3+32\alpha^2\beta-4\alpha^2\delta+14\alpha\beta^2+18\alpha\delta^2-4\alpha\beta\delta+11\beta^2\delta+27\beta\delta^2+17\delta^3}{96(b-a)}.
\end{aligned}
$$

If $\delta-\beta\geq 4\alpha$, *and* $(3\delta+\beta)/2\leq b-a<(\delta+3\beta)/2$, *then*

$$\begin{aligned}V[\xi]=&\ \tfrac{(b-a)^2}{24}+\tfrac{(\delta-\beta)(3a+5b)}{16}-\tfrac{(\delta-\beta)(3\delta+\beta)(a+b)}{16(b-a)}+\tfrac{16\alpha^2+5\beta^2+5\delta^2+32\alpha\beta-16\alpha\delta+6\beta\delta}{96}+\tfrac{(5\beta-\delta-8\alpha)(b-a)^2}{192(\beta-\alpha)}\\
&+\tfrac{12\alpha^3+5\beta^3+32\alpha^2\beta-4\alpha^2\delta+14\alpha\beta^2+18\alpha\delta^2-4\alpha\beta\delta+11\beta^2\delta+27\beta\delta^2+17\delta^3}{96(b-a)}-\tfrac{(3\beta-\delta+2\alpha)(\beta-\delta-4\alpha)^3}{384(\beta-\delta)(b-a)}+\tfrac{(\delta+3\beta)^2(b-a)}{256(\beta-\alpha)}\\
&+\tfrac{16\alpha^2-20\alpha\beta+36\alpha\delta+6\beta\delta+19\beta^2-9\delta^2}{192}-\tfrac{(3\delta+\beta)(16\alpha^2+32\alpha\beta-16\alpha\delta+6\beta\delta+23\delta^2+3\beta^2)}{192(b-a)}+\tfrac{(\beta-\delta)^2(11\beta-5\delta+10\alpha)}{6\times256(b-a)}\\
&+\tfrac{(\beta-\delta)^4}{256\times24(\beta-\alpha)(b-a)}-\tfrac{(\beta-\delta)^3(5\beta-\delta-8\alpha)}{384\times4(\beta-\alpha)(b-a)}-\tfrac{(\beta-\delta)^2(\delta+3\beta)^2}{256\times4(\beta-\alpha)(b-a)}-\tfrac{(16\alpha^2-20\alpha\beta+36\alpha\delta+6\beta\delta+19\beta^2-9\delta^2)(\beta-\delta)}{384(b-a)}\\
&+\tfrac{(4\alpha+\delta-\beta)^3(\beta-\delta)}{384\times4(\beta-\alpha)(b-a)}+\tfrac{(\beta-\delta-2\alpha)(\beta-\delta)^2(13\delta-\beta-24\alpha)}{384\times2(\delta-\alpha)(b-a)}+\tfrac{(\beta-\delta-4\alpha)^2(\beta-\delta)(\beta-9\delta+16\alpha)}{256\times4(\delta-\alpha)(b-a)}-\tfrac{(\beta-\delta)^3(11\beta-75\delta+128\alpha)}{256\times24(\delta-\alpha)(b-a)}\\
&-\tfrac{(\beta^2-2\alpha^2)(\beta-\delta)(5\beta-8\alpha-5\delta)}{64(\beta-\alpha)(b-a)}+\tfrac{(\beta^2-2\alpha^2)(\beta-\delta-4\alpha)^2}{64(\beta-\alpha)(b-a)}+\tfrac{(\beta^3-2\alpha^3)(\beta-\delta-2\alpha)}{12(\beta-\alpha)(b-a)}+\tfrac{9\alpha^2+4\beta^2-2\delta^2-5\alpha\delta+7\alpha\beta+6\beta\delta}{48(b-a)}\alpha\\
&-\tfrac{(\delta+\beta-2\alpha)^3(\beta-\delta-4\alpha)^4}{24\times256(\delta-\alpha)(\beta-\alpha)(\beta-\delta)^2(b-a)}+\tfrac{(\delta+\beta-2\alpha)^2(\beta-\delta-4\alpha)^3(\delta+\beta+2\alpha)}{384\times4(\delta-\alpha)(\beta-\alpha)(\beta-\delta)(b-a)}-\tfrac{(\delta+\beta-2\alpha)^2(\beta-\delta-4\alpha)^2(\delta+\beta+6\alpha)}{256\times4(\delta-\alpha)(\beta-\alpha)(b-a)}\\
&+\tfrac{(\beta-\delta-4\alpha)^3(19\delta-3\beta-36\alpha)}{256\times24(\delta-\alpha)(b-a)}+\tfrac{(\delta+\beta-2\alpha)^2(11\delta+11\beta+42\alpha)(\beta-\delta)^2}{24\times256(\delta-\alpha)(\beta-\alpha)(b-a)}+\tfrac{(\delta+\beta-2\alpha)^2(\beta-\delta-2\alpha)(\delta+\beta+10\alpha)(\beta-\delta)}{384\times2(\delta-\alpha)(\beta-\alpha)(b-a)}\\
&+\tfrac{(\delta+\beta-2\alpha)(\beta-\delta-4\alpha)^4}{6\times256(\beta-\delta)^2(b-a)}+\tfrac{(b-a)^3}{384(\beta-\alpha)}.\end{aligned}$$

If $\delta-\beta\geq 4\alpha$, *and* $(\beta-\delta+4\alpha)/2\leq b-a<(3\delta+\beta)/2$, *then*

$$\begin{aligned}V[\xi]=&\ \tfrac{\beta+\delta-2\alpha}{384(\delta-\alpha)(\beta-\alpha)}(b-a)^3+\left(\tfrac{5\delta-\beta-8\alpha}{192(\delta-\alpha)}+\tfrac{5\beta-\delta-8\alpha}{192(\beta-\alpha)}\right)(b-a)^2+\left(\tfrac{(3\delta+\beta)^2}{256(\delta-\alpha)}+\tfrac{(\delta+3\beta)^2}{256(\beta-\alpha)}\right)(b-a)\\
&-\tfrac{(4\alpha-\delta+\beta)^3}{384\times2(\delta-\alpha)}+\tfrac{(4\alpha-\delta+\beta)^4}{256\times8(\delta-\alpha)(b-a)}-\tfrac{(\beta-\delta)^4}{256\times24(\beta-\alpha)(b-a)}-\tfrac{(\beta-\delta)^3(5\beta-\delta-8\alpha)}{384\times4(\beta-\alpha)(b-a)}-\tfrac{(\delta+\beta-2\alpha)^2(\beta-\delta-4\alpha)^2(\delta+\beta+6\alpha)}{256\times4(\delta-\alpha)(\beta-\alpha)(b-a)}\\
&-\tfrac{(4\alpha-\delta+\beta)^3(5\delta-\beta-8\alpha)}{384\times4(\delta-\alpha)(b-a)}-\tfrac{(4\alpha+\delta-\beta)^3(\beta-\delta)}{384\times4(\beta-\alpha)(b-a)}-\tfrac{(4\alpha-\delta+\beta)^2(3\delta+\beta)^2}{256\times4(\delta-\alpha)(b-a)}-\tfrac{(\beta-\delta)(16\alpha^2-20\alpha\beta+36\alpha\delta+6\beta\delta+19\beta^2-9\delta^2)}{384(b-a)}\\
&-\tfrac{(\beta-\delta)^2(\delta+3\beta)^2}{256\times4(\beta-\alpha)(b-a)}+\tfrac{(\beta-\delta-4\alpha)^3(19\delta-3\beta-36\alpha)}{256\times24(\delta-\alpha)(b-a)}+\tfrac{(\beta-\delta-2\alpha)(\beta-\delta)^2(13\delta-\beta-24\alpha)}{384\times2(\delta-\alpha)(b-a)}+\tfrac{(\beta-\delta-4\alpha)^2(\beta-\delta)(\beta-9\delta+16\alpha)}{256\times4(\delta-\alpha)(b-a)}\\
&-\tfrac{(\beta^2-2\alpha^2)(\beta-\delta)(5\beta-8\alpha-5\delta)}{64(\beta-\alpha)(b-a)}-\tfrac{(4\alpha-\delta+\beta)(16\alpha^2+4\alpha\beta+12\alpha\delta+6\beta\delta+7\beta^2+3\delta^2)}{384(b-a)}-\tfrac{(\delta+\beta-2\alpha)^3(\beta-\delta-4\alpha)^4}{24\times256(\delta-\alpha)(\beta-\alpha)(\beta-\delta)^2(b-a)}\\
&+\tfrac{72\alpha^3+5\beta^3-3\delta^3+30\alpha\beta^2+6\alpha\delta^2+72\alpha^2\beta-32\alpha^2\delta-20\alpha\beta\delta-5\beta^2\delta+3\beta\delta^2}{96(b-a)}+\tfrac{(\beta^3-2\alpha^3)(\beta-\delta-2\alpha)}{12(\beta-\alpha)(b-a)}+\tfrac{(\beta^2-2\alpha^2)(\beta-\delta-4\alpha)^2}{64(\beta-\alpha)(b-a)}\\
&+\tfrac{(\beta-\delta)^2(11\beta-5\delta+10\alpha)}{6\times256(b-a)}-\tfrac{(3\beta-\delta+2\alpha)(\beta-\delta-4\alpha)^3}{384(\beta-\delta)(b-a)}-\tfrac{(\beta-\delta)^3(11\beta-75\delta+128\alpha)}{256\times24(\delta-\alpha)(b-a)}+\tfrac{(\delta+\beta-2\alpha)^2(\beta-\delta-4\alpha)^3(\delta+\beta+2\alpha)}{384\times4(\delta-\alpha)(\beta-\alpha)(\beta-\delta)(b-a)}\\
&+\tfrac{(\delta+\beta-2\alpha)^2(\beta-\delta-2\alpha)(\delta+\beta+10\alpha)(\beta-\delta)}{384\times2(\delta-\alpha)(\beta-\alpha)(b-a)}+\tfrac{(\delta+\beta-2\alpha)^2(11\delta+11\beta+42\alpha)(\beta-\delta)^2}{24\times256(\delta-\alpha)(\beta-\alpha)(b-a)}+\tfrac{9\alpha^2+4\beta^2-2\delta^2-5\alpha\delta+7\alpha\beta+6\beta\delta}{48(b-a)}\alpha\\
&+\tfrac{16\alpha^2-8\alpha\beta+24\alpha\delta+6\beta\delta+13\beta^2-3\delta^2}{96}+\tfrac{(\delta+\beta-2\alpha)(\beta-\delta-4\alpha)^4}{6\times256(\beta-\delta)^2(b-a)}.\end{aligned}$$

If $\delta-\beta\geq 4\alpha$, *and* $(\beta-\delta)/2\leq b-a<(\beta-\delta+4\alpha)/2$, *then*

$$\begin{aligned}V[\xi]=&\ \tfrac{32\alpha^2+5\beta^2+3\delta^2-6\alpha\delta+14\alpha\beta}{96}+\tfrac{(\delta-\beta)(32\alpha^2-3\alpha\delta+11\alpha\beta+\delta^2+6\beta^2+\beta\delta)}{192(b-a)}+\tfrac{(b-a)^3}{384(\beta-\alpha)}+\tfrac{(13\beta-\delta-16\alpha)(b-a)^2}{192(\beta-\alpha)}\\
&+\tfrac{(\delta+3\beta)^2(b-a)}{256(\beta-\alpha)}+\tfrac{16\alpha^2-20\alpha\beta+36\alpha\delta+6\beta\delta+19\beta^2-9\delta^2}{192}-\tfrac{(\beta-\delta)^4}{256\times24(\beta-\alpha)(b-a)}+\tfrac{(\beta-\alpha)(b-a)}{16}-\tfrac{(\beta-\delta)^3(5\beta-\delta-8\alpha)}{384\times4(\beta-\alpha)(b-a)}\\
&-\tfrac{(\beta-\delta)^2(\delta+3\beta)^2}{256\times4(\beta-\alpha)(b-a)}+\tfrac{(\delta-\beta+4\alpha)^4}{384\times4(\beta-\alpha)(b-a)}-\tfrac{(\delta+\beta-2\alpha)^3(\beta-\delta-4\alpha)^4}{24\times256(\delta-\alpha)(\beta-\alpha)(\beta-\delta)^2(b-a)}+\tfrac{(\delta+\beta-2\alpha)^2(\beta-\delta-4\alpha)^3(\delta+\beta+2\alpha)}{384\times4(\delta-\alpha)(\beta-\alpha)(\beta-\delta)(b-a)}\\
&+\tfrac{(\delta+\beta-2\alpha)^2(11\delta+11\beta+42\alpha)(\beta-\delta)^2}{24\times256(\delta-\alpha)(\beta-\alpha)(b-a)}+\tfrac{(\delta+\beta-2\alpha)^2(\beta-\delta-2\alpha)(\delta+\beta+10\alpha)(\beta-\delta)}{384\times2(\delta-\alpha)(\beta-\alpha)(b-a)}-\tfrac{(\delta+\beta-2\alpha)^2(\beta-\delta-4\alpha)^2(\delta+\beta+6\alpha)}{256\times4(\delta-\alpha)(\beta-\alpha)(b-a)}\\
&+\tfrac{(\beta-\delta-4\alpha)^3(19\delta-3\beta-36\alpha)}{256\times24(\delta-\alpha)(b-a)}+\tfrac{(\beta-\delta-2\alpha)(\beta-\delta)^2(13\delta-\beta-24\alpha)}{384\times2(\delta-\alpha)(b-a)}+\tfrac{(\beta^3-2\alpha^3)(\beta-\delta-2\alpha)}{12(\beta-\alpha)(b-a)}-\tfrac{(\beta-\delta)^3(11\beta-75\delta+128\alpha)}{256\times24(\delta-\alpha)(b-a)}\\
&-\tfrac{(\beta^2-2\alpha^2)(\beta-\delta)(5\beta-8\alpha-5\delta)}{64(\beta-\alpha)(b-a)}+\tfrac{(\beta^2-2\alpha^2)(\beta-\delta-4\alpha)^2}{64(\beta-\alpha)(b-a)}+\tfrac{(\beta-\delta-4\alpha)^2(\beta-\delta)(\beta-9\delta+16\alpha)}{256\times4(\delta-\alpha)(b-a)}+\tfrac{(\delta+\beta-2\alpha)(\beta-\delta-4\alpha)^4}{6\times256(\beta-\delta)^2(b-a)}\\
&-\tfrac{(\beta-\delta)(16\alpha^2-20\alpha\beta+36\alpha\delta+6\beta\delta+19\beta^2-9\delta^2)}{384(b-a)}-\tfrac{(3\beta-\delta+2\alpha)(\beta-\delta-4\alpha)^3}{384(\beta-\delta)(b-a)}+\tfrac{9\alpha^2+4\beta^2-2\delta^2-5\alpha\delta+7\alpha\beta+6\beta\delta}{48(b-a)}\alpha\\
&+\tfrac{(\beta-\delta)^2(11\beta-5\delta+10\alpha)}{6\times256(b-a)}.\end{aligned}$$

If $\delta-\beta\geq 4\alpha$, *and* $(\beta-\delta-4\alpha)/2\leq b-a<(\beta-\delta)/2$, *then*

$$\begin{aligned}V[\xi]=&\left(\tfrac{(\delta+\beta-2\alpha)^3}{384(\delta-\alpha)(\beta-\alpha)(\beta-\delta)^2}-\tfrac{1}{384(\delta-\alpha)}\right)\left(\tfrac{5a^4-3b^4+6a^2b^2+4a^3b-12ab^3}{b-a}\right)-\left(\tfrac{(\delta+\beta-2\alpha)^2(\delta+\beta+2\alpha)}{96(\delta-\alpha)(\beta-\alpha)(\beta-\delta)}-\tfrac{\beta-5\delta+8\alpha}{96(\delta-\alpha)}\right)\\
&\left(\tfrac{b^3-a^3+3ab^2-3a^2b}{b-a}\right)+\left(\tfrac{(\delta+\beta-2\alpha)^2(\delta+\beta+6\alpha)}{256(\beta-\alpha)(\delta-\alpha)}-\tfrac{(\beta-\delta)(\beta-9\delta+16\alpha)}{256(\delta-\alpha)}-\tfrac{\beta^2-2\alpha^2}{16(\beta-\alpha)}-\tfrac{\beta-3\delta-6\alpha}{64}\right)(b-a)-\left(\tfrac{1}{96(\delta-\alpha)}\right.\\
&\left.-\tfrac{(\delta+\beta-2\alpha)^3}{96(\beta-\alpha)(\delta-\alpha)(\beta-\delta)^2}+\tfrac{\delta+\beta-2\alpha}{24(\beta-\delta)^2}\right)(a+b)^3+\left(\tfrac{(\delta+\beta-2\alpha)^2(\delta+\beta+2\alpha)}{64(\beta-\alpha)(\delta-\alpha)(\beta-\delta)}+\tfrac{5\delta-8\alpha-\beta}{64(\delta-\alpha)}+\tfrac{3\beta-\delta+2\alpha}{16(\beta-\delta)}\right)(a+b)^2\end{aligned}$$

$$
\begin{aligned}
&+\frac{(\delta+\beta-2\alpha)^2(\delta+\beta+10\alpha)(\beta-\delta)}{384\times 2(\delta-\alpha)(\beta-\alpha)}-\frac{(\beta^2-2\alpha^2)(\beta-\delta)}{16(\beta-\alpha)}+\left(\frac{1}{24\times 256(\delta-\alpha)}-\frac{(\delta+\beta-2\alpha)^3}{24\times 256(\beta-\alpha)(\delta-\alpha)(\beta-\delta)^2}\right)\frac{(\beta-\delta-4\alpha)^4}{b-a}\\
&+\left(\frac{(\delta+\beta-2\alpha)^2(\delta+\beta+2\alpha)}{4\times 384(\beta-\alpha)(\delta-\alpha)(\beta-\delta)}-\frac{\beta-5\delta+8\alpha}{4\times 384(\delta-\alpha)}\right)\frac{(\beta-\delta-4\alpha)^3}{b-a}-\left(\frac{(\delta+\beta-2\alpha)^2(\delta+\beta+6\alpha)}{4\times 256(\beta-\alpha)(\delta-\alpha)}-\frac{(\beta-\delta)(\beta+16\alpha-9\delta)}{4\times 256(\delta-\alpha)}\right)\frac{(\beta-\delta-4\alpha)^2}{b-a}\\
&+\left(\frac{(\delta+\beta-2\alpha)^2(\delta+\beta+10\alpha)(\beta-\delta)}{4\times 384(\beta-\alpha)(\delta-\alpha)}+\frac{(13\delta-\beta-24\alpha)(\beta-\delta)^2}{4\times 384(\delta-\alpha)}+\frac{\beta^3-2\alpha^3}{24(\beta-\alpha)}+\frac{(\beta^2-2\alpha^2)(\delta-\beta-4\alpha)}{64(\beta-\alpha)(b-a)}\right)\frac{\beta-\delta-4\alpha}{b-a}\\
&+\frac{(\beta-3\delta-6\alpha)(\beta-\delta-4\alpha)^2}{256(b-a)}+\frac{(13\delta-\beta-24\alpha)(\beta-\delta)^2}{384\times 2(\delta-\alpha)}+\frac{(3\beta-\delta+2\alpha)(a^3-b^3+3a^2b-3ab^2)}{24(\beta-\delta)(b-a)}+\frac{(\delta+\beta-2\alpha)(\beta-\delta-4\alpha)^4}{256\times 6(\beta-\delta)^2(b-a)}+\frac{\beta^3-2\alpha^3}{12(\beta-\alpha)}\\
&+\frac{64\alpha^2+11\beta^2+5\delta^2-10\alpha\beta+26\alpha\delta}{192}-\frac{(64\alpha^2+5\delta^2+11\beta^2+26\alpha\delta-10\alpha\beta)(\beta-\delta-4\alpha)}{384(b-a)}+\frac{(\delta+\beta-2\alpha)(3a^4-5b^4-6a^2b^2+12a^3b-4ab^3)}{96(\beta-\delta)^2(b-a)}\\
&-\frac{(3\beta-\delta+2\alpha)(\beta-\delta-4\alpha)^3}{384(\beta-\delta)(b-a)}.
\end{aligned}
$$

If $\delta-\beta\geq 4\alpha$, and $b-a<(\beta-\delta-4\alpha)/2$, then

$$
\begin{aligned}
V[\xi]=&\left(\frac{(\delta+\beta-2\alpha)^2(\delta+\beta+2\alpha)}{96(\delta-\alpha)(\beta-\alpha)(\beta-\delta)}-\frac{\beta-5\delta+8\alpha}{96(\delta-\alpha)}\right)(b-a)^2+\frac{(\delta+\beta-2\alpha)^2(\beta-\delta)(\delta+\beta+10\alpha)}{384(\delta-\alpha)(\beta-\alpha)}\\
&+\frac{(13\delta-\beta-24\alpha)(\beta-\delta)^2}{384(\delta-\alpha)}-\frac{(\beta-\delta)(\beta^2-2\alpha^2)}{8(\beta-\alpha)}+\frac{\beta^3-2\alpha^3}{6(\beta-\alpha)}.
\end{aligned}
$$

Proof: The proof of the theorem is similar to that of Theorem 3.1. □

Example 3.3 *Let ξ be a trapezoidal fuzzy random variable. For each ω, $\xi(\omega)=(X(\omega)-2, X(\omega)-1, X(\omega)+1, X(\omega)+3)$ is a triangular fuzzy variable with $X\sim\mathcal{U}(0,1)$. Calculate $V[\xi]$. Since $\alpha=1,\beta=3,\delta=2,a=0,b=1$, $(\beta-\delta)/2=1/2<b-a=1<(\beta-\delta+4\alpha)/2=5/2$, by Theorem 3.3, we have*

$$
V[\xi]=\frac{(b-a)^2}{12}+\frac{(\beta+\delta-2\alpha)(b-a)}{16}+\frac{8\alpha^2+\beta^2+\delta^2+\alpha\beta+\alpha\delta}{12}+\frac{(\beta-\delta)(10\alpha\delta-10\alpha\beta+\beta^2-\delta^2)}{192(b-a)}=\frac{463}{192}.
$$

4 Concluding Remarks

The variance of a fuzzy random variable is often contained in the objectives or constraints of fuzzy random optimization problems. Due to the twofold uncertainty, the variance of a fuzzy random variable is difficult to compute. Motivated by this fact, this paper first presented some moment formulas for trapezoidal fuzzy variables, then established several variance formulas for trapezoidal fuzzy random variables, in which the randomness is characterized by uniform distributions.Three numerical examples were also provided to illustrate the useful applications of the obtained formulas.

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