Journal of Uncertain Systems
Vol.3, No.1, pp.64–80, 2009
Online at: www.jus.org.uk

An Outline for a Heuristic Approach to Possibilistic Clustering of the Three-Way Data

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Received 19 February 2008; Accepted 2 June 2008

Abstract. This paper deals in a preliminary way with the problem of clustering of three-way data. A method of the problem solving is based on the application of a direct possibilistic clustering algorithm based on the concept of allotment among fuzzy cluster to a matrix of feeble fuzzy tolerance, which represent a structure of the set of objects under uncertainty. The paper provides the description of basic ideas of the method of clustering and the plan of a direct possibilistic clustering algorithm. Basic ideas of the method of three-way data preprocessing for construction of a matrix of feeble fuzzy tolerance are also considered. An illustrative example of three-way data preprocessing and clustering is given and an analysis of the experimental results of the method's application to the Sato's three-way data is carried out. Preliminary conclusions are discussed.

Keywords: possibilistic clustering, allotment, feeble fuzzy tolerance, three-way data, type-two fuzzy sets

1 Introduction

The first subsection of this introduction includes a brief review of uncertain data clustering methods. The second subsection provides some remarks on author's preliminary results.

1.1 A Problem of Clustering of Uncertain Data

Clustering is a process aiming at grouping a set of objects into classes according to the characteristics of data so that objects within a cluster have high mutual similarity while objects in different clusters are dissimilar. In other words, cluster analysis refers to a spectrum of methods, which try to divide a set of objects $X = \{x_1, ..., x_n\}$ into subsets, called clusters, which are pairwise disjoint, all non empty and reproduce X via union. Fuzzy sets theory, which was proposed by Zadeh [30], gives an idea of uncertainty of belonging to a cluster, which is described by a membership function. Fuzzy clustering methods have been applied effectively in image processing, data analysis, symbol recognition and modeling. The idea of fuzzy approach application to clustering problems was proposed by Bellmann, Kalaba and Zadeh [1]. Heuristic methods of fuzzy clustering, hierarchical methods of fuzzy clustering and optimization methods of fuzzy clustering were proposed by different researchers. A review of some heuristic methods, hierarchical methods and optimization methods of fuzzy clustering was made by Viattchenin [17].

The most widespread approach in fuzzy clustering is the optimization approach and the traditional optimization methods of fuzzy clustering are based on the concept of fuzzy partition. The initial set $X = \{x_1, ..., x_n\}$ of n objects represented by the matrix of similarity coefficients, the matrix of dissimilarity coefficients or the matrix of object attributes, should be divided into c fuzzy clusters. Namely, the grade $\mu_{li},\ 1 \leq l \leq c,\ 1 \leq i \leq n$, to which an object x_i belongs to the fuzzy cluster A^l should be determined. For each object $x_i,\ i = 1, \ldots, n$ the grades of membership should satisfy the conditions of a fuzzy partition:

$$\sum_{l=1}^{c} \mu_{li} = 1,\ 1 \leq i \leq n;\ 0 \leq \mu_{li} \leq 1,\ 1 \leq l \leq c. \quad (1)$$

In other words, the family of fuzzy sets $P(X) = \{A^l | l = \overline{1, c},\ c \leq n\}$ is the fuzzy partition of the initial set of objects $X = \{x_1, ..., x_n\}$ if condition (1) is met. Fuzzy partition $P(X)$ may be described with the aid of

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a partition matrix $P_{c\times n} = [\mu_{li}],\ l = 1, \ldots, c,\ i = 1, \ldots, n$. The set of all fuzzy partitions will be denoted by Π. So, the fuzzy problem formulation in cluster analysis can be defined as the optimization task $Q \to \underset{P(X)\in\Pi}{extr}$ under the constraints (1), where Q is a fuzzy objective function.

The best known optimization approach to fuzzy clustering is the method of fuzzy c-means, developed by Bezdek [2]. The fuzzy c-means algorithm is the basis of the family of fuzzy clustering algorithms. The family of objective function-based fuzzy clustering algorithms includes

- fuzzy c-means algorithm (FCM): spherical clusters of approximately the same size;
- Gustafson-Kessel algorithm (GK): ellipsoidal clusters with approximately the same size; there are also axis-parallel variants of this algorithm; can also be used to detect lines;
- Gath-Geva algorithm (GG): ellipsoidal clusters with varying size; there are also axis-parallel variants of this algorithm; can also be used to detect lines;
- fuzzy c-varieties algorithm (FCV): detection of linear manifolds, that is infinite lines in 2D data;
- fuzzy c-shells algorithm (FCS): detection of circles;
- fuzzy c-rings algorithm (FCR): detection of circles;
- fuzzy c-quadric shells algorithm (FCQS): detection of ellipsoids.

These fuzzy clustering algorithms were proposed by different authors and they are described in Höppner, Klawonn, Kruse and Runkler [5] in detail.

If, on the other hand, condition

$$\sum_{l=1}^{c} \mu_{li} \geq 1,\ 1 \leq i \leq n;\ 0 \leq \mu_{li} \leq 1,\ 1 \leq l \leq c \tag{2}$$

is met for each object $x_i,\ 1 \leq i \leq n$, then the corresponding family of fuzzy sets $C(X) = \{A^l | l = \overline{1, c},\ c \leq n\}$ is the fuzzy coverage of the initial set of objects $X = \{x_1, ..., x_n\}$.

The concept of fuzzy coverage is used mainly in heuristic fuzzy clustering procedures. For example, an algorithm of Couturier and Fioleau [4] is very good illustration of the characterization. Moreover, the conditions (1) of fuzzy partition are very difficult from essential positions. So, a possibilistic approach to clustering was proposed by Krishnapuram and Keller [7]. A concept of possibilistic partition is a basis of possibilistic clustering methods and membership values $\mu_{li},\ \ i = 1, \ldots, n,\ \ l = 1, \ldots, c$ can be interpreted as the values of typicality degree. For each object $x_i,\ \ i = 1, \ldots, n$ the grades of membership should satisfy the conditions of a possibilistic partition:

$$\sum_{l=1}^{c} \mu_{li} > 0,\ 1 \leq i \leq n;\ 0 \leq \mu_{li} \leq 1,\ 1 \leq l \leq c. \tag{3}$$

So, the family of fuzzy sets $Y(X) = \{A^l | l = \overline{1, c},\ c \leq n\}$ is the possibilistic partition of the initial set of objects $X = \{x_1, ..., x_n\}$ if condition (3) is met. The possibilistic approach to clustering was developed by Łęski [8], Zhang and Leung [34], Yang and Wu [29], Xie, Wang, and Chung [26], and other researchers. This approach can be considered as a way in the optimization approach in fuzzy clustering because all methods of possibilistic clustering are objective function-based methods.

Most fuzzy clustering methods are designed for treating crisp data. However, we often have to deal with objects that cannot be described by the quantitative, large or binary signs. In other words, there exists a sign of the object that may assume several values at the same time or, for a given sign; there exists uncertainty in representing the values of this sign. Traditional fuzzy clustering methods cannot be applied directly to such types of objects. So, a problem of fuzzy clustering of uncertain data arises. Such a need occurs mostly in medicine, biology, chemistry, economy, sociology and some other domains.

Several kinds of uncertainty exist and a number of approaches to the uncertain data fuzzy clustering problem solving were proposed by different researchers. Firstly, Žák [32] uses fuzzy sets to describe the uncertainty of the data and introduces the concepts of fuzzy objects and fuzzy dissimilarity. Using these notions, Žák [32] proposed hierarchical and non-hierarchical clustering methods of fuzzy objects.

Secondly, Yang and Ko [27] proposed a class of fuzzy c-number clustering procedures for fuzzy data clustering. These so-called FCN-algorithms were developed by Yang and Liu [28] for conical fuzzy vector data. Notable that a methodology of fuzzy data clustering based on the D-AFC(c)-algorithm was proposed by Viattchenin [25].

Thirdly, Sato and Sato [10] consider a problem of fuzzy clustering for three-way data. Typical three-way data are composed of objects, attributes and situations, for instance, of height and weight of children at several ages. The purpose of clustering for three-way data is to reveal the latent structure through all the time situations by constructing clusters which take into account not only the similarity between the pair of objects at individual time instants but also the similarity between the patterns of change of observation in time. Fuzzy clustering procedure for three-way data is considered as a procedure of solving a multicriteria optimization problem. In other words, cluster structure of the set of objects $X = \{x_1, ..., x_n\}$ must be robust at each time. The method of fuzzy clustering of Sato and Sato [10] was developed by Sato-Ilic and Jain [11].

From the other hand, very interesting results for three-way data were obtained by Coppi and D'Urso [3]. In particular, fuzzy multivariate time trajectories are defined, three types of dissimilarity measures are introduced and three corresponding kinds of dynamic fuzzy clustering models are suggested in [3]. These models are based on a generalization of the Yang and Ko [27] objective functions for fuzzy clustering.

The problem of fuzzy clustering for three-way data is very important in medicine, sociology, economics, and military applications. Objective function-based fuzzy clustering methods are a basis of clustering for three-way data. However, the methods are complex from mathematical positions. Moreover, some methods have not a serious epistemological motivation. That is why a common method of clustering for three-way data must be developed. The fact is main motivation of the work.

1.2 Preliminary Results

Heuristic algorithms of fuzzy clustering display high level of essential clarity and low level of a complexity. Some heuristic clustering algorithms are based on a definition of a cluster concept and the aim of these algorithms is cluster detection conform to a given definition. Mandel [8] notes that such algorithms are called algorithms of direct classification or direct clustering algorithms. Direct heuristic algorithms of fuzzy clustering are simple and very effective in many cases.

Let us remember the preliminary results which are necessary in the rest of the paper. In the first place, an outline for a new heuristic method of fuzzy clustering was presented by Viattchenin [18], where concepts of fuzzy α-cluster and allotment among fuzzy α-clusters were introduced and a basic version of direct fuzzy clustering algorithm was described. The basic version of the algorithm can be called the D-AFC(c)-algorithm. The allotment of elements of the set of classified objects among fuzzy clusters can be considered as a special case of possibilistic partition. That is why the D-AFC(c)-algorithm can be considered as a direct algorithm of possibilistic clustering. The fact was noted by Viattchenin [24].

Secondly, an object cannot be similar to oneself at different time situations. That is why the three-way data structure cannot be represented by the a fuzzy similarity relation on the set of objects $X = \{x_1, ..., x_n\}$. The notions of powerful fuzzy tolerance, feeble fuzzy tolerance and strict feeble fuzzy tolerance were introduced by Viattchenin [12]. The fuzzy tolerances are considered also from philosophical positions by Viattchenin [15]. An application of the feeble fuzzy tolerance for the three-way data structure representation was proposed by Viattchenin [14]. Moreover, the D-AFC(c)-algorithm can be applied directly to the matrix of feeble fuzzy tolerance. The matrix of feeble fuzzy tolerance can be obtained after an application of a special technique to the three-way data. The special technique of the three-way data preprocessing was proposed by Viattchenin [21, 22].

The results will be described in detail in further considerations. The main goal of the present paper is consideration of the problem of possibilistic clustering of the three-way data. For this purpose, a short outline of the method of possibilistic clustering based on the concept of allotment of elements of the set of classified objects among fuzzy clusters is presented. Techniques of the three-way data preprocessing are outlined and described in detail. An illustrative example of the data preprocessing is shown and experimental results of application of the proposed method to Sato's three-way data are given. Concluding remarks are stated and perspectives for research are considered.

2 Outline of the Approach

The basic concepts of the heuristic method of possibilistic clustering based on the allotment concept and a plan of the direct clustering algorithm are considered in the first subsection. Techniques of the three-way data preprocessing are outlined in the second subsection of the section.

2.1 General Plan of the Clustering Procedure

Let us remind the basic concepts of the fuzzy clustering method based on the concept of allotment among fuzzy clusters, which was proposed by Viattchenin [18]. The concept of fuzzy tolerance is the basis for the concept of fuzzy α-cluster. That is why definition of fuzzy tolerance must be considered in the first place.

Let $X = \{x_1, ..., x_n\}$ be the initial set of elements and $T : X \times X \to [0, 1]$ some binary fuzzy relation on $X = \{x_1, ..., x_n\}$ with $\mu_T(x_i, x_j) \in [0, 1]$, $\forall x_i, x_j \in X$ being its membership function.

Definition 2.1 *Fuzzy tolerance is the fuzzy binary intransitive relation which possesses the symmetricity property*

$$\mu_T(x_i, x_j) = \mu_T(x_j, x_i),\ \forall x_i, x_j \in X, \tag{4}$$

and the reflexivity property

$$\mu_T(x_i, x_i) = 1,\ \forall x_i \in X. \tag{5}$$

The notions of powerful fuzzy tolerance, feeble fuzzy tolerance and strict feeble fuzzy tolerance were considered by Viattchenin [12, 18], as well. In this context the classical fuzzy tolerance in the sense of Definition 2.1 was called usual fuzzy tolerance and this kind of fuzzy tolerance was denoted by T_2.

The kind of the fuzzy tolerance imposed determines the real structure of the data, as demonstrated by Viattchenin [21]. So, the notions of powerful fuzzy tolerance, feeble fuzzy tolerance and strict feeble fuzzy tolerance must be considered too.

Definition 2.2 *The feeble fuzzy tolerance is the fuzzy binary intransitive relation which possesses the symmetricity property (4) and the feeble reflexivity property*

$$\mu_T(x_i, x_j) \leq \mu_T(x_i, x_i),\ \forall x_i, x_j \in X. \tag{6}$$

This kind of fuzzy tolerance is denoted by T_1.

Definition 2.3 *The strict feeble fuzzy tolerance is the feeble fuzzy tolerance with strict inequality in (6):*

$$\mu_T(x_i, x_j) < \mu_T(x_i, x_i),\ \forall x_i, x_j \in X. \tag{7}$$

This kind of fuzzy tolerance is denoted by T_0.

Definition 2.4 *The powerful fuzzy tolerance is the fuzzy binary intransitive relation which possesses the symmetricity property (4) and the powerful reflexivity property. The powerful reflexivity property is defined as the condition of reflexivity (5) together with the condition*

$$\mu_T(x_i, x_j) < 1,\ \forall x_i, x_j \in X,\ x_i \neq x_j. \tag{8}$$

This kind of fuzzy tolerance is denoted by T_3.

Fuzzy tolerances T_1 and T_0 are subnormal fuzzy relations if the condition $\mu_T(x_i, x_i) < 1$, $\forall x_i \in X$ is met. The fact was demonstrated by Viattchenin [13] and it is important for representation of the structure of the uncertain data. However, the essence of the method here considered does not depend on the kind of fuzzy tolerance. That is why the method herein is described for any fuzzy tolerance T.

Let us consider the general definition of fuzzy cluster, the concept of the fuzzy cluster's typical point and the concept of the fuzzy allotment of objects.

The number c of fuzzy clusters can be equal to the number of objects, n. This is taken into account in further considerations.

Let $X = \{x_1, ..., x_n\}$ be the initial set of objects. Let T be a fuzzy tolerance on X and α be α-level value of T, $\alpha \in (0, 1]$. Columns or lines of the fuzzy tolerance matrix are fuzzy sets $\{A^1, ..., A^n\}$. Let $\{A^1, ..., A^n\}$ be fuzzy sets on X, which are generated by a fuzzy tolerance T.

Definition 2.5 *The* α*-level fuzzy set* $A^l_{(\alpha)} = \{(x_i, \mu_{A^l}(x_i)) | \mu_{A^l}(x_i) \geq \alpha,\ x_i \in X,\ l \in [1, n]\}$ *is fuzzy* α*-cluster or, simply, fuzzy cluster. So* $A^l_{(\alpha)} \subseteq A^l, \alpha \in (0, 1], A^l \in \{A^1, \ldots, A^n\}$ *and* μ_{li} *is the membership degree of the element* $x_i \in X$ *for some fuzzy cluster* $A^l_{(\alpha)}, \alpha \in (0, 1], l \in \{1, \ldots, n\}$*. Value of* α *is the tolerance threshold of fuzzy clusters elements.*

The membership degree of the element $x_i \in X$ for some fuzzy cluster $A^l_{(\alpha)},\ \alpha \in (0, 1],\ l \in \{1, \ldots, n\}$ can be defined as a

$$\mu_{li} = \begin{cases} \mu_{A^l}(x_i), & x_i \in A^l_\alpha \\ 0, & otherwise \end{cases}, \tag{9}$$

where an α-level $A^l_\alpha = \{x_i \in X | \mu_{A^l}(x_i) \geq \alpha\},\ \alpha \in (0, 1]$ of a fuzzy set A^l is the support of the fuzzy cluster $A^l_{(\alpha)}$. So, the α-level A^l_α of a fuzzy set A^l is a crisp set and condition $A^l_\alpha = Supp(A^l_{(\alpha)})$ is met for each fuzzy cluster $A^l_{(\alpha)},\ \alpha \in (0, 1],\ l \in \{1, \ldots, n\}$.

Membership degree can be interpreted as a degree of typicality of an element to a fuzzy cluster. The value of a membership function of each element of the fuzzy cluster in the sense of Definition 2.5 is the degree of similarity of the object to some typical object of fuzzy cluster.

In other words, if columns or lines of fuzzy tolerance T matrix are fuzzy sets $\{A^1, ..., A^n\}$ on X then fuzzy clusters $\{A^1_{(\alpha)}, ..., A^n_{(\alpha)}\}$ are fuzzy subsets of fuzzy sets $\{A^1, ..., A^n\}$ for some value $\alpha,\ \alpha \in (0, 1]$. The value zero for a fuzzy set membership function is equivalent to non-belonging of an element to a fuzzy set. That is why values of tolerance threshold α are considered in the interval $(0, 1]$.

Definition 2.6 *If* T *is a fuzzy tolerance on* X*, where* X *is the set of elements, and* $\{A^1_{(\alpha)}, ..., A^n_{(\alpha)}\}$ *is the family of fuzzy clusters for some* $\alpha \in (0, 1]$*, then the point* $\tau^l_e \in A^l_\alpha$*, for which*

$$\tau^l_e = \arg\max_{x_i} \mu_{li},\ \forall x_i \in A^l_\alpha \tag{10}$$

is called a typical point of the fuzzy cluster $A^l_{(\alpha)},\ \alpha \in (0, 1],\ l \in \{1, \ldots, n\}$.

Obviously, a typical point of a fuzzy cluster does not depend on the value of tolerance threshold. Moreover, a fuzzy cluster can have several typical points. That is why symbol e is the index of the typical point.

Definition 2.7 *Let* $R^\alpha_z(X) = \{A^l_{(\alpha)} | l = \overline{1, c},\ 2 \leq c \leq n,\ \alpha \in (0, 1]\}$ *be a family of fuzzy clusters for some value of tolerance threshold* $\alpha,\ \alpha \in (0, 1]$*, which are generated by some fuzzy tolerance* T *on the initial set of elements* $X = \{x_1, ..., x_n\}$*. If condition*

$$\sum_{l=1}^{c} \mu_{li} > 0,\ \forall x_i \in X \tag{11}$$

is met for all fuzzy clusters $A^l_{(\alpha)},\ l = \overline{1, c},\ c \leq n$*, then the family is the allotment of elements of the set* $X = \{x_1, ..., x_n\}$ *among fuzzy clusters* $\{A^l_{(\alpha)} | l = \overline{1, c},\ 2 \leq c \leq n\}$ *for some value of the tolerance threshold* $\alpha,\ \alpha \in (0, 1]$.

It should be noted that several allotments $R^\alpha_z(X)$ can exist for some tolerance threshold $\alpha,\ \alpha \in (0, 1]$. That is why symbol z is the index of an allotment.

The condition (11) requires that every object $x_i,\ i = 1, \ldots, n$ must be assigned to at least one fuzzy cluster $A^l_{(\alpha)},\ l = 1, \ldots, c,\ c \leq n$ with the membership degree higher than zero. The condition $2 \leq c \leq n$ requires that the number of fuzzy clusters in $R^\alpha_z(X)$ must be more than two. Otherwise, the unique fuzzy cluster will contain all objects, possibly with different positive membership degrees.

Obviously, the definition of the allotment among fuzzy clusters (11) is similar to the definition of the possibilistic partition (3). So, the allotment among fuzzy clusters can be considered as the possibilistic partition and fuzzy clusters in the sense of definition 2.5 are elements of the possibilistic partition. However, the concept of allotment will be used in further considerations. The concept of allotment is the central point of the method. But the next concept introduced should be paid attention to, as well.

Definition 2.8 *Allotment* $R^\alpha_I(X) = \{A^l_{(\alpha)} | l = \overline{1, n},\ \alpha \in (0, 1]\}$ *of the set of objects among* n *fuzzy clusters for some tolerance threshold* $\alpha,\ \alpha \in (0, 1]$ *is the initial allotment of the set* $X = \{x_1, ..., x_n\}$.

In other words, if initial data are represented by a matrix of some fuzzy T then lines or columns of the matrix are fuzzy sets $A^l \subseteq X,\ l = \overline{1,n}$ and level fuzzy sets $A^l_{(\alpha)},\ l = \overline{1,n},\ \alpha \in (0,1]$ are fuzzy clusters. These fuzzy clusters constitute an initial allotment for some tolerance threshold and they can be considered as clustering components.

Thus, the problem of fuzzy cluster analysis can be defined in general as the problem of discovering the unique allotment $R^*(X)$, resulting from the classification process, which corresponds to either most natural allocation of objects among fuzzy clusters or to the researcher's opinion about classification. In the first case, the number of fuzzy clusters c is not fixed. In the second case, the researcher's opinion determines the kind of the allotment sought and the number of fuzzy clusters c can be fixed.

If some allotment $R^\alpha_z(X) = \{A^l_{(\alpha)} | l = \overline{1,c},\ c \le n,\ \alpha \in (0,1]\}$ corresponds to the formulation of a concrete problem, then this allotment is an adequate allotment. In particular, if condition

$$\bigcup_{l=1}^{c} A^l_\alpha = X, \tag{12}$$

and condition

$$card(A^l_\alpha \cap A^m_\alpha) = 0,\ \forall A^l_{(\alpha)}, A^m_{(\alpha)},\ l \ne m,\ \alpha \in (0,1] \tag{13}$$

are met for all fuzzy clusters $A^l_{(\alpha)},\ l = \overline{1,c}$ of some allotment $R^\alpha_z(X) = \{A^l_{(\alpha)} | l = \overline{1,c},\ c \le n,\ \alpha \in (0,1]\}$ then the allotment is the allotment among fully separate fuzzy clusters.

However, fuzzy clusters in the sense of Definition 2.5 can have an intersection area. This fact was demonstrated by Viattchenin [20]. If the intersection area of any pair of different fuzzy cluster is an empty set, then the condition (13) is met and fuzzy clusters are called fully separate fuzzy clusters. Otherwise, fuzzy clusters are called particularly separate fuzzy clusters and $w = \{0, \dots, n\}$ is the maximum number of elements in the intersection area of different fuzzy clusters. Obviously, for $w = 0$ fuzzy clusters are fully separate fuzzy clusters. So, the conditions (12) and (13) can be generalized for a case of particularly separate fuzzy clusters. Condition

$$\sum_{l=1}^{c} card(A^l_\alpha) \ge card(X),\ \forall A^l_{(\alpha)} \in R^\alpha_z(X),\ \alpha \in (0,1],\ card(R^\alpha_z(X)) = c, \tag{14}$$

and condition

$$card(A^l_\alpha \cap A^m_\alpha) \le w,\ \forall A^l_{(\alpha)}, A^m_{(\alpha)},\ l \ne m,\ \alpha \in (0,1], \tag{15}$$

are generalizations of conditions (12) and (13). Obviously, if $w = 0$ in conditions (14) and (15) then conditions (12) and (13) are met.

The adequate allotment $R^\alpha_z(X)$ for some value of tolerance threshold $\alpha,\ \alpha \in (0,1]$ is a family of fuzzy clusters which are elements of the initial allotment $R^\alpha_I(X)$ for the value of α and the family of fuzzy clusters should satisfy either the conditions (12) and (13) or the conditions (14) and (15). So, the construction of adequate allotments $R^\alpha_z(X) = \{A^l_{(\alpha)} | l = \overline{1,c},\ c \le n,\ \alpha \in (0,1]\}$ for every $\alpha,\ \alpha \in (0,1]$ is a trivial problem of combinatorics.

Several adequate allotments can exist. Thus, the problem consists in the selection of the unique adequate allotment $R^*(X)$ from the set B of adequate allotments, $B = \{R^\alpha_z(X)\}$, which is the class of possible solutions of the concrete classification problem and $B = \{R^\alpha_z(X)\}$ depends on the parameters the classification problem. The selection of the unique adequate allotment $R^*(X)$ from the set $B = \{R^\alpha_z(X)\}$ of adequate allotments must be made on the basis of evaluation of allotments. The criterion

$$F_1(R^\alpha_z(X), \alpha) = \sum_{l=1}^{c} \frac{1}{n_l} \sum_{i=1}^{n_l} \mu_{li} - \alpha \cdot c, \tag{16}$$

where c is the number of fuzzy clusters in the allotment $R^\alpha_z(X)$ and $n_l = card(A^l_\alpha),\ A^l_{(\alpha)} \in R^\alpha_z(X)$ is the number of elements in the support of the fuzzy cluster $A^l_{(\alpha)}$, can be used for evaluation of allotments. The criterion

$$F_2(R^\alpha_z(X), \alpha) = \sum_{l=1}^{c} \sum_{i=1}^{n_l} (\mu_{li} - \alpha), \tag{17}$$

can also be used for evaluation of allotments. Both criteria were proposed by Viattchenin [16].

Maximum of criterion (16) or criterion (17) corresponds to the best allotment of objects among c fuzzy clusters. So, the classification problem can be characterized formally as determination of the solution $R^*(X)$ satisfying

$$R^*(X) = \arg \max_{R_z^\alpha(X) \in B} F(R_z^\alpha(X), \alpha), \tag{18}$$

where $B = \{R_z^\alpha(X)\}$ is the set of adequate allotments corresponding to the formulation of a concrete classification problem and criteria (16) and (17) are denoted by $F(R_z^\alpha(X), \alpha)$.

The criterion (16) can be considered as the average total membership of objects in fuzzy clusters of the allotment $R_z^\alpha(X)$ minus $\alpha \cdot c$. The quantity $\alpha \cdot c$ regularizes with respect to the number of clusters c in the allotment $R_z^\alpha(X)$. The criterion (17) can be considered as the total membership of objects in fuzzy clusters of the allotment $R_z^\alpha(X)$ with an appreciation through the value α of tolerance threshold.

The condition (18) must be met for the some unique allotment $R_z^\alpha(X) \in B(c)$. Otherwise, the number c of fuzzy clusters in the allotment sought $R^*(X)$ is suboptimal. The condition was formulated by Viattchenin [19].

Detection of fixed c number of fuzzy clusters can be considered as the aim of classification. So, the adequate allotment $R_z^\alpha(X)$ is any allotment among c fuzzy clusters in the case. There is a seven-step procedure of classification:

1. Calculate α-level values of the fuzzy tolerance T and construct the sequence $0 < \alpha_0 < \alpha_1 < \ldots < \alpha_\ell < \ldots < \alpha_Z \leq 1$ of α-levels; let $\ell := 1$;
2. Construct the initial allotment $R_I^\alpha(X) = \{A_{(\alpha)}^l | l = \overline{1, n}\}$, $\alpha = \alpha_\ell$ for every value α_ℓ from the sequence $0 < \alpha_0 < \alpha_1 < \ldots < \alpha_\ell < \ldots < \alpha_Z \leq 1$;
3. Let $w := 0$;
4. Construct allotments $R_z^\alpha(X) = \{A_{(\alpha)}^l | l = \overline{1, c}, c \leq n\}$, $\alpha = \alpha_\ell$, which satisfy conditions (14) and (15) for every value α_ℓ from the sequence $0 < \alpha_0 < \alpha_1 < \ldots < \alpha_\ell < \ldots < \alpha_Z \leq 1$;
5. Construct the class of possible solutions of the classification problem $B(c) = \{R_z^\alpha(X)\}$, $\alpha \in \{\alpha_1, \ldots, \alpha_Z\}$ for the given number of fuzzy clusters c and different values of the tolerance threshold α, $\alpha \in \{\alpha_1, \ldots, \alpha_Z\}$ as follows:
   if for some allotment $R_z^\alpha(X)$, $\alpha \in \{\alpha_1, \ldots, \alpha_Z\}$ the condition $card(R_z^\alpha(X)) = c$ is met
   then $R_z^\alpha(X) \in B(c)$
   else let $w := w + 1$ and go to step 4;
6. Calculate the value of some criterion $F(R_z^\alpha(X), \alpha)$ for every allotment $R_z^\alpha(X) \in B(c)$;
7. The result $R^*(X)$ of classification is formed as follows:
   if for some unique allotment $R_z^\alpha(X)$ from the set $B(c)$ the condition (18) is met
   then the allotment is the result of classification
   else the number c of classes is suboptimal.

The allotment $R_z^\alpha(X) = \{A_{(\alpha)}^l | l = \overline{1, c},\ \alpha \in (0, 1]\}$ among the given number of fuzzy clusters and the corresponding value of tolerance threshold α, $\alpha \in (0, 1]$ are results of classification.

2.2 A Technique of the Three-Way Data Preprocessing

The problem of clustering of three-way data can be formulated as follows. Let $X = \{x_1, ..., x_n\}$ is set of objects, where objects are indexed i, $i = 1, \ldots, n$; each object x_i is described by m_1 attributes, indexed t^1, $t^1 = 1, \ldots, m_1$, so that an object x_i can be represented by vector $x_i = (x_i^1, \ldots, x_i^{t_1}, \ldots, x_i^{m_1})$; every attribute x^{t_1}, $t_1 = 1, \ldots, m_1$ can be characterized by m_2 values of 2-ary attributes, so that $x_i^{t_1} = (x_i^{t_1(1)}, \ldots, x_i^{t_1(t_2)}, \ldots, x_i^{t_1(m_2)})$. So, the three-way data can be presented by a poly-matrix as follows

$$X_{n \times m_1 \times m_2} = [x_i^{t_1(t_2)}],\ i = 1, \ldots, n;\ t_1 = 1, \ldots, m_1,\ t_2 = 1, \ldots, m_2. \tag{19}$$

In other words, the three-way data are the data, which are observed by the values of m_1 attributes with respect to n objects for m_2 situations. The purpose of the clustering is to classify the set $X = \{x_1, ..., x_n\}$ into c fuzzy clusters. So, an allotment $R^*(X)$ among c fuzzy clusters $A_{(\alpha)}^1, ..., A_{(\alpha)}^n$, $\alpha \in (0, 1]$ must be detected.

The D-AFC(c)-algorithm can be applied directly to the data given as a matrix of tolerance coefficients. This means that it can be used with the objects by attributes data by choosing a suitable metric to measure similarity. So, the main problem is the problem of the three-way data preprocessing. First of all a method for the traditional two-way data preprocessing must be considered.

The matrix of fuzzy tolerance $T = [\mu_T(x_i, x_j)],\ i, j = 1, \ldots, n$ is the matrix of initial data for the D-AFC(c)-algorithm. However, the data can be presented as a matrix of attributes $\hat{X}_{n\times m_1} = [\hat{x}_i^{t_1}],\ i = 1, \ldots, n,\ t_1 = 1, \ldots, m_1$, where the value $\hat{x}_i^{t_1}$ is the value of the t_1th attribute for ith object. In the first place, the data can be normalized as follows

$$x_i^{t_1} = \frac{\hat{x}_i^{t_1}}{\max\limits_i \hat{x}_i^{t_1}}. \tag{20}$$

In the second place, the data can be normalized using a formula

$$x_i^{t_1} = \frac{\hat{x}_i^{t_1} - \min\limits_i \hat{x}_i^{t_1}}{\max\limits_i \hat{x}_i^{t_1} - \min\limits_i \hat{x}_i^{t_1}}. \tag{21}$$

So, each object can be considered as a fuzzy set $x_i,\ i = 1, \ldots, n$ and $x_i^{t_1} = \mu_{x_i}(x^{t_1}) \in [0, 1],\ i = 1, \ldots, n,\ t_1 = 1, \ldots, m_1$ are their membership functions.

The matrix of coefficients of pair wise dissimilarity between objects $I = [\mu_I(x_i, x_j)],\ i, j = 1, \ldots, n$ can be obtained after application of some distance to the matrix of normalized data $X_{n\times m_1} = [\mu_{x_i}(x^{t_1})],\ i = 1, \ldots, n;\ t_1 = 1, \ldots, m_1$. The most widely used distances for fuzzy sets $x_i, x_j,\ i, j = 1, \ldots, n$ in $X = \{x_1, ..., x_n\}$ are:

The normalized Hamming distance

$$l(x_i, x_j) = \frac{1}{m_1}\sum_{t_1=1}^{m_1} |\mu_{x_i}(x^{t_1}) - \mu_{x_j}(x^{t_1})|,\ i, j = 1, \ldots, n. \tag{22}$$

The normalized Euclidean distance

$$e(x_i, x_j) = \sqrt{\frac{1}{m_1}\sum_{t_1=1}^{m_1} \left(\mu_{x_i}(x^{t_1}) - \mu_{x_j}(x^{t_1})\right)^2},\ i, j = 1, \ldots, n. \tag{23}$$

The squared normalized Euclidean distance

$$\varepsilon(x_i, x_j) = \frac{1}{m_1}\sum_{t_1=1}^{m_1} \left(\mu_{x_i}(x^{t_1}) - \mu_{x_j}(x^{t_1})\right)^2,\ i, j = 1, \ldots, n. \tag{24}$$

These distances were considered by Kaufmann [6] in detail. The matrix of fuzzy tolerance $T = [\mu_T(x_i, x_j)]$, $i, j = 1, \ldots, n$ can be obtained after application of complement operation

$$\mu_T(x_i, x_j) = 1 - \mu_I(x_i, x_j),\ \forall i, j = 1, \ldots, n \tag{25}$$

to the matrix of fuzzy intolerance $I = [\mu_I(x_i, x_j)],\ i, j = 1, \ldots, n$.

The three-way data can be normalized. For the purpose, (20) can be generalized as follows

$$x_i^{t_1(t_2)} = \frac{\hat{x}_i^{t_1(t_2)}}{\max\limits_{i,t_2} \hat{x}_i^{t_1(t_2)}}. \tag{26}$$

On the other hand, the three-way data can be normalized using a generalization of the formula (21), which can be written as follows

$$x_i^{t_1(t_2)} = \frac{\hat{x}_i^{t_1(t_2)} - \min\limits_{i,t_2} \hat{x}_i^{t_1(t_2)}}{\max\limits_{i,t_2} \hat{x}_i^{t_1(t_2)} - \min\limits_{i,t_2} \hat{x}_i^{t_1(t_2)}}. \tag{27}$$

So, each object $x_i, i = 1, \ldots, n$ from the initial set $X = \{x_1, \ldots, x_n\}$ can be considered as a type-two fuzzy set and $x_i^{t_1(t_2)} = \mu_{x_i}(x^{t_1(t_2)}),\ i = 1, \ldots, n\ ;\ t_1 = 1, \ldots, m_1,\ t_2 = 1, \ldots, m_2,\ x^{t_1(t_2)} = \mu_{t_1}(x^{t_2}) \in [0, 1]$, $t_1 = 1, \ldots, m_1,\ t_2 = 1, \ldots, m_2$ are its membership functions.

The concept of a type-two fuzzy set was introduced by Zadeh [31] as an extension of the concept of an ordinary fuzzy set, which was called type-one fuzzy set. The advances of type-two fuzzy sets for pattern recognition were considered by Zeng and Liu [33].

In the case of three-way data each object $x_i,\ i = 1, \ldots, n$ can be presented as a matrix $X_{(i)m_1 \times m_2} = [x_i^{t_1(t_2)}],\ t_1 = 1, \ldots, m_1,\ t_2 = 1, \ldots, m_2$. Dissimilarity coefficients between the objects can be constructed on a basis of generalizations of distances (22) – (24) between fuzzy sets and these generalizations must be taken into account dissimilarities between objects attributes as well as attributes situations.

So, generalizations of the distances for fuzzy sets are functions of dissimilarities and the functions can be written as follows:

A generalization of the normalized Hamming distance for the three-way data is described by the expression

$$l_{G_2}(x_i, x_j) = \frac{1}{m_1} \sum_{t_1=1}^{m_1} \left(\frac{1}{m_2^2} \sum_{u_1, v_1=1}^{m_2} |\mu_{x_i}(x^{t_1,u_1}) - \mu_{x_j}(x^{t_1,v_1})| \right),\ i, j = 1, \ldots, n. \tag{28}$$

A generalization of the normalized Euclidean distance for the three-way data is described by the expression

$$e_{G_2}(x_i, x_j) = \sqrt{\frac{1}{m_1} \sum_{t_1=1}^{m_1} \left(\frac{1}{m_2^2} \sum_{u_1, v_1=1}^{m_2} \left(\mu_{x_i}(x^{t_1,u_1}) - \mu_{x_j}(x^{t_1,v_1})\right)^2 \right)},\ i, j = 1, \ldots, n. \tag{29}$$

A generalization of the squared normalized Euclidean distance for the three-way data is described by the expression

$$\varepsilon_{G_2}(x_i, x_j) = \frac{1}{m_1} \sum_{t_1=1}^{m_1} \left(\frac{1}{m_2^2} \sum_{u_1, v_1=1}^{m_2} \left(\mu_{x_i}(x^{t_1,u_1}) - \mu_{x_j}(x^{t_1,v_1})\right)^2 \right),\ i, j = 1, \ldots, n. \tag{30}$$

These functions of dissimilarities were proposed by Viattchenin [21, 22]. Obviously, for $m_2 = 1$ a usual distance for fuzzy sets will be obtained in every case. A matrix of a feeble fuzzy tolerance T_1 will be obtained after an application of the formulae (28) – (30) and the complement operation (25) to the three-way data. The fact was demonstrated by Viattchenin [21].

However, a value m_2 can be different for different attributes $\hat{x}^{t_1},\ t_1 \in \{1, \ldots, m_1\}$, or a value m_2 of grades for a fixed attribute $\hat{x}^{t_1},\ t_1 \in \{1, \ldots, m_1\}$ can be different for different objects $x_i,\ i \in \{1, \ldots, n\}$. So, each object $x_i,\ i = 1, \ldots, n$ cannot be presented as a matrix $X_{(i)m_1 \times m_2} = [x_i^{t_1(t_2)}],\ t_1 = 1, \ldots, m_1,\ t_2 = 1, \ldots, m_2$, because a value m_2, which is general for all attributes $\hat{x}^{t_1},\ t_1 \in \{1, \ldots, m_1\}$, must be established. In these cases a general value m_2 can be defined as follows:

$$m_2 = \max_{t_1} m_2^{(t_1)},\ t_1 = 1, \ldots, m_1, \tag{31}$$

where the number of grades of every attribute $\hat{x}^{t_1},\ t_1 \in \{1, \ldots, m_1\}$ is denoted by $m_2^{(t_1)}$. However, values $x_i^{t_1(t_2)},\ i \in \{1, \ldots, n\}$ may not be known for some objects $x_i \in X,\ i \in \{1, \ldots, n\}$. In such a case, the unknown values $x_i^{t_1(t_2)},\ i \in \{1, \ldots, n\}$ can be defined as follows

$$x_i^{t_1(t_2)} = \max_{t_1} t_2^{(t_1)},\ i \in \{1, \ldots, n\},\ t_2 = 1, \ldots, m_2^{(t_1)}. \tag{32}$$

Obviously, the method of the three-way data preprocessing can be very simply generalized for the case of the p-way data.

3 The Experimental Results

The Sato's three-way data are described in the first subsection. Illustrative examples of the data preprocessing are considered also in the first subsection. The second subsection includes results of numerical experiments for three functions of distances.

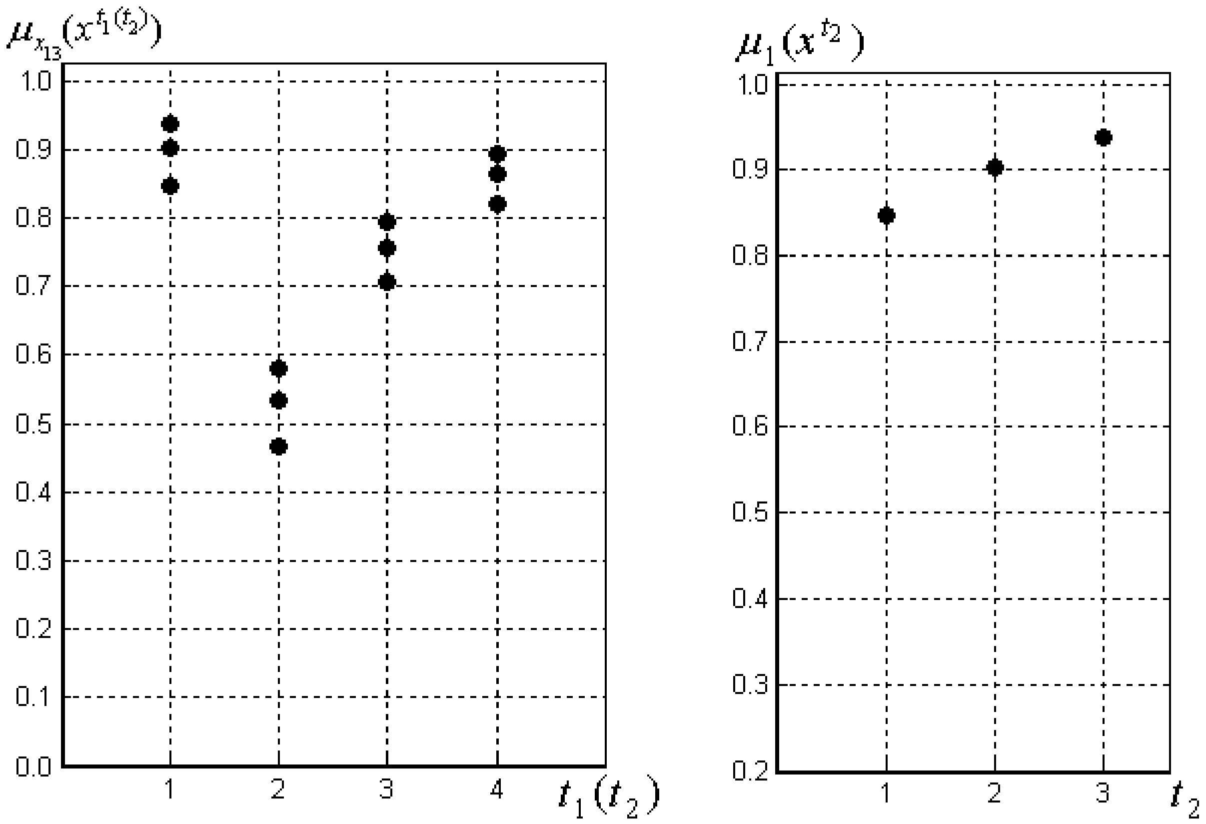
3.1 The Sato's Three-Way Data

The Sato's artificial three-way data are a follow-up of the survey of physical constitution, involving height, weight, chest girth and sitting height, which are the measurements of 38 boys at three instants, that is, when subjects are 13, 14 and 15 years old. These data originally appear in Sato and Sato [10]. The original data can be rewritten and the rewritten data are presented in Table 1.

Denote height by $\hat{x}^1$, weight by $\hat{x}^2$, chest girth by $\hat{x}^3$ and sitting height by $\hat{x}^4$. So, every attribute $\hat{x}^{t_1}$, $t_1 = 1, \ldots, 4$ is observed at three instants $t_2 = 1, \ldots, 3$. A value of the t_1th attribute in the t_2th moment for the ith object will be denoted by $\hat{x}_i^{t_1(t_2)}$, $i = 1, \ldots, 38$, $t_1 = 1, \ldots, 4$, $t_2 = 1, \ldots, 3$. The methodology of the three-way data preprocessing can be applied directly to the data.

The data can be normalized using formula (26) or formula (27). For example, the thirteenth object after normalization (26) will be presented as a matrix $X_{4\times 3} = [x_{13}^{t_1(t_2)}]$, $t_1 = 1, \ldots, 4$, $t_2 = 1, \ldots, 3$, shown in Table 2.

The matrix can be presented as a membership function of the type-two fuzzy set on the attributes and every attribute can be described by type-one fuzzy set. The membership function of a type-two fuzzy set which describes the thirteenth object and the membership function of a type-one fuzzy set which describes the first attribute of the thirteenth object are presented in Figure 1.



a) b)

Figure 1: (a) a membership function of the type-two fuzzy set which describes the thirteenth object after normalization using the formula (26) and (b) a membership function of the type-one fuzzy set describing the first attribute of the thirteenth object after normalization using the formula (26) at three time instants.

The thirteenth object after normalization (27) can be presented as a matrix $X_{4\times 3} = [x_{13}^{t_1(t_2)}]$, $t_1 = 1, \ldots, 4$, $t_2 = 1, \ldots, 3$, shown in Table 3. So, the membership function of a type-two fuzzy set which describes the thirteenth object and the membership function of a type-one fuzzy set which describes the first attribute of the thirteenth object are presented in Figure 2.

Table 1: Physical constitution of 38 boys

Boys	Height, cm			Weight, kg			Chest girth, cm			Sitting height, cm		
	13 years old	14 years old	15 years old	13 years old	14 years old	15 years old	13 years old	14 years old	15 years old	13 years old	14 years old	15 years old
1	147	157	162	40	47	54	70	76	81	80	85	87
2	161	166	167	49	50	52	75	75	79	85	87	88
3	153	159	161	45	48	51	72	75	75	86	90	92
4	155	163	168	51	58	66	77	82	87	85	87	92
5	160	165	167	51	56	61	75	77	82	86	88	89
6	153	159	167	38	43	44	67	70	71	81	84	87
7	166	169	172	67	72	79	86	89	92	89	90	95
8	168	174	175	55	60	65	76	79	81	91	93	95
9	142	149	157	35	39	46	69	68	75	75	78	82
10	151	160	165	44	51	57	72	78	80	79	85	89
11	164	167	169	55	58	65	77	79	80	88	89	93
12	153	163	168	42	46	53	70	73	78	83	88	91
13	148	158	164	41	47	51	72	77	81	78	82	85
14	164	169	171	75	84	88	92	97	102	90	93	95
15	145	151	162	34	39	45	65	68	72	76	80	84
16	151	159	162	51	57	64	80	83	87	81	85	87
17	145	153	162	50	55	59	82	84	82	79	81	86
18	154	163	169	47	53	56	71	75	80	82	86	89
19	156	166	171	48	50	56	73	72	75	81	86	89
20	144	149	157	30	33	37	60	62	66	73	75	79
21	154	164	169	41	49	56	69	76	77	82	88	91
22	155	165	169	43	52	57	71	75	79	82	87	90
23	155	162	166	48	58	60	76	85	84	82	86	89
24	155	162	172	49	55	57	73	76	76	80	84	87
25	156	163	164	48	53	54	76	79	82	81	86	87
26	156	164	172	50	53	56	74	76	79	81	84	87
27	162	168	170	45	48	52	71	71	75	84	88	89
28	147	154	163	37	43	50	71	75	80	79	82	86
29	149	157	166	40	47	53	71	79	78	80	83	87
30	148	155	162	37	41	47	69	70	74	78	81	85
31	156	163	166	52	57	62	75	79	81	83	87	89
32	141	151	159	35	42	48	68	74	79	73	77	82
33	140	147	157	30	34	43	67	70	73	76	77	83
34	146	153	161	49	52	53	76	78	76	80	79	84
35	162	168	161	53	58	53	74	78	76	86	79	84
36	146	158	165	36	44	51	68	75	73	77	85	89
37	141	151	158	41	46	51	71	75	76	76	80	83
38	158	167	171	65	71	79	93	93	90	85	90	91

Table 2: A description of an object as a matrix after normalization using the formula (26)

A number i of an object	A number of the situation t_2 of an attribute $x^{t_1(t_2)}$	Attributes $x^{t_1(t_2)}$ of the object x_i			
		$x^{1(t_2)}$	$x^{2(t_2)}$	$x^{3(t_2)}$	$x^{4(t_2)}$
13	1	0.8457	0.4659	0.7059	0.8211
	2	0.9029	0.5341	0.7549	0.8632
	3	0.9371	0.5795	0.7941	0.8947

Table 3: A description of an object as a matrix after normalization using the formula (27)

A number i of an object	A number of the situation t_2 of an attribute $x^{t_1(t_2)}$	Attributes $x^{t_1(t_2)}$ of the object x_i			
		$x^{1(t_2)}$	$x^{2(t_2)}$	$x^{3(t_2)}$	$x^{4(t_2)}$
13	1	0.2286	0.1897	0.2857	0.2273
	2	0.5143	0.2931	0.4048	0.4091
	3	0.6857	0.3621	0.5000	0.5455

So, a membership function of the type-two fuzzy set which is describes an object $x_i \in X,\ i \in \{1, \dots, 38\}$ depends on the method of normalization of the data. The fact is very important for the classification result.

3.2 The Results of Classification

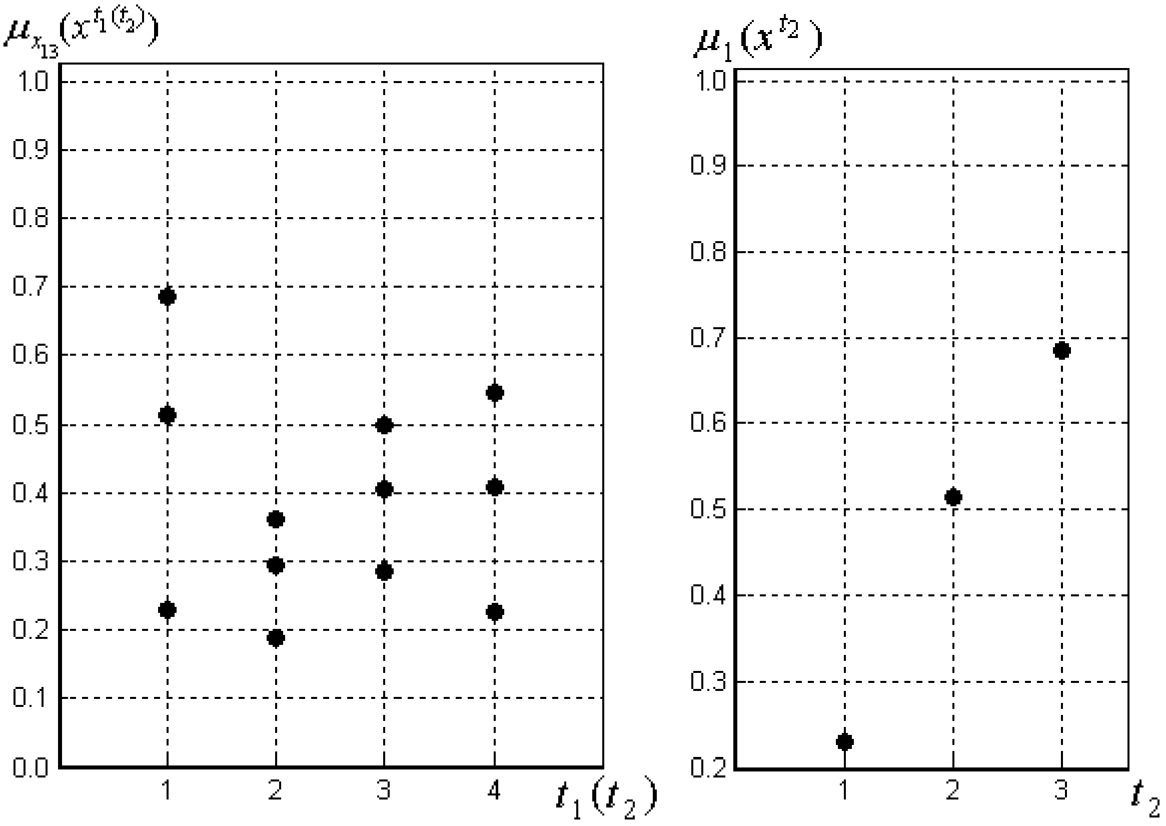
Let us consider the classification result which was presented by Sato and Sato [10]. A fuzzy partition is the result of application of their method to the three-way data. Membership functions of four classes of the fuzzy partition are presented in Figure 3.

Membership values of the first class are represented by $+$, membership values of the second class are represented by $\blacksquare$, membership values of the third class are represented by $\times$, and membership values of the fourth class are represented by $\circ$. The number of classes was determined so as to get the most reasonable interpretations of fuzzy clusters. The boys from the first class have a good constitution through all the years. The boys from the second class have a poor constitution, with a tendency of growing between 13 and 14 years of age. The boys from the third class and from the fourth class have the standard constitution, but there is a difference in the pattern of growth, namely the third class has the tendency of growing in both height and weight through all the years. On the other hand, the fourth class contains the boys who grow through all the years but not so conspicuously. Moreover, the boys who belong to the third class and the fourth class simultaneously have the tendency to growth in height from 13 to 14 years of age, and to the growth in weight from 14 to 15 years of age. So, the reasonable classes of physical constitution were found by taking into account the growth pattern.

For comparison, the D-AFC(c)-algorithm was applied to the data for all three kinds of proposed functions of distance. Formula (27) was used for data normalization in every experiment. Let us consider results of experiments.

By executing the D-AFC(c)-algorithm for four classes using the formula (28) in the process of the data preprocessing, we obtain that the first class is formed by 3 elements, the second class is composed of 4 elements, the third class is formed by 28 elements and the fourth class includes 6 elements. The second element belongs to the third class and to the fourth class, the seventh element belongs to the first class and to the fourth class, and the ninth element belongs to the second class and to the third class. The allotment $R^*(X)$, which corresponds to the result, was obtained for the tolerance threshold $\alpha = 0.810500$.

The value of the membership function of the fuzzy cluster, which corresponds to the first class is maximal for the thirty-eighth object and is equal 0.8937. So, the thirty-eighth object is the typical point of the fuzzy



a) b)

Figure 2: (a) a membership function of the type-two fuzzy set which describes the thirteenth object after normalization using the formula (27) and (b) a membership function of the type-one fuzzy set describing the first attribute of the thirteenth object after normalization using the formula (27) at three time instants.

cluster which corresponds to the first class. The membership value of the twentieth object is equal 0.8991 and the value is maximal for the fuzzy cluster which corresponds to the second class. Thus, the twentieth object is the typical point of the fuzzy cluster which corresponds to the second class. The membership function of the third fuzzy cluster is maximal for the thirteenth object and is equal 0.8709. That is why the thirteenth object is the typical point of the fuzzy cluster which corresponds to the third class. The membership function of the fourth fuzzy cluster is maximal for the eighth object and is equal 0.9252. That is why the eighth object is the typical point of the fourth fuzzy cluster.

Membership functions of four classes of the allotment are presented in Figure 4 and values which equal zero are not shown in the figure.

By executing the D-AFC(c)-algorithm for four classes using the formula (29) in the process of the data preprocessing, we obtain that the first class is formed by 3 elements, the second class is composed of 5 elements, the third class is formed by 28 elements and the fourth class includes 7 elements. The second and the fifth elements belong to the third class and to the fourth class, the seventh element belongs to the first class and to the fourth class, the ninth and the fifth elements belong to the second class and to the third class. The allotment $R^*(X)$, which is corresponds to the result, was obtained for the tolerance threshold $\alpha = 0.755300$.

The fourteenth object is the typical point of the first fuzzy cluster, the twentieth object is the typical point of the fuzzy cluster which corresponds to the second class, the thirteenth object is the typical point of the fuzzy cluster which corresponds to the third class, and the eighth object is the typical point of the fourth fuzzy cluster.

Membership functions of four classes of the allotment are presented in Figure 5.

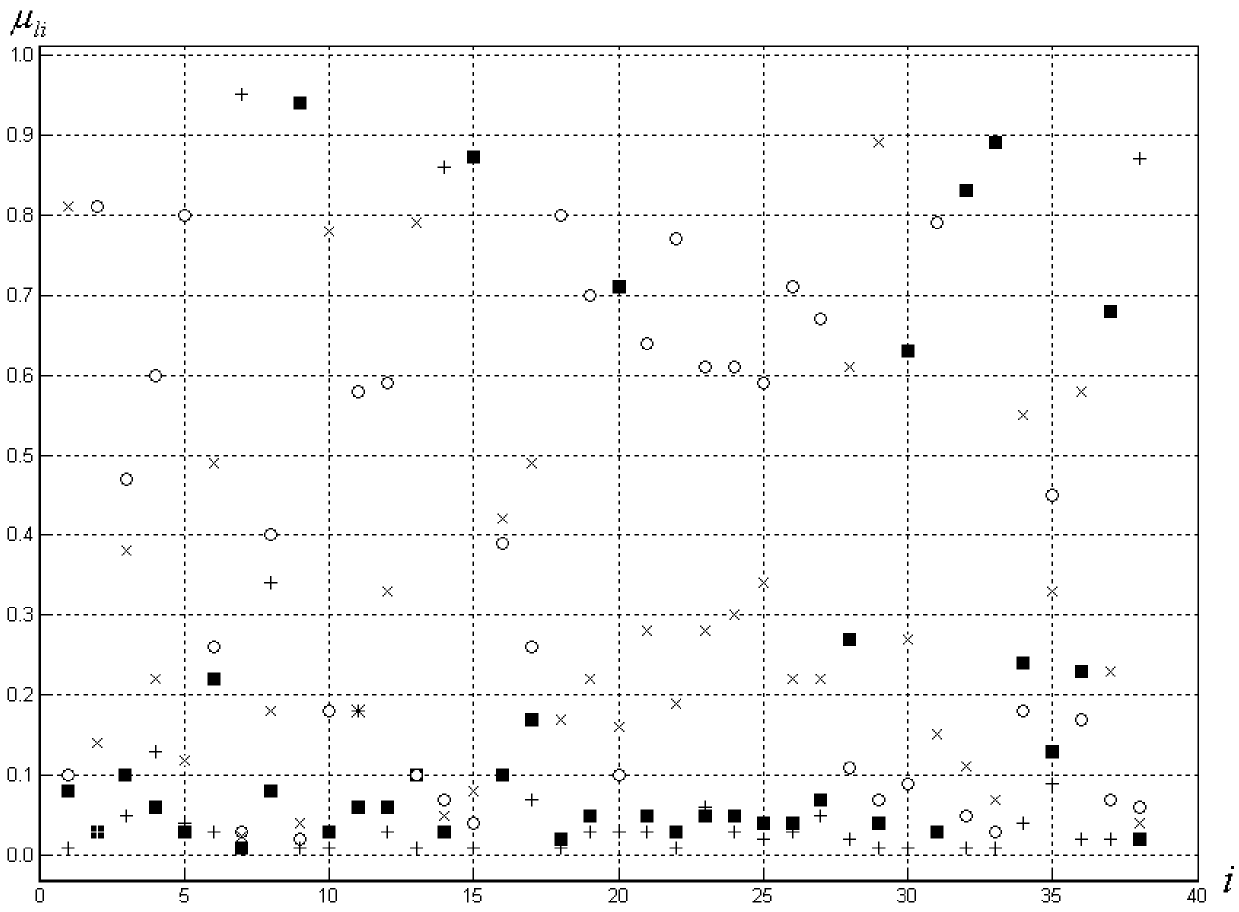


Figure 3: Membership function of four classes obtained from the Sato's classification method

Object assignments due to application of the D-AFC(c)-algorithm for four classes using the formula (30), we similar to the assignments, which were obtained using the formula (29) in the process of the data preprocessing. The allotment $R^*(X)$, which corresponds to the result in the case of using of the formula (30), was obtained for the tolerance threshold $\alpha = 0.940100$.

So, the results, which are obtained from the D-AFC(c)-algorithm using the proposed method of the three-way data preprocessing, are similar to the results, which were obtained by Sato and Sato [10] using their multicriteria optimization method. Moreover, the membership function from the proposed method is sharper than the membership function from the Sato's method.

4 Final Remarks

Preliminary conclusions are discussed in the first subsection of the section. The second subsection deals with the perspectives on future investigations.

4.1 Discussions

The results of application of the fuzzy clustering method based on the allotment concept can be very well interpreted. Moreover, the fuzzy clustering method based on the allotment concept depends on the set of adequate allotments only. That is why the clustering results are stable.

The methodology of fuzzy clustering of three-way data is outlined in the paper. The approach is based on the concept of the feeble fuzzy tolerance which is represents the structure of the three-way data. For construction the feeble tolerance matrix, the three-way data can be normalized and every object can be

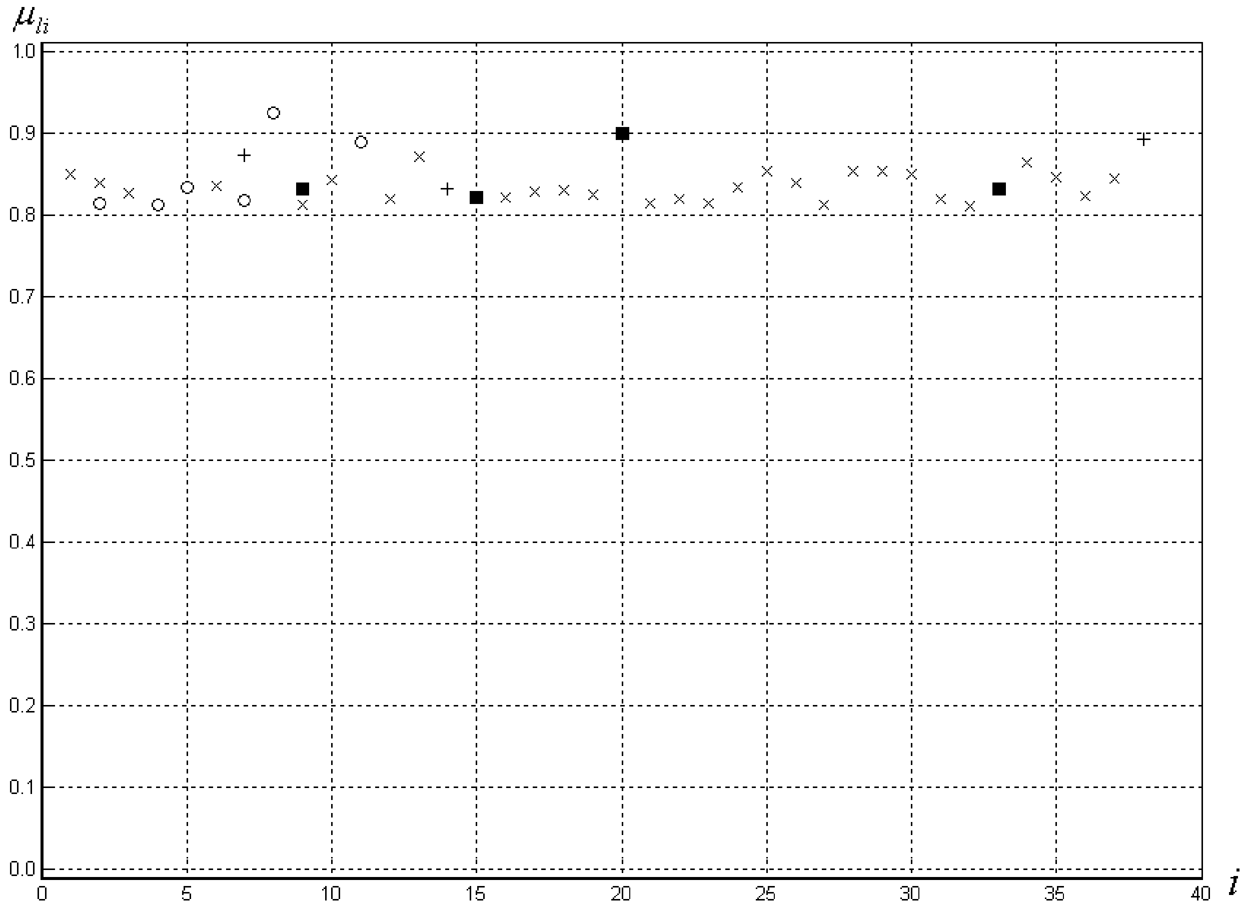


Figure 4: Membership functions of four classes obtained from the D-AFC(c)-algorithm using the formula (28).

represented as the type-two fuzzy set. A matrix of dissimilarity coefficients between type-two fuzzy set can be constructed using the proposed functions of dissimilarities. The results of application of the proposed methodology of the three-way data preprocessing and its processing by the D-AFC(c)-algorithm to the Sato's three-way data show that the methodology and the D-AFC(c)-algorithm are a precise and effective technique for the three-way data possibilistic clustering.

It is should be noted, that the proposed approach is more general and simple, than of the method of fuzzy clustering of Sato and Sato [10], because the latter method is more complex, than the here proposed method.

4.2 Perspectives

In the first place, the D-AFC(c)-algorithm is the basic version of the clustering procedure. Other parameters of a clustering procedure were introduced by Viattchenin [20]. Moreover, a heuristic for the detection of an unknown number of fuzzy clusters in the sought allotment was also proposed by Viattchenin [23]. So, the corresponding versions of the algorithm can be developed.

In the second place, the described approach of the three-way data can be generalized for a case of the multi-way data very simply. Moreover, the values of the grades of attributes of the objects can be represented by fuzzy numbers. So, a combination of the proposed approach with the method of Yang and Ko [27] of calculation of a distance between fuzzy numbers can be elaborated.

These perspectives for investigations are of great interest both from the theoretical point of view and from the practical one as well.

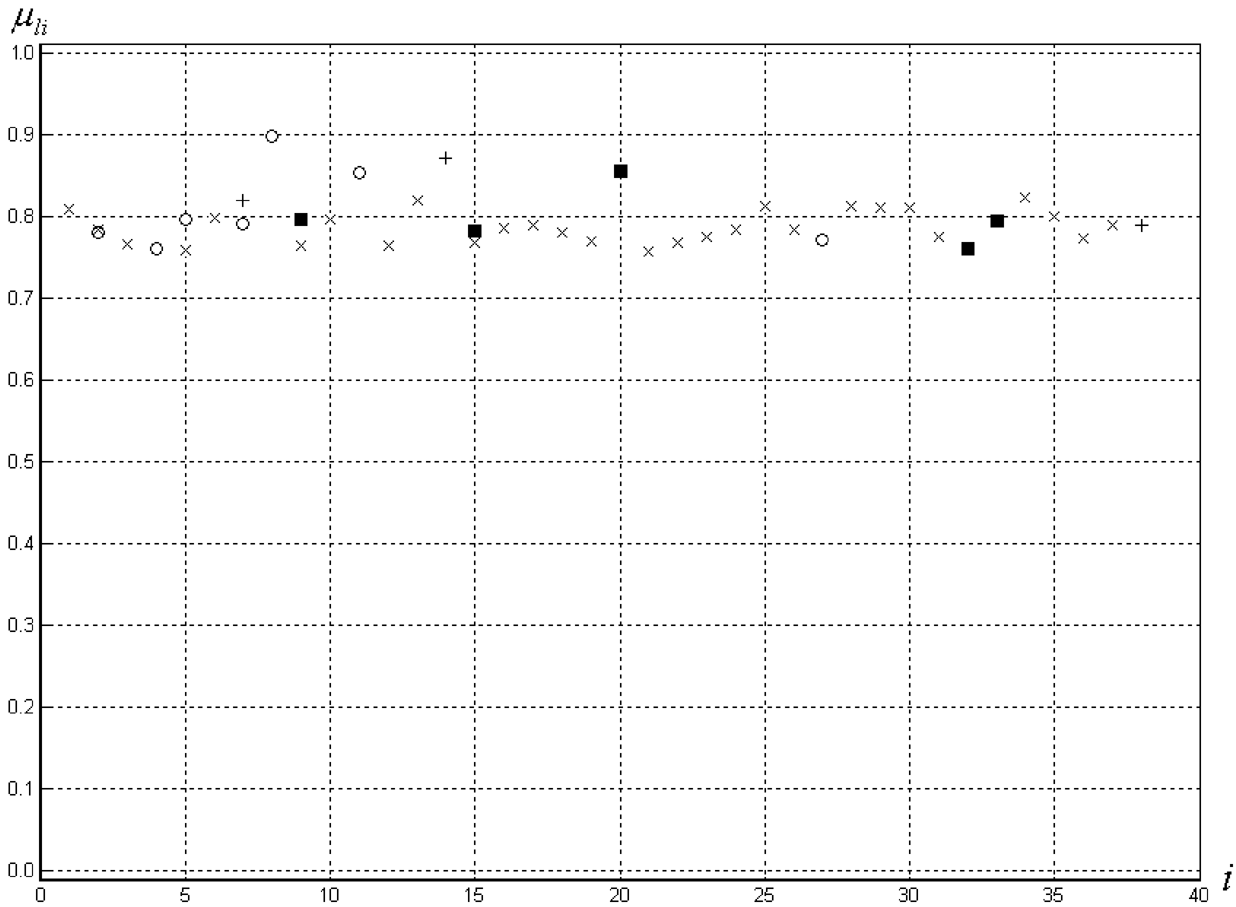


Figure 5: Membership functions of four classes obtained from the D-AFC(c)-algorithm using the formula (29).

Acknowledgements

I am grateful to Dr. Jan W. Owsinski and Prof. Janusz Kacprzyk for their useful remarks during the paper preparation. I would like to thank the Director of the Systems Research Institute of the Polish Academy of Sciences for the possibility of conducting the investigations and the Mianowski Fund and Siemens for financial support. I am grateful to Mr. Aliaksandr Damaratski and Mr. Valentin Ganchenko for elaborating the experimental software. I also thank the anonymous referees for their valuable comments.

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