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# Hybrid Logic and Uncertain Logic

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**Abstract**

Probabilistic logic and credibilistic logic are two branches of multi-valued logic for dealing with random knowledge and fuzzy knowledge, respectively. In this paper, a hybrid logic is introduced for dealing with random knowledge and fuzzy knowledge simultaneously. First, a hybrid formula is introduced on the basis of random proposition and fuzzy proposition. Furthermore, a hybrid truth value is defined by chance measure. As a generalization of probabilistic logic, credibilistic logic and hybrid logic, an uncertain logic is proposed to deal with the general uncertain knowledge. Within the framework of uncertainty theory, the uncertain truth value is defined by uncertain measure and some of its basic properties are studied including the law of excluded middle, the law of contradiction and the law of truth conservation.

**Keywords:** multi-valued logic, probabilistic logic, credibilistic logic, hybrid logic, uncertain logic

## 1 Introduction

Classical logic assumes that each proposition is either true or false. However, vague predicate presents a challenge to classical logic because proposition containing vague predicate can fail to be true or false and therefore such a proposition cannot be adequately represented in classical logic. For example, "tall" is a vague predicate and the proposition "$5'7''$ Tom is tall" is neither completely true nor completely false. In order to deal with vague predicate, a three-valued logic was proposed by Lukasiewicz, which was then extended to multi-valued logic by other researchers [1, 2, 16, 18]. In multi-valued logic, the connectives and the rules for constructing formula are those used in classical logic, and the disjunction, conjunction and negation of formulas are defined by max, min operations together with the complementation to 1, respectively. In 1972, Lee [4] defined the concept of satisfiability and studied the resolution principle. A definition of logical inference of one formula based upon the assertion of some premise formulas was introduced by Yager [17]. For more information concerning the theory of multi-valued logic, the interested readers may consult the book [13].

Although multi-valued logic is well developed, a practical interpretation of truth value is controversial. In 1976, Nilsson [14] considered the truth value as probability value and gave a probabilistic modus ponens, which constructe the foundation of probabilistic logic. In addition, the consistency between probabilistic logic and classical logic was also proved by Nilsson [14]. Recently, Li and Liu [7] introduced a credibilistic logic by explaining the truth value as credibility value. In fact, probabilistic logic and credibilistic logic are all branches of multi-valued logic. The difference is that the former is used to deal with random knowledge, and the latter is used to deal with fuzzy knowledge. However, random knowledge and fuzzy knowledge may appear simultaneously in a complex system. The purpose of this paper is to introduce a hybrid logic for dealing with random knowledge and fuzzy knowledge simultaneously. In addition, as a generalization of probabilistic logic, credibilistic logic and hybrid logic, a general uncertain logic will be introduced within the framework of uncertainty theory.

The rest part of this paper is organized as follows. For facilitating the understanding of the paper, Section 2 recalls some basic concepts and properties about probabilistic logic and credibilistic logic. In Section 3, a hybrid logic is introduced as an extension of probabilistic logic and credibilistic logic, and the consistency between hybrid logic and classical logic is also proved. For dealing with a general uncertain knowledge system, an uncertain logic is proposed in Section 4 within the framework of uncertainty theory. At the end of this paper, a brief summary about this paper is given.

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## 2 Preliminaries

In multi-valued logic, a *proposition* is defined as a statement with truth value belonging to $[0,1]$, and a *formula* is defined as a member of the minimal set $\mathcal{S}$ of finite sequence of primitive symbols $(\neg, \vee, q, q_1, q_2, \cdots)$ satisfying: (a) $q \in \mathcal{S}$ for each proposition $q$; (b) if $X \in \mathcal{S}$, then $\neg X \in \mathcal{S}$; (c) if $X \in \mathcal{S}$ and $Y \in \mathcal{S}$, then $X \vee Y \in \mathcal{S}$. The symbol $\neg$ means negation, and $\vee$ means disjunction. For example, if $X$ and $Y$ are formulas, then $\neg X$ means "the negation of $X$" and $X \vee Y$ means "$X$ or $Y$". In addition, conjunction symbol $\wedge$ and implication symbol $\rightarrow$ are defined as

$$X \wedge Y = \neg(\neg X \vee \neg Y), \quad X \rightarrow Y = \neg X \vee Y$$

where $X \wedge Y$ means "$X$ and $Y$", and $X \rightarrow Y$ means "if $X$ then $Y$". It is clear that $X \wedge Y$ and $X \rightarrow Y$ are formulas.

Let $X$ be a formula containing propositions $q_1, q_2, \cdots, q_n$. Its *truth function* is defined as a function $f : \{0,1\}^n \rightarrow \{0,1\}$ such that $f(x_1, x_2, \cdots, x_n) = 1$ if and only if $X = 1$ with respect to $q_i = x_i$, where $X = 1$ means $X$ is true and $X = 0$ means $X$ is false. For example, the truth function of $q_1 \vee q_2$ is

$$f(1,1) = 1, \quad f(1,0) = 1, \quad f(0,1) = 1, \quad f(0,0) = 0,$$

and the truth function of $q_1 \rightarrow q_2$ is

$$f(1,1) = 1, \quad f(1,0) = 0, \quad f(0,1) = 1, \quad f(0,0) = 1.$$

### Probabilistic Logic

Probabilistic logic was proposed by Nilsson [14] which defines truth value as probability value. In probabilistic logic, we use the terms *random proposition* and *random formula* instead of proposition and formula, respectively. For example,

"It rains in Beijing with probability 0.9"

is a random proposition, where "It rains in Beijing" is a statement, and its truth value is 0.9 in probability. Generally speaking, we use $\eta$ to express a random proposition and use $p$ to express its probability value. In fact, a random proposition $\eta$ is essentially a random variable defined as

$$\eta = \begin{cases} 1, & \text{with probability } p \\ 0, & \text{with probability } 1-p, \end{cases}$$

where $\eta = 1$ means $\eta$ is true and $\eta = 0$ means $\eta$ is false.

Let $X$ be a random formula containing random propositions $\eta_1, \eta_2, \cdots, \eta_n$. It is clear that $X$ is a random variable taking values 0 or 1 defined by its truth function $f$ as

$$X = f(\eta_1, \eta_2, \cdots, \eta_n).$$

In this equation, the symbols $\eta_1, \eta_2, \cdots, \eta_n$ are considered as random variables. For any random formula $X$, its truth value was defined by Nilsson [14] as

$$T(X) = \Pr\{X = 1\}. \tag{1}$$

### Credibilistic Logic

Recently, Li and Liu [7] proposed a credibilistic logic as another branch of multi-valued logic within the framework of credibility theory (See Appendix 1). In credibilistic logic, we use the terms *fuzzy proposition* and *fuzzy formula* instead of proposition and formula, respectively. For example,

"Tom lives in Beijing with credibility 0.8" is a fuzzy proposition, where "Tom lives in Beijing" is a statement, and its truth value is 0.8 in credibility. Generally speaking, we use $\xi$ to express a fuzzy proposition and use $c$ to express its credibility value. If we use $\xi = 1$ to express $\xi$ is true, and use $\xi = 0$ to express $\xi$ is false, then $\xi$ is essentially a fuzzy variable

$$\xi = \begin{cases} 1, & \text{with credibility } c \\ 0, & \text{with credibility } 1 - c. \end{cases}$$

Let $X$ be a fuzzy formula containing fuzzy propositions $\xi_1, \xi_2, \cdots, \xi_n$. It is clear that $X$ is a fuzzy variable taking values 0 or 1 defined by its truth function $f$ as

$$X = f(\xi_1, \xi_2, \cdots, \xi_n).$$

In this equation, the symbols $\xi_1, \xi_2, \cdots, \xi_n$ are considered as fuzzy variables. For any fuzzy formula $X$, its truth value was defined by Li and Liu [7] as

$$T(X) = \mathrm{Cr}\{X = 1\}. \tag{2}$$

# 3 Hybrid Logic

Probabilistic logic and credibilistic logic are used to deal with random knowledge and fuzzy knowledge, respectively. In this section, a hybrid logic will be introduced within the framework of chance theory for dealing with random knowledge and fuzzy knowledge simultaneously. A brief introduction about chance theory may be found in Appendix 2.

**Definition 3.1** *A* hybrid formula *is defined as a member of the minimal set $\mathcal{S}$ of finite sequence of primitive symbols satisfying:*
(a) *$\xi \in \mathcal{S}$ for each fuzzy proposition $\xi$;*
(b) *$\eta \in \mathcal{S}$ for each random proposition $\eta$;*
(c) *if $X \in \mathcal{S}$, then $\neg X \in \mathcal{S}$;*
(d) *if $X \in \mathcal{S}$ and $Y \in \mathcal{S}$, then $X \vee Y, X \wedge Y, X \to Y \in \mathcal{S}$.*

**Example 3.1** Let $\xi$ be a fuzzy proposition "Tom lives in Beijing with credibility $\alpha$", $\eta$ a random proposition "It rains in Beijing with probability $\beta$" and $\tau$ a random proposition "Tom stays home with probability $\gamma$". Then the sentence "If Tom lives in Beijing and it rains in Beijing, then Tom stays home" is a hybrid formula, denoted by $\xi \wedge \eta \to \tau$.

Assume that $X$ is a hybrid formula containing fuzzy propositions $\xi_1, \xi_2, \cdots, \xi_n$ and random propositions $\eta_1, \eta_2, \cdots, \eta_m$. Then $X$ is essentially a hybrid variable defined by its truth function $f$ as

$$X = f(\xi_1, \xi_2, \cdots, \xi_n, \eta_1, \eta_2, \cdots, \eta_m).$$

In this equation, the symbols $\xi_i$ and $\eta_i$ are considered as fuzzy variable and random variable, respectively.

**Definition 3.2** *Let $X$ be a hybrid formula. Then its truth value is defined as*

$$T(X) = \mathrm{Ch}\{X = 1\}. \tag{3}$$

In addition, if $X = f(\xi_1, \xi_2, \cdots, \xi_n, \eta_1, \eta_2, \cdots, \eta_m)$, then it follows from Definition 3.2 that

$$T(X) = \mathrm{Ch}\{f(\xi_1, \xi_2, \cdots, \xi_n, \eta_1, \eta_2, \cdots, \eta_m) = 1\}.$$

**Remark 3.1** If the hybrid formula $X$ degenerates to a random formula, then it follows from the chance composition theorem (10) that

$$T(X) = \mathrm{Ch}\{X = 1\} = \mathrm{Pr}\{X = 1\}.$$

On the other hand, if $X$ degenerates to a fuzzy formula, then we have

$$T(X) = \mathrm{Ch}\{X = 1\} = \mathrm{Cr}\{X = 1\}.$$

Hence, hybrid logic is consistent with the probabilistic logic and credibilistic logic.

**Example 3.2** Let $\xi$ be a fuzzy proposition with credibility $\alpha$, and let $\eta$ be a random proposition with probability $\beta$. Then
(i) $\xi \wedge \eta$ is a hybrid formula with truth value

$$T(\xi \wedge \eta) = \mathrm{Ch}\{\xi \wedge \eta = 1\} = \mathrm{Ch}\{\xi = 1, \eta = 1\} = \mathrm{Cr}\{\xi = 1\} \wedge \Pr\{\eta = 1\} = \alpha \wedge \beta;$$

(ii) $\xi \vee \eta$ is a hybrid formula with truth value

$$T(\xi \vee \eta) = \mathrm{Ch}\{\xi \vee \eta = 1\} = \mathrm{Ch}\{\xi = 1 \text{ or } \eta = 1\} = \mathrm{Cr}\{\xi = 1\} \vee \Pr\{\eta = 1\} = \alpha \vee \beta;$$

(iii) $\xi \to \eta$ is a hybrid formula with truth value

$$\begin{aligned} T(\xi \to \eta) &= \mathrm{Ch}\{\xi \to \eta = 1\} = 1 - \mathrm{Ch}\{\xi = 1, \eta = 0\} \\ &= 1 - \mathrm{Cr}\{\xi = 1\} \wedge (1 - \Pr\{\eta = 1\}) \\ &= 1 - \alpha \wedge (1 - \beta) \\ &= (1 - \alpha) \vee \beta. \end{aligned}$$

**Example 3.3** Let $\xi$ be a fuzzy proposition with credibility $\alpha$, and let $\eta$ and $\tau$ be random propositions with probabilities $\beta$ and $\gamma$, respectively. Then $\xi \wedge \eta \to \tau$ is a hybrid formula with truth value

$$\begin{aligned} T(\xi \wedge \eta \to \tau) &= 1 - \mathrm{Ch}\{\xi = 1, \eta = 1, \tau = 0\} = 1 - \mathrm{Cr}\{\xi = 1\} \wedge \Pr\{\eta = 1, \tau = 0\} \\ &= 1 - \mathrm{Cr}\{\xi = 1\} \wedge (\Pr\{\eta = 1\} \times \Pr\{\tau = 0\}) \\ &= 1 - \alpha \wedge (\beta \times (1 - \gamma)). \end{aligned}$$

**Example 3.4** Suppose that $\xi$ and $\tau$ are fuzzy propositions with credibilities $\alpha$ and $\gamma$, respectively, and $\eta$ is a random proposition with probability $\beta$. Then $\xi \wedge \eta \to \tau$ is a hybrid formula with truth value

$$\begin{aligned} T(\xi \wedge \eta \to \tau) &= 1 - \mathrm{Ch}\{\xi = 1, \eta = 1, \tau = 0\} = 1 - \mathrm{Cr}\{\xi = 1, \tau = 0\} \wedge \Pr\{\eta = 1\} \\ &= 1 - \mathrm{Cr}\{\xi = 1\} \wedge \Pr\{\tau = 0\} \wedge \Pr\{\eta = 1\} \\ &= 1 - \alpha \wedge \beta \wedge (1 - \gamma). \end{aligned}$$

**Theorem 3.1** *Let $X$ be a hybrid formula containing fuzzy propositions $\xi_1, \xi_2, \cdots, \xi_n$ and random propositions $\eta_1, \eta_2, \cdots, \eta_m$. If its truth function is $f(x_1, x_2, \cdots, x_n, y_1, y_2, \cdots, y_m)$, then its truth value is*

$$T(X) = \begin{cases} \max\limits_{\boldsymbol{x}} \left( \dfrac{\mu(\boldsymbol{x})}{2} \wedge \sum\limits_{f(\boldsymbol{x},\boldsymbol{y})=1} \phi(\boldsymbol{y}) \right), & \text{if } \max\limits_{\boldsymbol{x}} \left( \dfrac{\mu(\boldsymbol{x})}{2} \wedge \sum\limits_{f(\boldsymbol{x},\boldsymbol{y})=1} \phi(\boldsymbol{y}) \right) < 0.5 \\ 1 - \max\limits_{\boldsymbol{x}} \left( \dfrac{\mu(\boldsymbol{x})}{2} \wedge \sum\limits_{f(\boldsymbol{x},\boldsymbol{y})=0} \phi(\boldsymbol{y}) \right), & \text{if } \max\limits_{\boldsymbol{x}} \left( \dfrac{\mu(\boldsymbol{x})}{2} \wedge \sum\limits_{f(\boldsymbol{x},\boldsymbol{y})=1} \phi(\boldsymbol{y}) \right) \geq 0.5, \end{cases}$$

*where $\boldsymbol{x} = (x_1, x_2, \cdots, x_n)$, $\boldsymbol{y} = (y_1, y_2, \cdots, y_m)$, $\mu$ is the joint membership function of $(\xi_1, \xi_2, \cdots, \xi_n)$ and $\phi$ is the joint probability mass function of $(\eta_1, \eta_2, \cdots, \eta_n)$.*

**Proof:** Since $X = f(\xi_1, \xi_2, \cdots, \xi_n, \eta_1, \eta_2, \cdots, \eta_m)$, we have

$$T(X) = \mathrm{Ch}\{X = 1\} = \mathrm{Ch}\{f(\xi_1, \xi_2, \cdots, \xi_n, \eta_1, \eta_2, \cdots, \eta_m) = 1\}.$$

The theorem follows immediately from the definition of chance measure.

**Remark 3.2** Let $\mathbf{x}$ be an $n$-dimensional vector $(x_1, x_2, \cdots, x_n)$ with $x_i \in \{0, 1\}$ for all $1 \leq i \leq n$, and $\mathbf{y}$ an $m$-dimensional vector $(y_1, y_2, \cdots, y_m)$ with $y_j \in \{0, 1\}$ for all $1 \leq j \leq m$. Since fuzzy variable $\xi_i$ is defined as

$$\xi_i = \begin{cases} 1, & \text{with credibility } c_i \\ 0, & \text{with credibility } 1 - c_i, \end{cases}$$

its membership function is

$$\mu_i(x_i) = \begin{cases} 2c_i \wedge 1, & \text{if } x_i = 1 \\ 2(1-c_i) \wedge 1, & \text{if } x_i = 0, \end{cases}$$

and the joint membership function of fuzzy vector $(\xi_1, \xi_2, \cdots, \xi_n)$ is

$$\mu(\mathbf{x}) = \min_{1 \le i \le n} \mu_i(x_i).$$

In addition, since random variable $\eta_j$ is defined as

$$\eta_j = \begin{cases} 1, & \text{with probability } p_j \\ 0, & \text{with probability } 1 - p_j, \end{cases}$$

its probability mass function is

$$\phi_j(y_j) = \begin{cases} p_j, & \text{if } y_j = 1 \\ 1 - p_j, & \text{if } y_j = 0, \end{cases}$$

and the joint probability mass function of random vector $(\eta_1, \eta_2, \cdots, \eta_m)$ is

$$\phi(\mathbf{y}) = \prod_{1 \le j \le m} \phi_j(y_j).$$

**Theorem 3.2** *(Law of Excluded Middle) For any hybrid formula $X$, we have*

$$T(X \vee \neg X) = 1.$$

**Proof:** It follows from the normality of chance measure that

$$T(X \vee \neg X) = \mathrm{Ch}\{X \vee \neg X = 1\} = \mathrm{Ch}\{\{X = 1\} \cup \{X = 0\}\} = 1.$$

The proof is complete.

**Theorem 3.3** *(Law of Contradiction) For any hybrid formula $X$, we have*

$$T(X \wedge \neg X) = 0.$$

**Proof:** It follows from the law of excluded middle that

$$T(X \wedge \neg X) = 1 - T\{\neg X \vee \neg\neg X\} = 1 - 1 = 0.$$

The proof is complete.

**Theorem 3.4** *(Law of Truth Conservation) For any hybrid formula $X$, we have*

$$T(\neg X) = 1 - T(X).$$

**Proof:** It follows from the self-duality of chance measure (11) that

$$T(\neg X) = \mathrm{Ch}\{\neg X = 1\} = \mathrm{Ch}\{X = 0\} = 1 - \mathrm{Ch}\{X = 1\} = 1 - T(X).$$

The proof is complete.

Generally speaking, the higher the truth value is, the more true the hybrid formula is. For any hybrid formula $X$, $T(X) = 1$ means $X$ is certainly true and $T(X) = 0$ means $X$ is certainly false.

**Theorem 3.5** *For any hybrid formulas $X$ and $Y$, we have*

$$T(X) \vee T(Y) \le T(X \vee Y) \le T(X) + T(Y).$$

**Proof:** It follows from the monotonicity of chance measure (12) that

$$\begin{aligned} T(X \vee Y) &= \mathrm{Ch}\{X \vee Y = 1\} = \mathrm{Ch}\{\{X = 1\} \cup \{Y = 1\}\} \\ &\geq \mathrm{Ch}\{X = 1\} \vee \mathrm{Ch}\{Y = 1\} \\ &= T(X) \vee T(Y). \end{aligned}$$

Furthermore, it follows from the subadditivity of chance measure (13) that

$$T(X \vee Y) = \mathrm{Ch}\{\{X = 1\} \cup \{Y = 1\}\} \leq \mathrm{Ch}\{X = 1\} + \mathrm{Ch}\{Y = 1\} = T(X) + T(Y).$$

The proof is complete.

**Remark 3.3** The hybrid logic and classical logic are consistent. First, for any hybrid formula $X$, if $T(X) = 1$, it follows from Theorem 3.4 that

$$T(\neg X) = 1 - T(X) = 0,$$

and if $T(X) = 0$, we have $T(\neg X) = 1$. Furthermore, for any hybrid formulas $X$ and $Y$, if $T(X) = 1$ or $T(Y) = 1$, it follows from Theorem 3.5 that

$$T(X \vee Y) \geq T(X) \vee T(Y) = 1,$$

which implies that $T(X \vee Y) = 1$. If $T(X) = 0$ and $T(Y) = 0$, it follows from Theorem 3.5 that

$$T(X \vee Y) \leq T(X) + T(Y) = 0.$$

That is, $T(X \vee Y) = 0$.

**Theorem 3.6** *For any hybrid formula $X$, we have*

$$T(X \vee X) = T(X).$$

**Proof:** It follows from Definition 3.2 that

$$T(X \vee X) = \mathrm{Ch}\{X \vee X = 1\} = \mathrm{Ch}\{X = 1\} = T(X).$$

The proof is complete.

**Theorem 3.7** *For any hybrid formulas $X$ and $Y$, we have*

$$T(X) + T(Y) - 1 \leq T(X \wedge Y) \leq T(X) \wedge T(Y).$$

**Proof:** It follows from Theorem 3.5 that

$$T(X \wedge Y) = 1 - T(\neg X \vee \neg Y) \leq 1 - T(\neg X) \vee T(\neg Y) = T(X) \wedge T(Y),$$

$$T(X \wedge Y) = 1 - T(\neg X \vee \neg Y) \geq 1 - (T(\neg X) + T(\neg Y)) = T(X) + T(Y) - 1.$$

The proof is complete.

**Theorem 3.8** *For any hybrid formula $X$, we have*

$$T(X \wedge X) = T(X).$$

**Proof:** It follows from Theorem 3.6 that

$$T(X \wedge X) = 1 - T(\neg X \vee \neg X) = 1 - T(\neg X) = T(X).$$

The proof is complete.

**Theorem 3.9** *For any hybrid formulas $X$ and $Y$, we have*

$$(1 - T(X)) \vee T(Y) \leq T(X \to Y) \leq 1 - T(X) + T(Y).$$

**Proof:** It follows from Theorem 3.5 and the fact $X \to Y = \neg X \vee Y$ that

$$T(X \to Y) = T(\neg X \vee Y) \geq T(\neg X) \vee T(Y) = (1 - T(X)) \vee T(Y),$$

$$T(X \to Y) = T(\neg X \vee Y) \leq T(\neg X) + T(Y) = 1 - T(X) + T(Y).$$

The proof is complete.

**Theorem 3.10** *For any hybrid formula $X$, we have*

$$T(X \to X) = 1, \quad T(X \to \neg X) = 1 - T(X).$$

**Proof:** It follows from Theorem 3.6 that

$$T(X \to X) = T(\neg X \vee X) = 1,$$

$$T(X \to \neg X) = T(\neg X \vee \neg X) = T(\neg X) = 1 - T(X).$$

The proof is complete.

**Theorem 3.11** *For any hybrid formulas $X$ and $Y$, we have*

$$T(\neg Y \to \neg X) = T(X \to Y).$$

**Proof:** Since $X \to Y = \neg X \vee Y$, we have

$$T(\neg Y \to \neg X) = T(\neg\neg Y \vee \neg X) = T(\neg X \vee Y) = T(X \to Y).$$

The proof is complete.

**Theorem 3.12** *For any fuzzy formula $X$ and random formula $Y$, we have*
(a) $T(X \wedge Y) = T(X) \wedge T(Y)$;
(b) $T(X \vee Y) = T(X) \vee T(Y)$;
(c) $T(X \to Y) = (1 - T(X)) \vee T(Y)$,
*where $T(X)$ represents the credibilistic truth value and $T(Y)$ represents the probabilistic truth value.*

**Proof:** Part (a). It follows from chance composition theorem (10) that

$$T(X \wedge Y) = \mathrm{Ch}\{X \wedge Y = 1\} = \mathrm{Ch}\{X = 1, Y = 1\} = \mathrm{Cr}\{X = 1\} \wedge \Pr\{Y = 1\} = T(X) \wedge T(Y).$$

Part (b). It follows from the self-duality theorem and part (a) that

$$T(X \vee Y) = T(\neg(\neg X \wedge \neg Y)) = 1 - T(\neg X) \wedge T(\neg Y) = T(X) \vee T(Y).$$

Part (c). It follows from the self-duality theorem and part (b) that

$$T(X \to Y) = T(\neg X \vee Y) = T(\neg X) \vee T(Y) = (1 - T(X)) \vee T(Y).$$

The proof is complete.

# 4 Uncertain Logic

In this section, we introduce an uncertain logic within the framework of uncertainty theory as a generalization of probabilistic logic, credibilistic logic and hybrid logic. A brief knowledge about uncertainty theory may be found in Appendix 3.

In uncertain logic, we define the truth value for each proposition as uncertain measure. Hence, we use the terms *uncertain proposition* and *uncertain formula* instead of proposition and formula, respectively. Generally speaking, we use $\tau$ to express an uncertain proposition and use $u$ to express its uncertainty value. If we use

$\tau = 1$ to express $\tau$ is true, and use $\tau = 0$ to express $\tau$ is false, then $\tau$ is essentially an uncertain variable defined as

$$\tau = \begin{cases} 1, & \text{with uncertainty } u \\ 0, & \text{with uncertainty } 1 - u. \end{cases}$$

Let $X$ be an uncertain formula containing uncertain propositions $\tau_1, \tau_2, \cdots, \tau_n$. It is clear that $X$ is essentially an uncertain variable taking values 0 or 1 defined by its truth function $f$ as

$$X = f(\tau_1, \tau_2, \cdots, \tau_n).$$

In this equation, the symbols $\tau_1, \tau_2, \cdots, \tau_n$ are considered as uncertain variables.

**Definition 4.1** *For each uncertain formula $X$, its truth value is defined as*

$$T(X) = \mathcal{M}\{X = 1\}.$$

For any uncertain proposition $\tau$, it is easy to prove that $\mathcal{M}\{\tau = 1\} = u$. That is, the truth value of each uncertain proposition is just its uncertainty value. In addition, if $X = f(\tau_1, \tau_2, \cdots, \tau_n)$, then

$$T(X) = \mathcal{M}\{f(\tau_1, \tau_2, \cdots, \tau_n) = 1\}.$$

**Example 4.1** If $\tau$ is an uncertain proposition with uncertainty value $\alpha$, then $\neg\tau$ is an uncertain formula and its truth value is

$$T(\neg\tau) = \mathcal{M}\{\neg\tau = 1\} = 1 - \mathcal{M}\{\tau = 1\} = 1 - T(X) = 1 - \alpha.$$

**Example 4.2** Suppose that $\xi$ and $\eta$ are two uncertain propositions with uncertainty values $\alpha$ and $\beta$, respectively. Then we have

(i) $\xi \vee \eta$ is an uncertain formula with truth value belonging to $[\alpha \vee \beta, (\alpha + \beta) \wedge 1]$ because

$$T(\xi \vee \eta) = \mathcal{M}\{\xi = 1 \text{ or } \eta = 1\} \geq \mathcal{M}\{\xi = 1\} \vee \mathcal{M}\{\eta = 1\} = \alpha \vee \beta,$$

$$T(\xi \vee \eta) = \mathcal{M}\{\xi = 1 \text{ or } \eta = 1\} \leq \mathcal{M}\{\xi = 1\} + \mathcal{M}\{\eta = 1\} = \alpha + \beta;$$

(ii) $\xi \wedge \eta$ is an uncertain formula with truth value belonging to $[(\alpha + \beta - 1) \vee 0, \alpha \wedge \beta]$ because

$$T(\xi \wedge \eta) = \mathcal{M}\{\xi = 1, \eta = 1\} \geq \mathcal{M}\{\xi = 1\} + \mathcal{M}\{\eta = 1\} - 1 = \alpha + \beta - 1,$$

$$T(\xi \wedge \eta) = \mathcal{M}\{\xi = 1, \eta = 1\} \leq \mathcal{M}\{\xi = 1\} \wedge \mathcal{M}\{\eta = 1\} = \alpha \wedge \beta;$$

(iii) $\xi \to \eta$ is an uncertain formula with truth value belonging to $[(1 - \alpha) \vee \beta, (1 - \alpha + \beta) \wedge 1]$ because

$$T(\xi \to \eta) = T(\neg\xi \vee \eta) \geq (1 - \alpha) \vee \beta, \quad T(\xi \to \eta) = T(\neg\xi \vee \eta) \leq 1 - \alpha + \beta.$$

**Theorem 4.1** *(Law of Excluded Middle) For any uncertain formula $X$, we have*

$$T(X \vee \neg X) = 1.$$

**Proof:** It follows from the normality of uncertain measure that

$$T(X \vee \neg X) = \mathcal{M}\{X \vee \neg X = 1\} = \mathcal{M}\{\{X = 1\} \cup \{X = 0\}\} = 1.$$

The proof is complete.

**Theorem 4.2** *(Law of Contradiction) For any uncertain formula $X$, we have*

$$T(X \wedge \neg X) = 0.$$

**Proof:** It follows from the law of excluded middle that

$$T(X \wedge \neg X) = 1 - T\{\neg X \vee \neg\neg X\} = 1 - 1 = 0.$$

The proof is complete.

**Theorem 4.3** *(Law of Truth Conservation) For any uncertain formula $X$, we have*

$$T(\neg X) = 1 - T(X).$$

**Proof:** It follows from the self-duality of uncertain measure that

$$T(\neg X) = \mathcal{M}\{\neg X = 1\} = 1 - \mathcal{M}\{X = 1\} = 1 - T(X).$$

The proof is complete.

**Theorem 4.4** *For any uncertain formulas $X$ and $Y$, we have*

$$T(X) \vee T(Y) \le T(X \vee Y) \le T(X) + T(Y).$$

**Proof:** It follows from the monotonicity of uncertain measure that

$$T(X \vee Y) = \mathcal{M}\{X \vee Y = 1\} \ge \mathcal{M}\{X = 1\} \vee \mathcal{M}\{Y = 1\} = T(X) \vee T(Y).$$

On the other hand, it follows from the subadditivity of uncertain measure that

$$T(X \vee Y) = \mathcal{M}\{X \vee Y = 1\} \le \mathcal{M}\{X = 1\} + \mathcal{M}\{Y = 1\} = T(X) + T(Y).$$

The proof is complete.

**Remark 4.1** It follows from the self-duality, monotonicity and subadditivity of truth value that uncertain logic and classical logic are consistent.

**Theorem 4.5** *For any uncertain formulas $X$ and $Y$, we have*
*(a) $T(X \vee X) = T(X)$;*
(b) $T(X \wedge X) = T(X)$;
(c) $T(X) + T(Y) - 1 \le T(X \wedge Y) \le T(X) \wedge T(Y)$;
(d) $T(\neg Y \to \neg X) = T(X \to Y)$;
(e) $T(X \to X) = 1,\ T(X \to \neg X) = 1 - T(X)$;
(f) $(1 - T(X)) \vee T(Y) \le T(X \to Y) \le 1 - T(X) + T(Y)$.

**Proof:** Part (a). It follows from Definition 4.1 that

$$T(X \vee X) = \mathcal{M}\{X \vee X = 1\} = \mathcal{M}\{X = 1\} = T(X).$$

Part (b). It follows from Definition 4.1 that

$$T(X \wedge X) = \mathcal{M}\{X \wedge X = 1\} = \mathcal{M}\{X = 1\} = T(X).$$

Part (c). For any formulas $X$ and $Y$, it follows from Theorem 4.4 that

$$T(X \wedge Y) = 1 - T(\neg X \vee \neg Y) \le 1 - T(\neg X) \vee T(\neg Y) = 1 - (1 - T(X)) \vee (1 - T(Y)) = T(X) \wedge T(Y),$$

$$T(X \wedge Y) = 1 - T(\neg X \vee \neg Y) \ge 1 - T(\neg X) - T(\neg Y) = 1 - (1 - T(X)) - (1 - T(Y)) = T(X) + T(Y) - 1.$$

The other parts may be proved by a similar way of hybrid case.

# 5 Conclusions

In a complex knowledge system, random knowledge and fuzzy knowledge may occur simultaneously. For dealing with this case, a hybrid logic was introduced in this paper as an extension of probabilistic logic and credibilistic logic. First, a hybrid formula was defined and its truth value was defined by chance measure. Furthermore, the consistency between hybrid logic and classical logic was proved. For dealing with a general uncertain knowledge system, an uncertain logic was also proposed within the framework of uncertainty theory. In addition, the uncertain logic was also proved to be consistent with the classical logic based on the law of excluded middle, the law of contradiction and the law of truth conservation for uncertain truth value.

# Appendix 1: Credibility Measure

Credibility theory [9, 11] is a branch of mathematics for studying the behavior of fuzzy phenomena. The central concept of credibility theory is credibility measure, which was first proposed by Liu and Liu [12] in 2002. In addition, a sufficient and necessary condition for credibility measure was given by Li and Liu [5].

Let $\xi$ be a fuzzy variable with membership function $\mu$. Then for any set $B$ of real numbers, we have

$$\mathrm{Cr}\{\xi \in B\} = \frac{1}{2}\left(\sup_{x\in B}\mu(x) + 1 - \sup_{x\in B^c}\mu(x)\right). \tag{4}$$

This formula is also called the *credibility inversion theorem.* Different from possibility measure and necessity measure, credibility measure has self-duality property. That is, for any event $A$, we have

$$\mathrm{Cr}\{A\} + \mathrm{Cr}\{A^c\} = 1. \tag{5}$$

In addition, credibility measure is increasing and subadditive. That is, for any events $A$ and $B$, we have

$$\mathrm{Cr}\{A\} \le \mathrm{Cr}\{B\}, \text{if } A \subset B, \tag{6}$$

$$\mathrm{Cr}\{A \cup B\} \le \mathrm{Cr}\{A\} + \mathrm{Cr}\{B\}. \tag{7}$$

# Appendix 2: Chance Measure

Generally speaking, randomness and fuzziness are two kinds of basic uncertainties in the real world. Probability theory and credibility theory are two branches of mathematics for dealing with random phenomena and fuzzy phenomena, respectively. As the improvement of understanding about uncertain phenomena, some researchers began to study the complex system which includes both randomness and fuzziness. In 1978, Kwakernaak [3] introduced a fuzzy random variable which was redefined by Puri and Raslescu [15] as a measurable function from a probability space to the set of fuzzy variables. Similarly, Liu [8] proposed a random fuzzy variable as a function from a credibility space to the set of random variables. More generally, Liu [11] proposed the concepts of chance space and hybrid variable. In fact, both fuzzy random variable and random fuzzy variable can be described by hybrid variable. In 2008, Li and Liu [6] defined a chance measure for events on the basis of probability measure and credibility measure.

Suppose that $(\Theta, \mathcal{P}, \mathrm{Cr})$ is a credibility space and $(\Omega, \mathcal{A}, \mathrm{Pr})$ is a probability space. The product $(\Theta, \mathcal{P}, \mathrm{Cr}) \times (\Omega, \mathcal{A}, \mathrm{Pr})$ was called a chance space by Liu [10]. For any subset $\Lambda$ of $\Theta \times \Omega$, it was called an event by Liu [11] if $\Lambda(\theta) \in \mathcal{A}$ for each $\theta \in \Theta$, where

$$\Lambda(\theta) = \left\{\omega \in \Omega \;\middle|\; (\theta, \omega) \in \Lambda\right\}.$$

Let $\mathcal{L}$ be the collection of all events, i.e.,

$$\mathcal{L} = \left\{\Lambda \subset \Theta \times \Omega \;\middle|\; \Lambda(\theta) \in \mathcal{A}, \forall \theta \in \Theta\right\}. \tag{8}$$

Liu [11] proved that $\mathcal{L}$ is a $\sigma$-algebra over $\Theta \times \Omega$, and called it the product $\sigma$-algebra $\mathcal{P} \times \mathcal{A}$. For any event $\Lambda$, its chance measure was defined by Li and Liu [6] as

$$\mathrm{Ch}\{\Lambda\} = \begin{cases} \sup\limits_{\theta\in\Theta}(\mathrm{Cr}\{\theta\} \wedge \mathrm{Pr}\{\Lambda(\theta)\}), & \text{if } \sup\limits_{\theta\in\Theta}(\mathrm{Cr}\{\theta\} \wedge \mathrm{Pr}\{\Lambda(\theta)\}) < 0.5 \\ 1 - \sup\limits_{\theta\in\Theta}(\mathrm{Cr}\{\theta\} \wedge \mathrm{Pr}\{\Lambda^c(\theta)\}), & \text{if } \sup\limits_{\theta\in\Theta}(\mathrm{Cr}\{\theta\} \wedge \mathrm{Pr}\{\Lambda(\theta)\}) \ge 0.5. \end{cases} \tag{9}$$

For any $X \in \mathcal{P}$ and $Y \in \mathcal{A}$, Li and Liu [6] proved the chance composition theorem, that is,

$$\mathrm{Ch}\{X \times Y\} = \mathrm{Cr}\{X\} \wedge \mathrm{Pr}\{Y\}, \tag{10}$$

which implies that the chance measure is consistent with credibility measure and probability measure because

$$\mathrm{Ch}\{X \times \Omega\} = \mathrm{Cr}\{X\} \wedge \mathrm{Pr}\{\Omega\} = \mathrm{Cr}\{X\},$$
$$\mathrm{Ch}\{\Theta \times Y\} = \mathrm{Cr}\{\Theta\} \wedge \mathrm{Pr}\{Y\} = \mathrm{Pr}\{Y\}.$$

Furthermore, Li and Liu [6] proved that chance measure is self-dual. That is,

$$\mathrm{Ch}\{\Lambda\} + \mathrm{Ch}\{\Lambda^c\} = 1 \tag{11}$$

for any event $\Lambda$. In addition, chance measure is also increasing and subadditive. That is, for any events $\Lambda_1$ and $\Lambda_2$, we have

$$\mathrm{Ch}\{\Lambda_1\} \leq \mathrm{Ch}\{\Lambda_2\}, if\ \Lambda_1 \subseteq \Lambda_2, \tag{12}$$

$$\mathrm{Ch}\{\Lambda_1 \cup \Lambda_2\} \leq \mathrm{Ch}\{\Lambda_1\} + \mathrm{Ch}\{\Lambda_2\}. \tag{13}$$

A hybrid variable was defined by Liu [10] as a measurable function from a chance space $(\Theta, \mathcal{P}, \mathrm{Cr}) \times (\Omega, \mathcal{A}, \mathrm{Pr})$ to the set of real numbers, i.e., for any Borel set $B$ of real numbers, we have

$$\{(\theta, \omega) \in \Theta \times \Omega \mid \xi(\theta, \omega) \in B\} \in \mathcal{L}. \tag{14}$$

Suppose that $\xi_1, \xi_2, \cdots, \xi_n$ are fuzzy variables on credibility space $(\Theta, \mathcal{P}, \mathrm{Cr})$ and $\eta_1, \eta_2, \cdots, \eta_m$ are random variables on probability space $(\Omega, \mathcal{A}, \mathrm{Pr})$. If $f$ is a measurable function from $\Re^{n+m}$ to $\Re$, then we have $f(\xi_1, \xi_2, \cdots, \xi_n, \eta_1, \eta_2, \cdots, \eta_m)$ is a hybrid variable on chance space $(\Theta, \mathcal{P}, \mathrm{Cr}) \times (\Omega, \mathcal{A}, \mathrm{Pr})$.

# Appendix 3: Uncertain Measure

In order to deal with the general uncertain phenomena, Liu [11] defined an uncertain measure, which is neither a completely additive measure nor a completely nonadditive measure.

Let $\Gamma$ be a nonempty set, and let $\mathcal{L}$ be a $\sigma$-algebra over $\Gamma$. Each element $\Lambda \in \mathcal{L}$ is called an event. In 2007, Liu [11] defined an uncertain measure as a set function $\mathcal{M}$ on $\mathcal{L}$ satisfying:
(i) (Normality) $\mathcal{M}\{\Gamma\} = 1$;
(ii) (Monotonicity) $\mathcal{M}\{\Lambda_1\} \leq \mathcal{M}\{\Lambda_2\}$ whenever $\Lambda_1 \subset \Lambda_2$;
(iii) (Self-Duality) $\mathcal{M}\{\Lambda\} + \mathcal{M}\{\Lambda^c\} = 1$ for each event $\Lambda$;
(iv) (Countable Subadditivity) For any countable consequence of events $\{\Lambda_i\}$, we have

$$\mathcal{M}\left\{\bigcup_{i=1}^{\infty} \Lambda_i\right\} \leq \sum_{i=1}^{\infty} \mathcal{M}\{\Lambda_i\}.$$

It is clear that probability measure, credibility measure and chance measure are all instances of uncertain measure. The triplet $(\Gamma, \mathcal{L}, \mathcal{M})$ is called an uncertainty space. Furthermore, an uncertain variable $\tau$ was defined by Liu [11] as a measurable function from $(\Gamma, \mathcal{L}, \mathcal{M})$ to the set of real numbers, i.e., for any Borel set $B$ of real numbers, the set $\{\tau \in B\} = \{\gamma \in \Gamma | \tau(\gamma) \in B\}$ is an event.

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# References

[1] Adams, E., and H. Levine, On the uncertainties transimitted from premises to conclusions in deductive inferences, *Synthese*, vol.30, pp.429–460, 1975.

[2] Dempster, A., A generalization of Bayesian inference, *Journal of the Royal Statistical Society: Series B*, vol.30, pp.205–247, 1968.

[3] Kwakernaak, H., Fuzzy random variables-I: Definitions and theorems, *Information Sciences*, vol.15, pp.1–29, 1978.

[4] Lee, R., Fuzzy logic and the resolution principle, *Journal of the Association for Computing Machinery*, vol.19, no.1, pp.109–119, 1972.

[5] Li, X., and B. Liu, A sufficient and necessary condition for credibility measures, *International Journal of Uncertainty, Fuzziness & Knowledge-Based Systems*, vol.14, no.5, pp.527–535, 2006.

[6] Li, X., and B. Liu, Chance measure for hybrid events with both fuzziness and randomness, *Soft Computing*, vol.13, no.2, 105–115, 2009.

[7] Li, X., and B. Liu, Foundation of credibilistic logic, *Fuzzy Optimization and Decision Making*, vol.8, no.1, pp.91–102, 2009.

[8] Liu, B., *Theory and Practice of Uncertain Programming*, Physica-Verlag, Heidelberg, 2002.

[9] Liu, B., *Uncertainty Theory*, Springer-Verlag, Berlin, 2004.

[10] Liu, B., A survey of credibility theory, *Fuzzy Optimization and Decision Making*, vol.5, no.4, pp.387–408, 2006.

[11] Liu, B., *Uncertainty Theory*, 2nd ed., Springer-Verlag, Berlin, 2007.

[12] Liu, B., and Y.K. Liu, Expected value of fuzzy variable and fuzzy expected value models, *IEEE Transactions on Fuzzy Systems*, vol.10, no.4, pp.445–450, 2002.

[13] Merrie, B., *An Introduction to Many-Valued and Fuzzy Logic: Semantics, Algebras, and Derivation Systems*, Cambridge University, New York, 2008.

[14] Nilsson, N., Probabilistic logic, *Artificial Intelligence*, vol.28, pp.71–87, 1986.

[15] Puri, M.L., and D. Ralescu, Fuzzy random variables, *Journal of Mathematical Analysis and Applications*, vol.114, pp.409–422, 1986.

[16] Shafer, G., *Mathematical Theory of Evidence*, Princeton University Press, Princeton, New Jersey, 1979.

[17] Yager, R., Inference in a multivalued logic system, *International Journal of Man-Machine Studies*, vol.23, pp.27–44, 1985.

[18] Zadeh, L.A., Fuzzy logic and approximate reasoning, *Synthese*, vol.30, pp.407–428, 1975.