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A Robust Approach to Location-Allocation Problem under Uncertainty

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Abstract

This paper presents a new mathematical model for location-allocation problem considering uncertain parameter. In real-world cases, demand, distance, traveling time or any parameters in classical models may change over the period of time. So, considering uncertainty yields more flexibility for the results and its applications. In our study, environmental uncertainty is described by discrete scenarios where probability of occurrence each of them is not known. So, we use robust optimization technique to analyze the model. Therefore, we introduce a formulation of the robust location-allocation problem in which we have budget constraint. Also, we present mean value model where each uncertain parameter is replaced by its mean to compare with robust model. Finally, some numerical examples are illustrated to show effectiveness of the robust solutions.
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1 Introduction

Facility location decisions are costly and difficult to change, and they have impact for long horizon. These strategic decisions are critical determinant of whether materials will flow efficiently through the network. Because of the fact that each parameter of this problem can change sensitively in the period of time–such as time, demands and distances, so decisions related to the design can be made in effect.

Location-allocation (LA) problem in facility location problem is to locate a set of new facilities such that the total distance from facilities to customers is minimized. LA problem has been considered for half a century because of its extensively realistic application, such as facility layout problem and design of distribution networks. In real-world, we must consider the uncertain LA problem because some factors such as demands, allocations, distances even locations of customers or facilities may be shifted. LA problem was studied in detail by Gen [11, 12]. Badri *et al.* [2] and Hodey *et al.* [15] presented many models of different cases in LA. To solve these models, several algorithms such as branch-and-bound algorithms [16], simulated annealing [21], tabu search [6, 14, 22] have been developed. Recently, Daskin *et al.* [7] studied inventory–location models that considered expected cost of inventory with cost of location and allocation simultaneously. Aikens [1] had studied previous location-allocation models. Some researches in uncertainty are done in the area of multi product and multi echelon in supply chain system by Tsiakis *et al.* [25]. Gabor *et al.* [10] presented an approximation algorithm for a facility location problem with stochastic demand. They presented an expected value of a constraint that the probability that an arbitrary request lost was at most α.

In certain situations, all parameters are deterministic and known, while uncertain situations involve arbitrariness. In uncertain situations, parameters are unsure, and in addition, in some cases no information about probabilities is known but in the other probability of distribution function is known. Problems in the first level are identified as robust optimization problems and often aim to optimize the worst-case performance of the system. The problems in the second situations are specified as stochastic optimization problems; an ordinary goal is to optimize the expected value of objective functions. The goal of both stochastic and robust optimization is to determine a solution that will present well under any possible realization of the uncertain parameters. In this way, we can describe random parameters

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either by continuous distribution functions or discrete scenarios. Scenario based planning is an approach in which uncertainty is described by determining a number of possible future conditions by decision makers. The scenario based approach has generally advantage to allow parameters to be statistically dependent, which is often not practical when they are described by continuous probability distributions. In robust optimization problems continuous parameters are generally assumed to lie in some pre-determined intervals, because it is often impossible to consider a worst-case scenario when parameter values are unlimited. We will describe this case of uncertainty as "interval uncertainty" and describe parameters modeled this manner as "interval-uncertain" parameters [23].

A considerable step for extending a theory for robust optimization was taken independently by Ben-Tal and Nemirovski [3, 4, 5], El-Ghaoui and Lebret [8] and El-Ghaoui *et al.* [9]. To address the feature of over conservatism, these papers introduced less conservative models by considering uncertain linear problems with ellipsoidal uncertainties, which involve solving the robust counterparts of the nominal problem in the form of conic quadratic problems (see [15]). With suitably chosen ellipsoids, such a formulation can be used as a reasonable approximation to more complex uncertainty sets. However, a practical weakness of such an approach is that it leads to nonlinear, even though convex models, which are more difficult computationally than the earlier linear models [24].

One of the common measure solutions in robust optimization is considering regret of the solution, which is defined as a difference between the costs of a solution in a given scenario and the cost of the optimal solution for that scenario. Regret is sometimes defined as opportunity loss: the difference between the quality of a given solution and the quality of the solution that would have been selected had one known what the future held [14]. Models that aim to minimize the maximum regret across all scenarios are named minimax regret models.

Minimax regret models are usually developed in the literature. Such problems are generally solved using problem specific algorithms. Mausser *et al.* [17] introduced general-function algorithms for minimax regret linear models considering interval-uncertain objective function coefficients for problems and absolute regret and also, problems with relative regret defined by [18]. The algorithms are based on the fact that for a given solution, each random parameter is set either to its lower or its upper bound in the regret-maximizing scenario. To identify this scenario, for each uncertain parameter one binary variable and a few constraints are added to the main model which makes a MIP to be solved. This approach is practical for small- to moderate-size linear models. Mausser *et al.* [19] applied a greedy heuristic for the absolute regret case that used some methods for diversification to avoid local optima. Velarde *et al.* [13] presented notation of the robust capacitated international sourcing problem. They assumed a finite capacity for facilities that separate their work from the other related works. Demand and exchange rate were the uncertain parameters which described by scenarios. Also, the objective functions included a measure of risk.

In this research, environmental uncertainty is described by discrete scenarios where probability of occurrence each of them is not known. Therefore, we use robust optimization technique to analyze this condition. Indeed, formulation of the robust location-allocation problem in which we have budget constraint is introduced. Also, mean value model in which each uncertain parameter is replaced by its mean will be proposed to illustrate effectiveness of the robust approach.

The structure of this paper is as follows. Robust model for location-allocation model is presented in Section 2. We present computational results in Section 3. In Section 4, we summarize our conclusions and discuss avenues for future research.

2 Problem Definition

2.1 Robust Location Model

In this section, we illustrate details of the model which we are interested. The Robust Location-Allocation Problem (RLAP) consists of selection a set of warehouses to cover all customers that aims to minimize total distances from warehouses to customers. We model uncertainty in distances where is described by discrete scenarios and probability of occurrence each scenario is not known. Our model uses the following definitions:

Sets and parameters:

N: set of customers $\{1, 2, \ldots, n\}$;
M: set of warehouses $\{1, 2, \ldots, m\}$;
S: set of scenarios $\{1, 2, \ldots, s\}$;
C_{ij} : Cost of transporting demand for unit distance from warehouse j to customer i.
d_{ijs} : total distance from warehouse j to customer i in scenario s;
P : number of warehouses to be located;
B_{Max} : maximum available budget.

Decision variables:

$$X_{ijs} = \begin{cases} 1, \text{if customer } i \text{ assigned to warehouse } j \text{ in scenario } s \\ 0, \text{otherwise;} \end{cases}$$

$$U_j = \begin{cases} 1, \text{if warehouse j is located} \\ 0, \text{otherwise.} \end{cases}$$

We now describe the constraints in our model. We consider that for each scenario s, the budget constraint must be satisfied:

$$\sum_{i \in N} \sum_{j \in M} C_{ij} d_{ijs} X_{ijs} \le B_{Max}, \forall s. \tag{1}$$

The *X* and *U* variables are binary. In the robust optimization terminology (Mulvey *et al.* [20]), the *U*s are the design variables; in other words, they must be determined before the uncertainty has been realized. The *X*s are named the control decision variables, whose value can be adjusted according to the realization of each scenario (these variables are determined after knowing which scenario has occurred). The general robust optimization framework considers the infeasibility of the scenario in subproblems. Solutions that are feasible among all scenarios are named "model" robust. In our description, a given solution may be infeasible under some scenarios because of its inability to satisfy budget restriction. Our robust optimization formulation would consider the worst case scenario across all scenarios, in other words, it minimizes maximum objective in all scenarios.

A possible objective function for this problem can be characterized as follows:

$$Min\ Z = \max\{\sum_{i \in N} \sum_{j \in M} d_{ijs} X_{ijs}\} \tag{2}$$

The mentioned objective function is nonlinear but if we replace $Y = \max\{\sum_{i \in N} \sum_{j \in M} d_{ijs} X_{ijs}\}$ and adding suitable constraints, the model will be linear. This objective function results in a mixed integer programming formulation that can be solved using standard optimization software.

2.1.2 Proposed Robust Model

$$\min Z = Y \tag{3}$$

Subject to:

$$Y \ge \sum_{i \in N} \sum_{j \in M} d_{ijs} X_{ijs}, \forall s \tag{4}$$

$$\sum_{j \in M} X_{ijs} = 1, \forall i, s \tag{5}$$

$$\sum_{j \in M} U_j = p \tag{6}$$

$$X_{ijs} \le U_j, \forall i, j, s \tag{7}$$

$$\sum_{i \in N} \sum_{j \in M} C_{ij} d_{ijs} X_{ijs} \le B_{Max}, \forall s \tag{8}$$

$$U_j, X_{ijs} \sim (0,1), Y \ge 0, \forall i, j, s. \tag{9}$$

The objective function and constraint (4) show linearization of the proposed objective. Constraint set (5) guarantees that each customer in each scenario must be assigned to a single warehouse. Set constraint (6) forces that number of located warehouses is not exceeded the pre-determined number. Constraint set (7) says that a customer can be allocated to a warehouse if a warehouse is opened. Constraint set (8) states that total shipment cost must not be exceeded maximum available budget. Constraint set (9) determines type of decision variables.

2.2 Mean Value Model

In this section, we specify another model named it mean value model for location-allocation problem where each uncertain parameter is replaced by its mean. This model minimizes total expected distance of the problem. The aim of proposing this model is to compare robust model with mean value model which minimizes expected value of the objective function.

So, description of the mean value model is as follows:

$\bar{d}_{ij}$ = Average of distance between customer i and warehouse j $(\bar{d}_{ij} = \sum_s d_{ijs} / s)$.

$$X_{ij} = \begin{cases} 1, & \text{if customer } i \text{ assigned to warehouse } j \\ 0, & \text{otherwise.} \end{cases}$$

$$\min Z = \sum_{i \in N} \sum_{j \in M} \bar{d}_{ij} X_{ij} \quad (10)$$

Subject to:

$$\sum_{j \in M} X_{ij} = 1, \forall i \quad (11)$$

$$\sum_{j \in M} U_j = p \quad (12)$$

$$X_{ij} \le U_j, \forall i, j \quad (13)$$

$$\sum_{i \in N} \sum_{j \in M} C_{ij} \bar{d}_{ij} X_{ij} \le B_{Max} \quad (14)$$

$$U_j, X_{ij} \sim (0,1), \forall i, j \quad (15)$$

The objective function (10) minimizes total expected distance between customers and warehouses. Constraints (11)-(15) have same definitions such constraints (5)-(9) with eliminating scenarios concept, respectively.

3 Computational Results

In this section, to verify performance of the robust model, some numerical examples are given randomly. Each one is solved by both mentioned models and the results are compared. We begin solving problems and increase size of problems based on specific rule. Consider a company who wants to locate new warehouses. Assume that there are 10 up to 22 customers. Also, there are 3 scenarios. Suppose that a decision maker needs to select distribution centers from 8 up to 11 potential warehouses to serve customers. The capacities are unlimited and budget capacity is determined. In these examples, our aim is to compare worst-case solutions which are “robust” with mean value solutions. Problem information and the results are shown in Table 1. All examples are solved by branch-and-bound algorithm using Lingo 8 solver.

Table 1: Comparison between Robust solutions and Mean value solutions

Problem No.	No. of Customers	P	No. of scenarios	No. of Warehouses	Robust Objective Function	Mean Value Objective Function	Improvement Percent
1	10	3	1	8	16	19.35	20.94%
2	12	3	2	8	19	23.99	26.22%
3	14	3	2	9	22	27.01	22.77%
4	17	3	3	10	28	35.33	26.18%
5	20	3	1	10	33	41.65	26.21%
6	22	3	1	11	35	45	28.57%
7	25	3	1	12	42	51.75	23.21%
8	28	3	2	13	49	62.38	27.31%
9	31	3	2	14	56	69.76	24.58%
10	35	3	3	15	60	73.63	22.71%
Average					**22.71%**		

Table 1 shows comparison between the solutions obtained from robust model and the model which the values of uncertain parameters are replaced their mean. As we can see, in each example robust solution is better than expected value solution. So, performance evaluation is based on the best objective function found by robust model. The last column shows the percentage of gap between these two solutions and we name in improvement percent which is computed with the ratio: (mean value objective – robust objective value) / robust objective value×100. As shown in the last row, the average value of gap is 24.87% which implicates to a better performance of robust solutions rather than expected solutions ($Z_{Robust} \le Z_{Mean}$). These comparisons show efficiency of robust modeling rather than mean value modeling. As mentioned before, this robust formulation retains linearity and can be solved with mathematical programming solvers readily.

In Figure 1, objective values of robust model against mean value model are presented.

As shown in Figure 1, robust model curve is located below the mean value model, in other words, solutions obtained from mean value model are greater than robust model that yields to more efficiency for robust solutions where in large scale problem this efficiency can be more sensible. Data set for an important parameter for problem number 6 in Table 1 is illustrated in Table 2.

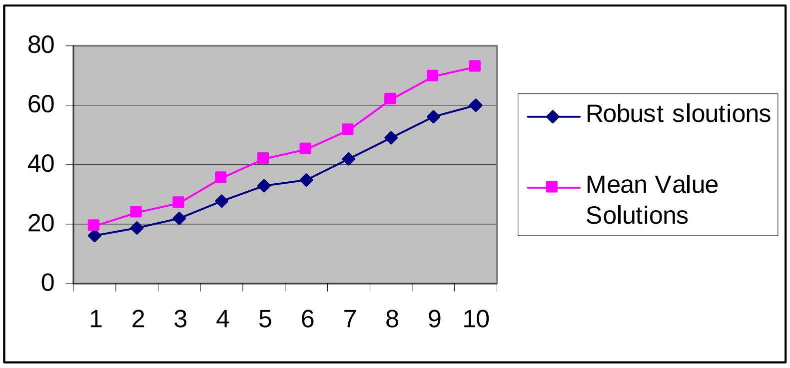

Figure 1: Comparison between robust solutions and mean value solutions

Table 2: Data set for parameter d_{ijs} per example number 6.

Warehouse No.	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**
Customer 1	3.67	1.67	4.00	4.33	2.67	2.33	3.00	3.33	3.33	1.67	1.67
Customer 2	1.33	2.67	4.00	1.67	3.00	2.33	4.67	4.00	3.00	3.00	3.67
Customer 3	2.67	4.33	2.67	2.67	3.67	3.33	2.67	3.33	3.67	3.00	1.67
Customer 4	2.67	2.33	3.00	2.00	2.33	2.67	3.00	3.33	1.67	3.33	2.67
Customer 5	4.00	2.33	2.33	4.33	3.00	3.33	3.33	1.67	4.33	3.33	2.67
Customer 6	2.67	2.33	4.00	3.00	3.67	3.67	2.33	2.67	3.33	3.00	3.33
Customer 7	4.00	2.33	3.67	2.67	4.00	4.00	2.67	2.33	3.67	2.33	2.33
Customer 8	3.33	3.00	2.00	3.67	2.67	2.67	3.00	3.67	1.33	3.00	3.67
Customer 9	3.33	1.67	2.33	2.00	4.00	3.33	1.67	3.33	2.33	4.33	2.33
Customer 10	5.00	2.67	2.67	4.00	4.33	1.00	2.67	1.67	1.67	2.00	3.00
Customer 11	2.33	5.00	2.33	1.67	2.33	3.33	2.00	2.33	4.67	3.67	2.33
Customer 12	2.33	4.33	2.67	1.67	4.67	2.67	4.00	3.67	3.67	2.67	3.33
Customer 13	2.67	2.33	2.33	2.67	2.33	2.00	3.00	2.33	3.00	2.00	3.33
Customer 14	4.33	2.33	3.67	3.00	4.33	2.33	2.00	3.00	2.00	4.67	3.33
Customer 15	2.67	2.00	4.33	4.00	4.67	3.67	2.67	3.00	3.67	4.00	2.67
Customer 16	3.67	2.67	1.33	2.33	4.67	3.00	4.33	3.67	2.33	2.67	3.33
Customer 17	3.67	4.33	3.00	3.33	3.00	4.67	2.33	3.33	4.00	2.33	2.67
Customer 18	3.00	4.33	2.00	2.00	2.67	4.00	2.67	3.33	3.33	2.67	2.67
Customer 19	2.33	2.00	4.33	2.33	1.33	3.33	3.33	2.67	3.67	1.67	2.00
Customer 20	1.67	3.67	2.33	2.33	3.33	2.33	3.00	3.33	1.33	2.67	2.67
Customer 21	3.33	4.00	3.00	2.00	2.67	1.67	3.67	4.00	2.67	2.33	3.33
Customer 22	1.00	3.00	2.67	3.67	1.33	3.33	2.33	3.00	3.33	1.67	1.67

4 Conclusion and Future Directions

In this paper, we proposed a new mathematical model for robust location-allocation problem. In the mentioned model, uncertainty of the parameters was described by discrete scenarios in that probability of the occurrence each scenario was not known. For this reason, we considered worst-case methodology to analyze the computational results and compare it with mean value model. Computational results showed effectiveness of robust modeling.

For future works, we suggest considering robust coverage radius and robust capacity constraint that yields to more flexibility for the model and results. Also, applying this contribution in supply chain design and considering risk analysis can be an interesting area to developed proposed model.

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