Journal of Uncertain Systems
Vol.3, No.1, pp.52-63, 2009
Online at: www.jus.org.uk

Itô Type Set-Valued Stochastic Differential Equation*

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Received 20 October 2008; Accepted 16 November 2008

Abstract

In this paper, we firstly illustrate why we should introduce the Itô type set-valued stochastic differential equation and then recall some basic results about the Lebesgue integral of a set-valued stochastic process with respect to time t. Secondly we obtain some new properties of the set-valued Lebesgue integral, especially inequality of the set-valued Lebesgue integrals. Finally we prove a theorem of existence and uniqueness of solution of Itô type set-valued stochastic differential equation.

Keywords: set-valued stochastic process, set-valued stochastic differential equation, set-valued Lebesgue integral, Itô integral

1 Introduction

Stochastic differential inclusions as a special form of stochastic differential equations appear in a natural way as a theoretical description of stochastic control problems (cf. [15]). Stochastic differential inclusion is

$$dx_t \in F(t, x_t)dt + G(t, x_t)dB_t, \quad x_0 = \xi, \tag{1}$$

which can be written as the following stochastic integral form

$$x_t - x_s \in \mathrm{cl}_{L^2}\Big(\int_s^t F(\tau, x_\tau)d\tau + G(\tau, x_\tau)dB_\tau\Big), \quad s, t \in [0, T], \tag{2}$$

where F, G are set-valued stochastic processes, $B = (B_t)_{t\in I}$ is a Brownian motion. In (1), there are two parts: one part is $F(t, x_t)dt$, which is related to the integral of a set-valued stochastic process with respect to time t, and the other part is $G(t, x_t)dB_t$, which is related to the Itô integral of a set-valued stochastic process with respect to the Brownian motion B_t.

In [12], Kim used the definition of stochastic integral of a set-valued stochastic process with respect to the Brownian motion introduced by Kisielewicz in [14] and discussed its properties. We called it the Aumann type Itô integral since the idea came from the Aumann integral of a set-valued function [2]. In [10], Jung and Kim gave a new definition with basic space being R by taking fixed time t. It may be more suitable to treat a set-valued stochastic process as a whole. In [23], Li and Ren introduced a new way to define the Itô integral of set-valued stochastic processes and discussed its properties.

There are many related former works about set-valued Lebesgue integral. Based on the work of Richter [29] and Kudo [19], Aumann introduced Aumann type Lebesgue integral of set-valued functions and discussed its properties in [2]. Kisielewicz introduced Aumann type Lebesgue integral of set-valued stochastic processes in [13]. Kisielewicz with his colleagues discussed stochastic differential inclusions, especially their solutions in [13]–[17]. In [20], Li and Li discussed more properties of the Lebesgue integral of set-valued stochastic processes. We would like to refer to related works such as [5], [24], [26], [32] and so on. In this paper, we shall continue to discuss the properties of the Lebesgue integral of set-valued stochastic processes, especially the inequality of the Lebesgue integrals, which is necessary to discuss set-valued stochastic differential equations.

*Research partially supported by NSFC(No.10771010), Research Fund of Beijing Educational Committee, PHR(IHLB) and 111 Talent Project Fund of BJUT, P.R. China.

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It is well known that classical Itô type stochastic differential equations have been widely used in the stochastic control (e.g. [25]) and financial mathematics (e.g. [4], [11]). The Itô type set-valued stochastic differential equation is

$$dX_t = b(t, X_t)dt + \sigma(t, X_t)dB_t, \tag{3}$$

where $b(t, X_t)$ takes values in the space $\mathbf{K}(R^d)$ (the set of all nonempty closed subsets of R^d), $\sigma(t, X_t)$ takes values in the space $\mathbf{K}(R^d \otimes R^m)$ (the set of all nonempty closed subsets of matrix space $R^d \otimes R^m$) and B_t is an m-dimensional Brownian motion. (3) can be written as set-valued stochastic integral form

$$X_t = X_0 + (L)\int_0^t b(t, X_t)dt + (I)\int_0^t \sigma(t, X_t)dB_t, \tag{4}$$

where $(L)\int_0^t b(t, X_t)dt$ is the set-valued Lebesgue integral and $(I)\int_0^t \sigma(t, X_t)dB_t$ is the set-valued Itô integral. If $\sigma(t, X_t) \in R^d \otimes R^m$, then we have

$$X_t = X_0 + (L)\int_0^t b(t, X_t)dt + \int_0^t \sigma(t, X_t)dB_t, \tag{5}$$

where $\int_0^t \sigma(t, X_t)dB_t$ is the classical Itô integral.

There are few papers about the Itô type set-valued stochastic differential equations, even in the special case (5). But we know there is a paper about the Itô type fuzzy stochastic differential equations. In [7], Hu *et. al.* used Hukuhara difference to define the differentiability and to discuss the Itô type fuzzy stochastic differential equations in the special case $\sigma(t, X_t) \in R^d \otimes R^m$, i.e. the equation (5). But since it is well-known that the space of all closed subsets of even R (the space of all real numbers) is not linear with respect to the addition and scalar multiplication, it leads to a big problem: under what conditions does the Hukuhara difference exist? It is a difficult problem so that they simply assume that the Hukuhara difference of a stochastic process at any two different times always exists. In this paper, we shall use selection method to consider the same type problem as in [7] without using the Hukuhara difference. We shall consider the Itô type set-valued stochastic integral equation (5), discuss the existence and uniqueness of its solution. By using level set method [28], we may easily extend the set-valued case to fuzzy set-valued case.

We organize our paper as follows. In Section 2, we introduce some necessary notations, definitions and results about set-valued stochastic processes and set-valued Lebesgue integral, and then we shall prove some new properties, especially inequality of set-valued Lebesgue integrals. In Section 3, we give a set-valued stochastic differential equation of Itô type, and prove the theorem of existence and uniqueness of solution to this kind of set-valued stochastic differential equation.

2 Stochastic Integral of Set-Valued Stochastic Processes and its Properties

Throughout this paper, assume that $(\Omega, \mathcal{A}, \mu)$ is a complete probability space, the σ-field filtration $\{\mathcal{A}_t : t \in I\}$ satisfies the usual conditions (i.e. containing all null sets, non-decreasing and right continuous), $I = [0, T]$ with $T > 0$, R is the set of all real numbers, N is the set of all natural numbers, R^d is the d-dimensional Euclidean space with usual norm $\|\cdot\|$, $\mathcal{B}(E)$ is the Borel field of the space E. Let $f = \{f(t), \mathcal{A}_t : t \in I\}$ be a R^d-valued adapted stochastic process. It is said that f is progressively measurable if for any $t \in I$, the mapping $(s, \omega) \mapsto f(s, \omega)$ from $[0, t] \times \Omega$ to R^d is $\mathcal{B}([0, t]) \times \mathcal{A}_t$-measurable. If let

$$\mathcal{C} = \{A \subseteq I \times \Omega : \forall t \in I, A \cap ([0, t] \times \Omega) \in \mathcal{B}([0, t]) \times \mathcal{A}_t\},$$

then f is progressively measurable if and only if f is $\mathcal{C}$-measurable. Each right continuous (left continuous) adapted process is progressively measurable.

Assume that $\mathcal{L}^p(R^d)$ $(p \geq 1)$ denotes the set of R^d-valued stochastic processes $f = \{f(t), \mathcal{A}_t : t \in I\}$ such that f satisfying (a) f is progressively measurable; and (b)

$$|||f|||_p = \Big[E\Big(\int_0^T \|f(t, \omega)\|^p ds\Big)\Big]^{1/p} < \infty. \tag{6}$$

Let $f, f' \in \mathcal{L}^p(R^d)$, $f = f'$ if and only if $|||f - f'|||_p = 0$. Then $(\mathcal{L}^p(R^d), ||| \cdot |||_p)$ is complete.

Now we review notation and concepts of set-valued stochastic processes.

Assume that $\mathbf{K}(R^d)$ is the family of all nonempty, closed subsets of R^d, and $\mathbf{K}_c(R^d)$ (*resp.* $\mathbf{K}_k(R^d)$, $\mathbf{K}_{kc}(R^d)$) is the family of all nonempty closed convex (*resp.* compact, compact convex) subsets of R^d. For any $x \in R^d$, A is a nonempty subset of R^d, define the distance between x and A as $d(x, A) = \inf_{y \in A} \|x - y\|$. The *Hausdorff metric* on $\mathbf{K}(R^d)$ is defined as

$$d_H(A, B) = \max\{\sup_{a \in A} d(a, B), \sup_{b \in B} d(b, A)\} \tag{7}$$

for $A, B \in \mathbf{K}(R^d)$. For $B \in \mathbf{K}(R^d)$, define $\|B\|_{\mathbf{K}} = d_H(\{0\}, B) = \sup_{a \in B} \|a\|$.

For a set-valued random variable F (cf. [6], [21]), define the set

$$S_F^p = \{f \in L^p[\Omega; R^d] : f(\omega) \in F(\omega) \ a.e.\},$$

where $L^p[\Omega; R^d]$ is the set of all R^d-valued random variables f such that $\|f\|_p = [E(\|f\|^p)]^{1/p} < \infty$. The *expectation* of F is defined as $E[F] = \{E[f] : f \in S_F^1\}$. It is called Aumann integral introduced by Aumann in 1965 (cf. [2]). A set-valued random variable $F : \Omega \to \mathbf{K}(R^d)$ is called *integrable* if S_F^1 is non-empty. F is called L^p*-bounded* if $\int_\Omega \|F(\omega)\|_{\mathbf{K}}^p d\mu < \infty$. Let $L^p[\Omega; \mathbf{K}(R^d)]$ (resp. $L^p[\Omega; \mathbf{K}_c(R^d)]$, $L^p[\Omega; \mathbf{K}_{kc}(R^d)]$) denote the family of $\mathbf{K}(R^d)$-valued (resp. $\mathbf{K}_c(R^d)$, $\mathbf{K}_{kc}(R^d)$-valued) L^p-bounded random variables. For any two set-valued random variables $F_1, F_2 \in L^p[\Omega; \mathbf{K}(R^d)]$, define

$$\Delta_p(F_1, F_2) = \Big(\int_\Omega d_H^p(F_1(\omega), F_2(\omega)) d\mu \Big)^{1/p},$$

then $(L^p[\Omega; \mathbf{K}(R^d)], \Delta_p)$ is a complete space. Concerning more definitions and more results of set-valued random variables, readers could refer to [6] or [21].

Definition 1 *A set-valued stochastic process $F = \{F(t) : t \in I\}$ is called* progressively measurable*, if it is $\mathcal{C}$-measurable, i.e., for any $A \in \mathcal{B}(R^d), \{(s, \omega) \in I \times \Omega : F(s, \omega) \cap A \neq \emptyset\} \in \mathcal{C}$. F is called $\mathcal{L}^p$-bounded, if the real stochastic process $\{\|F(t)\|_{\mathbf{K}}, \mathcal{A}_t : t \in I\} \in \mathcal{L}^p(R)$.*

Definition 2 *A R^d-valued process $\{f(t), \mathcal{A}_t : t \in I\} \in \mathcal{L}^p(R^d)$ is called an $\mathcal{L}^p$-selection of $F = \{F(t), \mathcal{A}_t : t \in I\}$ if $f(t, \omega) \in F(t, \omega)$ a.e.$(t, \omega) \in I \times \Omega$.*

Let $S^p(\{F(\cdot)\})$ or $S^p(F)$ denote the family of all $\mathcal{L}^p$-selections of $F = \{F(t), \mathcal{A}_t : t \in I\}$, i.e.

$$S^p(F) = \Big\{ \{f(t)\} \in \mathcal{L}^p(R^d) : f(t, \omega) \in F(t, \omega), \quad a.e. \ (t, \omega) \in I \times \Omega \Big\}.$$

Let $\mathcal{L}^p(\mathbf{K}(R^d))$ denote the set of all $\mathcal{L}^p$-bounded progressively measurable $\mathbf{K}(R^d)$-valued stochastic processes. Similarly, we have notations $\mathcal{L}^p(\mathbf{K}_c(R^d))$, $\mathcal{L}^p(\mathbf{K}_k(R^d))$ and $\mathcal{L}^p(\mathbf{K}_{kc}(R^d))$. Take $F_i = \{F_i(t) : t \in I\} \in \mathcal{L}^p(\mathbf{K}(R^d))$, $i = 1, 2$, define

$$\boldsymbol{\Delta}_p(F_1, F_2) = \Big[E\Big(\int_0^T d_H^p(F_1(s, \omega), F_2(s, \omega)) ds \Big) \Big]^{1/p}.$$

F_1 and F_2 are said to be *equivalent*, if $\boldsymbol{\Delta}_p(F_1, F_2) = 0$, denoted by $F_1 = F_2$. We have that $(\mathcal{L}^p(\mathbf{K}(R^d)), \boldsymbol{\Delta}_p)$ is complete, $\mathcal{L}^p(\mathbf{K}_c(R^d))$, $\mathcal{L}^p(\mathbf{K}_k(R^d))$ and $\mathcal{L}^p(\mathbf{K}_{kc}(R^d))$ are closed subsets of $(\mathcal{L}^p(\mathbf{K}(R^d)), \boldsymbol{\Delta}_p)$. Denote $|||F|||_p = \Big[E\Big(\int_0^T \|F(s)\|_{\mathbf{K}}^p ds \Big) \Big]^{1/p}$.

Now we introduce the concept of decomposability.

Definition 3 *A non-empty set $\Gamma \subseteq \mathcal{L}^p(R^d)$ is called decomposable with respect to the progressively measurable σ-field $\mathcal{C}$, if for any $f, g \in \Gamma$, any $U \in \mathcal{C}$, we have $I_U f + I_{U^c} g \in \Gamma$.*

Firstly, we know that for any set-valued progressively measurable stochastic process $F \in \mathcal{L}^p(\mathbf{K}(R^d))$, $S^p(F)$ is decomposable with respect to σ-field $\mathcal{C}$. Furthermore we have the following Theorem.

Theorem 1 [20] *Assume that $\Gamma \subseteq \mathcal{L}^p(R^d)$ is a non-empty closed set of R^d-valued progressively measurable stochastic processes, then Γ is decomposable with respect to progressively measurable σ-field $\mathcal{C}$ if and only if there exists a progressively measurable set-valued stochastic process $F \in \mathcal{L}^p(\mathbf{K}(R^d))$ such that $\Gamma = S^p(F)$. Furthermore, Γ is convex if and only if $F \in \mathcal{L}^p(\mathbf{K}_c(R^d))$.*

Now we consider the integral of set-valued stochastic process. To avoid trouble of dealing with almost every problem, we assume that $\mathcal{A}$ is μ-separable in the following. In this case, for any $p \geq 1$, $L^p[I \times \Omega, \mathcal{B}(I) \times \mathcal{A}, \lambda \times \mu; R^d]$ is a separable Banach space (cf. [31]), $\mathcal{L}^p(R^d)$ can be considered as its closed subset so that it is separable with respect to $|||\cdot|||_p$. Thus, For any $F = \{F(t) : t \in I\} \in \mathcal{L}^p(\mathbf{K}(R^d))$, $S^p(F)$ is separable. We may ignore almost everywhere problem and assume that the following definition is well-defined for all $(t, \omega) \in I \times \Omega$ rather than for almost everywhere $(t, \omega) \in I \times \Omega$.

Definition 4 *Let a set-valued stochastic process $F = \{F(t) : t \in I\} \in \mathcal{L}^p(\mathbf{K}(R^d))$, $1 \leq p < +\infty$. For any $\omega \in \Omega$, $t \in I$, define*

$$(A)\int_0^t F(s,\omega)ds := \left\{ \int_0^t f(s,\omega)ds : f \in S^p(F) \right\},$$

where $\int_0^t f(s,\omega)ds$ is the Lebesgue integral. $(A)\int_0^t F(s,\omega)ds$ is called the Aumann type Lebesgue integral of set-valued stochastic process F with respect to time t introduced in [14]. For any $0 \leq u < t < T$,

$$(A)\int_u^t F(s,\omega)ds := (A)\int_0^t I_{[u,t]}(s)F(s,\omega)ds.$$

Remark 1 In the Definition 4, the set of selections is $S^p(F)$. As a matter of fact, if we only consider the Lebesgue integral, we can use $S^1(F)$. But we often consider the sum of integral of a set-valued stochastic process with respect to time t and integral of a set-valued stochastic process with respect to a Brownian motion, where we have to use $S^2(F)$. Thus we here use $S^p(F)$ for more general case.

Remark 2 If a set-valued stochastic process $F = \{F(t) : t \in I\} \in \mathcal{L}^p(\mathbf{K}(R^d))$, then for any $t \in I$, $\Gamma(t) =: (A)\int_0^t F(s)ds$ is a non-empty subset of $L^p[\Omega, \mathcal{A}_t, \mu; R^d]$. Furthermore, if $F \in \mathcal{L}^p(\mathbf{K}_c(R^d))$, then we can prove that $(A)\int_0^t F(s)ds$ is a non-empty convex subset of $L^p[\Omega, \mathcal{A}_t, \mu; R^d]$. However, it is natural to hope that the result of integral is a set-valued stochastic process taking values in $\mathbf{K}(R^d)$ rather than in $L^p[\Omega, \mathcal{A}_t, \mu; R^d]$. If for any fixed $t \in I$, let $\Gamma(t)(\omega) =: (A)\int_0^t F(s,\omega)ds$ $(\omega \in \Omega)$, we also do not know whether $\Gamma(t)(\omega)$ is a closed subset or not, whether it is measurable or not. So it is necessary to give a new definition so that the integral is still a set-valued stochastic process. Since we can not prove directly that $\{\Gamma(t) : t \in I\}$ is decomposable with respect to $\mathcal{C}$, we firstly give the definition of decomposable closure.

Definition 5 *For any non-empty subset $\Gamma \subseteq L^p[I \times \Omega, \mathcal{C}, \lambda \times \mu; R^d]$, define the decomposable closure $\overline{de}\Gamma$ of Γ with respect to $\mathcal{C}$ as*

$$\overline{de}\Gamma = \Big\{ g = \{g(t,\omega) : t \in I\} : \text{ for any } \varepsilon > 0, \text{ there exists a } \mathcal{C}\text{-measurable finite partition}$$
$$\{A_1, \cdots, A_n\} \text{ of } I \times \Omega \text{ and } f_1, \cdots, f_n \in \Gamma \text{ such that } |||g - \sum_{i=1}^n I_{A_i} f_i|||_p < \varepsilon \Big\}.$$

Theorem 2 ([20]) *Assume that $F = \{F(t) : t \in I\} \in \mathcal{L}^p(\mathbf{K}(R^d))$, $\Gamma(t) = (A)\int_0^t F(s)ds$, then there exists a $\mathcal{C}$-measurable set-valued stochastic process $L(F) = \{L_t(F) : t \in I\} \in \mathcal{L}^p(\mathbf{K}(R^d))$ such that $S^p(L(F)) = \overline{de}\{\Gamma(t) : t \in I\}$. Furthermore, if $F \in \mathcal{L}^p(\mathbf{K}_c(R^d))$, then $\{L_t(F) : t \in I\} \in \mathcal{L}^p(\mathbf{K}_c(R^d))$.*

The set-valued stochastic process $L(F) = \{L_t(F) : t \in I\}$ defined in Theorem 2 is called the Lebesgue integral of a set-valued stochastic process $F = \{F(t) : t \in I\} \in \mathcal{L}^p(\mathbf{K}(R^d))$ with respect to the time t, and denoted as $L_t(F) = (L)\int_0^t F(s)ds$.

Theorem 3 ([20]) *Let $F = \{F(t) : t \in I\} \in \mathcal{L}^p(\mathbf{K}(R^d))$, then there exists a sequence of R^d-valued stochastic processes $\{f^i = \{f^i(t) : t \in I\} : i \geq 1\} \subseteq S^p(F)$ such that*

$$F(t,\omega) = \text{cl}\{f^i(t,\omega) : i \geq 1\}, \quad a.e.\ (t,\omega) \in I \times \Omega,$$

and

$$L_t(F) = \text{cl}\Big\{ \int_0^t f^i(s,\omega)ds : i \geq 1 \Big\} \quad a.e.\ (t,\omega) \in I \times \Omega.$$

Theorem 4 *Let set-valued stochastic process* $\{F(t) : t \in I\} \in \mathcal{L}^2(\mathbf{K}(R^d))$*. Then there exists a measurable subset* $A \subseteq I \times \Omega$ *with* $(\lambda \times \mu)(A) = 0$*, so that the following holds*

$$L_t(F)(\omega) = \text{cl}\Big\{L_{t_1}(F)(\omega) + (L)\int_{t_1}^{t} F(s,\omega)ds\Big\} \quad \textit{for any } (t,\omega),(t_1,\omega) \in I\times\Omega\setminus A, \quad t_1 \le t,$$

where the closure is taken in R^d*.*

Proof: From Theorem 3, there exist a sequence $\{(f^i(t))_{t\in I} : i = 1, 2, \cdots\} \subseteq S^2(F(\cdot))$ and a measurable subset $A \subseteq I \times \Omega$ with $(\lambda \times \mu)(A) = 0$ such that for each $(t,\omega) \in I \times \Omega \setminus A$, we have

$$F(t,\omega) = \text{cl}\{(f^i(t,\omega)) : i = 1, 2, \cdots\},$$

and

$$L_t(F)(\omega) = \text{cl}\Big\{\int_0^t f^i(s,\omega)ds : i = 1, 2, \cdots\Big\}. \tag{8}$$

Then for $0 \le t_1 < t$ with $(t_1,\omega) \in I \times \Omega \setminus A$, we have

$$L_{t_1}(F)(\omega) = \text{cl}\Big\{\int_0^{t_1} f^i(s,\omega)ds : i = 1, 2, \cdots\Big\}, \tag{9}$$

$$(L)\int_{t_1}^{t} F(s,\omega)ds = \text{cl}\Big\{\int_{t_1}^{t} f^i(s,\omega)ds : i = 1, 2, \cdots\Big\}. \tag{10}$$

It is obvious that

$$L_t(F)(\omega) \subseteq \text{cl}\Big\{L_{t_1}(F)(\omega) + (L)\int_{t_1}^{t} F(s,\omega)ds\Big\}.$$

Conversely, take $a \in \text{cl}\{L_{t_1}(F)(\omega) + (L)\int_{t_1}^{t} F(s,\omega)ds\}$, by (9) and (10) for any given $\epsilon > 0$, we can find $m(\epsilon), k(\epsilon) \in N$, such that

$$\Big\|a - \Big(\int_0^{t_1} f^{m(\epsilon)}(s,\omega)ds + \int_{t_1}^{t} f^{k(\epsilon)}(s,\omega)ds\Big)\Big\| < \frac{\epsilon}{2}. \tag{11}$$

Let $g(s,\omega) = f^{m(\epsilon)}(s,\omega)I_{[0,t_1]}(s) + f^{k(\epsilon)}(s,\omega)I_{[t_1,t]}(s)$, where $I_{[0,t_1]}(s)$ and $I_{[t_1,t]}(s)$ are indicator functions. Then $\int_0^t g(s,\omega)ds \in L_t(F)(\omega)$. From (8), there exists $n(\epsilon) \in N$, such that

$$\Big\|\int_0^t g(s,\omega)ds - \int_0^t f^{n(\epsilon)}(s,\omega)ds\Big\| < \frac{\epsilon}{2}. \tag{12}$$

By (11) and (12), we obtain

$$\Big\|a - \int_0^t f^{n(\epsilon)}(s,\omega)ds\Big\| < \epsilon,$$

which implies $a \in L_t(F)(\omega)$. Thus $L_t(F)(\omega) \supseteq \text{cl}\{L_{t_1}(F)(\omega) + (L)\int_{t_1}^{t} F(s,\omega)ds\}$.

Now we prove an inequality of set-valued Legesgue integrals which will be used in the next section.

Theorem 5 *Let set-valued stochastic processes* $F = \{F(t) : t \in I\}, G = \{G(t) : t \in I\} \in \mathcal{L}^2(\mathbf{K}(R^d))$*, then there exists a measurable subset* $A \subseteq I \times \Omega$ *with* $(\lambda \times \mu)(A) = 0$ *so that the following holds*

$$d_H^2(L_t(F)(\omega),\, L_t(G)(\omega)) \le t\int_0^t d_H^2(F(s,\,\omega),\, G(s,\,\omega))ds, \ \textit{for any } (t,\omega) \in I \times \Omega \setminus A.$$

Proof: Suppose $\Phi(t) = \int_0^t F(s)ds$, $\Psi(t) = \int_0^t G(s)ds$. From Theorem 3, there exist $\{f^i = \{f^i(t) : t \in I\}, i \ge 1\} \subseteq S^2(F)$, $\{g^j = \{g^j(t) : t \in I\}, j \ge 1\} \subseteq S^2(G)$, and a measurable subset $A \subseteq I \times \Omega$ with $(\lambda \times \mu)(A) = 0$ such that for each $(t,\omega) \in I \times \Omega \setminus A$,

$$F(t,\omega) = \text{cl}\{f^i(t,\omega) : i \ge 1\}, \quad G(t,\omega) = \text{cl}\{g^j(t,\omega) : j \ge 1\},$$

and

$$\begin{aligned}\Phi(t)(\omega) &= \text{cl}\Big\{\int_0^t f^i(s,\omega)ds : i \geq 1\Big\},\\ \Psi(t)(\omega) &= \text{cl}\Big\{\int_0^t g^j(s,\omega)ds : j \geq 1\Big\}.\end{aligned}$$

Hence, we have

$$\begin{aligned}\inf_{y\in L_t(G)(\omega)} \Big\|\int_0^t f^i(s,\omega)ds - y\Big\|^2 &= \inf_{j\geq 1} \Big\|\int_0^t f^i(s,\omega)ds - \int_0^t g^j(s,\omega)ds\Big\|^2\\ &\leq \inf_{j\geq 1} t\int_0^t \|f^i(s,\,\omega) - g^j(s,\,\omega)\|^2 ds.\end{aligned}$$

Further, we can show along the same arguments as in the proof of [21, Lemma 1.3.12]

$$\begin{aligned}\inf_{j\geq 1}\int_0^t \|f^i(s,\,\omega) - g^j(s,\,\omega)\|^2 ds &= \int_0^t \inf_{y\in G(s,\,\omega)} \|f^i(s,\,\omega) - y\|^2 ds\\ &\leq \int_0^t d_H^2(F(s,\,\omega),\,G(s,\,\omega))ds.\end{aligned}$$

Noticing that

$$\sup_{x\in L_t(F)(\omega)} \inf_{y\in L_t(G)(\omega)} \|x-y\| = \sup_{i\geq 1} \inf_{y\in L_t(G)(\omega)} \Big\|\int_0^t f^i(s,\,\omega)ds - y\Big\|,$$

we obtain

$$\sup_{x\in L_t(F)(\omega)} \inf_{y\in L_t(G)(\omega)} \|x-y\|^2 \leq t\int_0^t d_H^2(F(s,\,\omega),\,G(s,\,\omega))ds.$$

Similarly, we have

$$\sup_{x\in L_t(G)(\omega)} \inf_{y\in L_t(F)(\omega)} \|x-y\|^2 \leq t\int_0^t d_H^2(F(s,\,\omega),\,G(s,\,\omega))ds.$$

Hence, by the definition of Hausdorff distance, we arrive at the result.

3 The Existence and Uniqueness of the Solution of Itô Type Set-Valued Stochastic Differential Equation

We consider the following Itô type set-valued stochastic differential equation

$$dF(t) = f(t, F(t))dt + g(t, F(t))dB_t, \tag{13}$$

where the set-valued stochastic process $F \in \mathcal{L}^2(\mathbf{K}(R^d))$ with initial condition $F(0)$ being an L^2-bounded set-valued random variable, $f : I \times \mathbf{K}(R^d) \to \mathbf{K}(R^d)$ is measurable, $g : I \times \mathbf{K}(R^d) \to R^d \otimes R^m$ is measurable, B_t is an m-dimensional Brown motion. If $f \in \mathcal{L}^2(\mathbf{K}(R^d))$ and $g \in \mathcal{L}^2(R^d \otimes R^m)$, then equation (13) is equivalent to the integral form:

$$F(t) = F(0) + (L)\int_0^t f(s, F(s))ds + \int_0^t g(s, F(s))dB_s. \tag{14}$$

Theorem 6 (***Existence and uniqueness Theorem***) *Assume that $f(t, F), g(t, F), t \in I, F, F_1, F_2 \in \mathbf{K}(R^d)$ satisfy the following conditions:*
(i) Linear increasing condition

$$\|f(t, F)\|_{\mathbf{K}}^2 + \|g(t, F)\|^2 \leq K^2(1 + \|F\|_{\mathbf{K}}^2),$$

where K is a positive constant.
(ii) Lipschitz continuous condition

$$d_H(f(t, F_1), f(t, F_2)) + \|g(t, F_1) - g(t, F_2)\| \leq K d_H(F_1, F_2).$$

Then for any given initial L^2-bounded set-valued random variable $F(0)$, there is a solution to the equation (13), and the solution is unique in the space of $(\mathcal{L}^2(\mathbf{K}(R^d)), \mathbf{\Delta}_2)$.

Proof: Without loss of generality, we assume that Theorems 4 and 5 are right for all t, t_1. If $F \in \mathcal{L}^2(\mathbf{K}(R^d))$, then for any $t \in I$,

$$E\|f(t,F(t))\|_{\mathbf{K}}^2 + E\|g(t,F(t))\|^2 \le K^2(1 + E\|F(t)\|_{\mathbf{K}}^2).$$

We have $f \in \mathcal{L}^2(\mathbf{K}(R^d))$, $g \in \mathcal{L}^2(R^d \otimes R^m)$.

Step 1. We prove the existence by successively approaching. For simplification, we omit the character "(L)" before the symbol of the set-valued Lebesgue integral in the proof of this theorem.

For any $t \in I$, define

$$\begin{aligned} F_0(t) &= F(0), \\ F_{n+1}(t) &= F(0) + \int_0^t f(s, F_n(s))ds + \int_0^t g(s, F_n(s))dB_s, \quad n \ge 0. \end{aligned}$$

We firstly prove that for any $n \ge 0$, F_n is well-defined and satisfies:
(α) $F_n \in \mathcal{L}^2(\mathbf{K}(R^d))$;
(β) $\lim_{s\to t} Ed_H^2(F_n(t), F_n(s)) = 0$.

For $n = 0$, it is obviously right. Suppose that F_n has properties $(\alpha), (\beta)$ for any fixed n, we shall prove so does F_{n+1}. Indeed, since $F_n \in \mathcal{L}^2(\mathbf{K}(R^d))$, $f \in \mathcal{L}^2(\mathbf{K}(R^d))$, let

$$Y(t) := \int_0^t f(s, F_n(s))ds,$$

we have that $Y \in \mathcal{L}^2(\mathbf{K}(R^d))$ by the definition of set-valued Legesgue integral. For any $s, t \in I$, by using triangular inequality and Hölder inequality, we have

$$\begin{aligned} &\Big|E\|F_n(t)\|_{\mathbf{K}}^2 - E\|F_n(s)\|_{\mathbf{K}}^2\Big| \\ &= \Big|Ed_H^2(F_n(t),0) - Ed_H^2(F_n(s),0)\Big| \\ &\le E\Big|(d_H(F_n(t),0) + d_H(F_n(s),0))(d_H(F_n(t),0) - d_H(F_n(s),0))\Big| \\ &= E[(d_H(F_n(t),0) + d_H(F_n(s),0))|d_H(F_n(t),0) - d_H(F_n(s),0)|] \\ &\le E[(d_H(F_n(t),0) + d_H(F_n(s),0))d_H(F_n(t),F_n(s))] \\ &\le \Big(E[(d_H(F_n(t),0) + d_H(F_n(s),0))^2]Ed_H^2(F_n(t),F_n(s))\Big)^{1/2} \\ &\le \Big(2E[d_H^2(F_n(t),0) + d_H^2(F_n(s),0)]Ed_H^2(F_n(t),F_n(s))\Big)^{1/2}. \end{aligned}$$

Thus, we know that $E\|F_n(t)\|_{\mathbf{K}}^2$ is continuous in I by the assumptions.

By virtue of Theorems 4 and 5 and the assumptions of theorems, we obtain

$$\begin{aligned} Ed_H^2(Y(t),Y(s)) &= Ed_H^2\Big(\int_0^t f(s_1,F_n(s_1))ds_1, \int_0^s f(s_1,F_n(s_1))ds_1\Big) \\ &= Ed_H^2\Big(\mathrm{cl}(L_s(f) + \int_s^t f(s_1,F_n(s_1))ds_1), L_s(f)\Big) \\ &\le Ed_H^2\Big(\int_s^t f(s_1,F_n(s_1))ds_1, 0\Big) \\ &= E\Big\|\int_s^t f(s_1,F_n(s_1))ds_1\Big\|_{\mathbf{K}}^2 \\ &\le E(\int_s^t \|f(s_1,F_n(s_1))\|_{\mathbf{K}}ds_1)^2 \\ &\le (t-s)E\int_s^t \|f(s_1,F_n(s_1))\|_{\mathbf{K}}^2 ds_1 \\ &\le (t-s)E\int_s^t K^2(1 + \|F_n(s_1)\|_{\mathbf{K}}^2)ds_1. \end{aligned}$$

Thus, Y satisfies properties (α) and (β).

Now we investigate the Itô integral part. Since $g(t, F_n(t))$ is square integrable, the Itô integral

$$Z(t) := \int_0^t g(s, F_n(s))dB_s$$

exists and $Z(t)$ is square integrable. By using the properties of the classical Itô integral, we have

$$\begin{aligned} E\|Z(t) - Z(s)\|^2 &= E\Big\| \int_0^t g(s_1, F_n(s_1))dB_{s_1} - \int_0^s g(s_1, F_n(s_1))dB_{s_1} \Big\|^2 \\ &= E\Big\| \int_s^t g(s_1, F_n(s_1))dB_{s_1} \Big\|^2 \\ &= E\int_s^t \|g(s_1, F_n(s_1))\|^2 ds_1 \\ &\leq \int_s^t K^2(1 + E\|F_n(s_1)\|^2_{\mathbf{K}})ds_1. \end{aligned}$$

Hence, Z satisfies (α), (β). Since

$$F_{n+1}(t) = F(0) + Y(t) + Z(t),$$

F_{n+1} exists and satisfies (α) and (β).

Next we prove that F_n converges to F in $\mathcal{L}^2(\mathbf{K}(R^d))$. Let

$$\begin{aligned} F_0(t) &= F(0), \\ F_{n+1}(t) &= F(t_0) + \int_0^t f(s, F_n(s))ds + \int_0^t g(s, F_n(s))dB_s, \quad n \geq 0. \end{aligned}$$

Then due to the property of d_H and triangular inequality, we have

$$\begin{aligned} d_H(F_1(t), F_0(t)) &= d_H\Big(F_0(t) + \textstyle\int_0^t f(s, F_0(s))ds + \int_0^t g(s, F_0(s))dB_s, F_0(t)\Big) \\ &\leq d_H\Big(\textstyle\int_0^t f(s, F_0(s))ds, 0\Big) + d_H\Big(\int_0^t g(s, F_0(s))dB_s, 0\Big) \\ &= \Big\|\textstyle\int_0^t f(s, F_0(s))ds\Big\|_{\mathbf{K}} + \Big\|\int_0^t g(s, F_0(s))dB_s\Big\|, \end{aligned} \tag{15}$$

and for the first part of (15), by the Hölder inequality and the assumptions of theorem, we obtain

$$\begin{aligned} E\Big\|\textstyle\int_0^t f(s, F_0(s))ds\Big\|^2_{\mathbf{K}} &\leq E\Big(\textstyle\int_0^t \|f(s, F_0(s))\|_{\mathbf{K}}ds\Big)^2 \\ &\leq tE(\textstyle\int_0^t \|f(s, F_0(s))\|^2_{\mathbf{K}}ds) \\ &= t\textstyle\int_0^t E\|f(s, F_0(s))\|^2_{\mathbf{K}}ds \\ &\leq tK^2\textstyle\int_0^t(1 + E\|F_0(s)\|^2_{\mathbf{K}})ds \\ &\leq A^2t, \end{aligned} \tag{16}$$

where $A^2 = K^2(1 + E\|F(0)\|^2_{\mathbf{K}})T$. For the second part of (15), from classical Itô isometric property and the assumptions of theorem, we have

$$\begin{aligned} E\Big\|\textstyle\int_0^t g(s, F_0(s))dB_s\Big\|^2 &= E\textstyle\int_0^t \|g(s, F_0(s))\|^2 ds \\ &\leq E\textstyle\int_0^t K^2(1 + \|F_0(s)\|^2_{\mathbf{K}})ds \\ &= tK^2(1 + E\|F(0)\|^2_{\mathbf{K}}) \\ &\leq B^2t, \end{aligned} \tag{17}$$

where $B^2 = K^2(1 + E\|F(0)\|^2_{\mathbf{K}})$. Put (16) (17) into (15), we get

$$\begin{aligned} Ed^2_H(F_1(t), F_0(t)) &\leq 2E\Big\|\int_0^t f(s, F_0(s))ds\Big\|^2_{\mathbf{K}} + 2E\Big\|\int_0^t g(s, F_0(s))dB_s\Big\|^2 \\ &\leq 2(A + B)^2t. \end{aligned}$$

By the same way, we have

$$
\begin{aligned}
&Ed_H^2(F_{n+1}(t),F_n(t))\\
&=E\Big(d_H(F(0)+\int_0^t f(s,F_n(s))ds+\int_0^t g(s,F_n(s))dB_s,F(0)\\
&\qquad+\int_0^t f(s,F_{n-1}(s))ds+\int_0^t g(s,F_{n-1}(s))dB_s)\Big)^2\\
&\le E\Big[d_H(F(0),F(0))+d_H\Big(\int_0^t f(s,F_n(s))ds,\int_0^t f(s,F_{n-1}(s))ds\Big)\\
&\qquad+\Big\|\int_0^t g(s,F_n(s))dB_s-\int_0^t g(s,F_{n-1}(s))dB_s\Big\|\Big]^2\\
&\le 2Ed_H^2\Big(\int_0^t f(s,F_n(s))ds,\int_0^t f(s,F_{n-1}(s))ds\Big)\\
&\qquad+2E\Big\|\int_0^t g(s,F_n(s))dB_s-\int_0^t g(s,F_{n-1}(s))dB_s\Big\|^2\\
&\le 2tE\int_0^t d_H^2(f(s,F_n(s)),f(s,F_{n-1}(s)))ds\\
&\qquad+2E\int_0^t \|g(s,F_n(s))-g(s,F_{n-1}(s))\|^2ds\\
&\le 2tE\int_0^t K^2d_H^2(F_n(s),F_{n-1}(s))ds+2E\int_0^t K^2d_H^2(F_n(s),F_{n-1}(s))ds\\
&=2(t+1)K^2E\int_0^t d_H^2(F_n(s),F_{n-1}(s))ds.
\end{aligned}
$$

Iterating the above process, we obtain

$$Ed_H^2(F_{n+1}(t),F_n(t))\le K^{2n}2^{n+1}(A+B)^2\frac{(t+1)^n}{(n+1)!},$$

then

$$(Ed_H^2(F_{n+1}(t),F_n(t)))^{1/2}\le [K^{2n}2^{n+1}(A+B)^2\frac{(T+1)^{n+1}}{(n+1)!}]^{1/2}. \tag{18}$$

Since the sum of the right of (18) is a series which is convergent and not dependent on t, we have that for any $t\in I$,

$$
\begin{aligned}
\Delta_2(F_n(t),F_m(t)) &= \Big(Ed_H^2(F_n(t),F_m(t))\Big)^{1/2}\\
&\le \sum_{k=m}^{n-1}\Delta_2(F_{k+1}(t),F_k(t))\\
&= \sum_{k=m}^{n-1}\big(Ed_H^2(F_{k+1}(t),F_k(t))\big)^{1/2}\to 0 \quad (m,n\to\infty),
\end{aligned}
$$

by noticing the triangular inequality with respect to the metric Δ_2. Noting for any $m<n$, $\Delta_2(F_n(t),F_m(t))$ is also bounded in I. Thus, by using Fubini Theorem and bounded dominated theorem, we have

$$\mathbf{\Delta}_2(F_n,F_m)=\Big[E\Big(\int_0^T d_H^2(F_n(s,\omega),F_m(s,\omega))ds\Big)\Big]^{1/2}\to 0,\ \text{as } m,n\to\infty,$$

that is, F_n is convergent to F in $\mathcal{L}^2(\mathbf{K}(R^d))$. By triangular inequality, we have

$$Ed_H^2(F(s),F(t))$$

$$
\begin{aligned}
&\le E\Big(d_H(F(s),F_n(s))+d_H(F_n(s),F_n(t))+d_H(F_n(t),F(t))\Big)^2\\
&=Ed_H^2(F(s),F_n(s))+Ed_H^2(F_n(s),F_n(t))+Ed_H^2(F_n(t),F(t))\\
&\quad+2E(d_H(F(s),F_n(s))d_H(F_n(s),F_n(t)))+2E(d_H(F(s),F_n(s))d_H(F_n(t),F(t)))\\
&\qquad+2E(d_H(F_n(s),F_n(t))d_H(F_n(t),F(t)))\\
&\le Ed_H^2(F(s),F_n(s))+Ed_H^2(F_n(s),F_n(t))+Ed_H^2(F_n(t),F(t))\\
&\quad+2\sqrt{Ed_H^2(F(s),F_n(s))Ed_H^2(F_n(s),F_n(t))}+2\sqrt{Ed_H^2(F(s),F_n(s))Ed_H^2(F_n(t),F(t))}\\
&\quad+2\sqrt{Ed_H^2(F_n(s),F_n(t))Ed_H^2(F_n(t),F(t))}.
\end{aligned}
$$

By property (β) and the convergence of F_n, we have $\lim_{s\to t} Ed_H^2(F(s),F(t))=0$. Thus the solution of the set-valued stochastic integral equation is continuous.

Step 2. We prove the uniqueness. Let F and G are two solutions of the equation (14). Similar to the proof of the existence, we have

$$Ed_H^2(F(t),G(t))\le 2(t+1)K^2E\int_0^t d_H^2(F(s),G(s))ds. \tag{19}$$

Since the solutions $F,G\in\mathcal{L}^2(\mathbf{K}(R^d))$, we have

$$
\begin{aligned}
E\int_0^t d_H^2(F(s),G(s))ds &\le E\int_0^T d_H^2(F(s),G(s))ds\\
&\le 2E\int_0^T(\|F(s)\|_{\mathbf{K}}^2+\|G(s)\|_{\mathbf{K}}^2)ds\\
&= C^2<\infty.
\end{aligned}
$$

Together with (19) once, we have

$$Ed_H^2(F(t),G(t))\le 2(t+1)K^2C^2.$$

Together with (19) twice, we have

$$
\begin{aligned}
Ed_H^2(F(t),G(t)) &\le 2(t+1)K^2E\int_0^t d_H^2(F(s),G(s))ds\\
&= 2(t+1)K^2\int_0^t 2(s+1)K^2C^2ds\\
&\le 2^2(t+1)^2K^{2\cdot 2}\int_0^t C^2ds\\
&= (2K^2)^2(t+1)^2C^2t.
\end{aligned}
$$

Iterating the above process, we get

$$Ed_H^2(F(t),G(t))\le K^{2(n+1)}(2(t+1))^{n+1}C^2\frac{t^n}{n!}. \tag{20}$$

Let $n\to\infty$, the right of (20) converges to 0. By using Fubini theorem and classical bounded dominated theorem, we have

$$\mathbf{\Delta}_2(F,G)=\Big[E\Big(\int_0^T d_H^2(F(s,\omega),F(s,\omega))ds\Big)\Big]^{1/2}\le 0,\ \text{as } n\to\infty,$$

the uniqueness is proved.

Acknowledgments

We would like to thank referees for their valuable remarks and kind help.

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