Itô Type Set-Valued Stochastic Differential Equation*

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Abstract

In this paper, we firstly illustrate why we should introduce the Itô type set-valued stochastic differential equation and then recall some basic results about the Lebesgue integral of a set-valued stochastic process with respect to time t. Secondly we obtain some new properties of the set-valued Lebesgue integral, especially inequality of the set-valued Lebesgue integrals. Finally we prove a theorem of existence and uniqueness of solution of Itô type set-valued stochastic differential equation.

Keywords: set-valued stochastic process, set-valued stochastic differential equation, set-valued Lebesgue integral, Itô integral

1 Introduction

Stochastic differential inclusions as a special form of stochastic differential equations appear in a natural way as a theoretical description of stochastic control problems (cf. [15]). Stochastic differential inclusion is

\[ dx_t \in F(t, x_t)dt + G(t, x_t)dB_t, \quad x_0 = \xi, \]

which can be written as the following stochastic integral form

\[ x_t - x_s \in \text{cl}\ell L^2 \left( \int_s^t F(\tau, x_\tau) d\tau + G(\tau, x_\tau) dB_\tau \right), \quad s, t \in [0, T], \]

where \( F, G \) are set-valued stochastic processes, \( B = (B_t)_{t \in I} \) is a Brownian motion. In (1), there are two parts: one part is \( F(t, x_t)dt \), which is related to the integral of a set-valued stochastic process with respect to time \( t \), and the other part is \( G(t, x_t)dB_t \), which is related to the Itô integral of a set-valued stochastic process with respect to the Brownian motion \( B_t \).

In [12], Kim used the definition of stochastic integral of a set-valued stochastic process with respect to the Brownian motion introduced by Kisielewicz in [14] and discussed its properties. We called it the Aumann type Itô integral since the idea came from the Aumann integral of a set-valued function [2]. In [10], Jung and Kim gave a new definition with basic space being \( R \) by taking fixed time \( t \). It may be more suitable to treat a set-valued stochastic process as a whole. In [23], Li and Ren introduced a new way to define the Itô integral of set-valued stochastic processes and discussed its properties.

There are many related former works about set-valued Lebesgue integral. Based on the work of Richter [29] and Kudo [19], Aumann introduced Aumann type Lebesgue integral of set-valued functions and discussed its properties in [2]. Kisielewicz introduced Aumann type Lebesgue integral of set-valued stochastic processes in [13]. Kisielewicz with his colleagues discussed stochastic differential inclusions, especially their solutions in [13]–[17]. In [20], Li and Li discussed more properties of the Lebesgue integral of set-valued stochastic processes. We would like to refer to related works such as [5], [24], [26], [32] and so on. In this paper, we shall continue to discuss the properties of the Lebesgue integral of set-valued stochastic processes, especially the inequality of the Lebesgue integrals, which is necessary to discuss set-valued stochastic differential equations.

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It is well known that classical Itô type stochastic differential equations have been widely used in the stochastic control (e.g. [25]) and financial mathematics (e.g. [4], [11]). The Itô type set-valued stochastic differential equation is
\[ dX_t = b(t, X_t)dt + \sigma(t, X_t)dB_t, \tag{3} \]
where \( b(t, X_t) \) takes values in the space \( K(R^d) \) (the set of all nonempty closed subsets of \( R^d \)), \( \sigma(t, X_t) \) takes values in the space \( K(R^d \otimes R^m) \) (the set of all nonempty closed subsets of matrix space \( R^d \otimes R^m \)) and \( B_t \) is an \( m \)-dimensional Brownian motion. (3) can be written as set-valued stochastic integral form
\[ X_t = X_0 + (L) \int_0^t b(t, X_t)dt + (I) \int_0^t \sigma(t, X_t)dB_t, \tag{4} \]
where \((L) \int_0^t b(t, X_t)dt\) is the set-valued Lebesgue integral and \((I) \int_0^t \sigma(t, X_t)dB_t\) is the set-valued Itô integral. If \( \sigma(t, X_t) \in R^d \otimes R^m \), then we have
\[ X_t = X_0 + (L) \int_0^t b(t, X_t)dt + \int_0^t \sigma(t, X_t)dB_t, \tag{5} \]
where \( \int_0^t \sigma(t, X_t)dB_t \) is the classical Itô integral.

There are few papers about the Itô type set-valued stochastic differential equations, even in the special case (5). But we know there is a paper about the Itô type fuzzy stochastic differential equations. In [7], Hu et al. used Hukuhara difference to define the differentiability and to discuss the Itô type fuzzy stochastic differential equations in the special case \( \sigma(t, X_t) \in R^d \otimes R^m \), i.e. the equation (5). But since it is well-known that the space of all closed subsets of even \( R \) (the space of all real numbers) is not linear with respect to the addition and scalar multiplication, it leads to a big problem: under what conditions does the Hukuhara difference exist? It is a difficult problem so that they simply assume that the Hukuhara difference of a stochastic process at any two different times always exists. In this paper, we shall use selection method to consider the same type problem as in [7] without using the Hukuhara difference. We shall consider the Itô type set-valued stochastic integral equation (5), discuss the existence and uniqueness of its solution. By using level set method [28], we may easily extend the set-valued case to fuzzy set-valued case.

We organize our paper as follows. In Section 2, we introduce some necessary notations, definitions and results about set-valued stochastic processes and set-valued Lebesgue integral, and then we shall prove some new properties, especially inequality of set-valued Lebesgue integrals. In Section 3, we give a set-valued stochastic differential equation of Itô type, and prove the theorem of existence and uniqueness of solution to this kind of set-valued stochastic differential equation.

2 Stochastic Integral of Set-Valued Stochastic Processes and its Properties

Throughout this paper, assume that \((\Omega, \mathcal{A}, \mu)\) is a complete probability space, the \(\sigma\)-field filtration \(\{\mathcal{A}_t : t \in I\}\) satisfies the usual conditions (i.e. containing all null sets, non-decreasing and right continuous), \(I = [0, T]\) with \(T > 0\), \(R\) is the set of all real numbers, \(N\) is the set of all natural numbers, \(R^d\) is the \(d\)-dimensional Euclidean space with usual norm \(\| \cdot \|\), \(\mathcal{B}(E)\) is the Borel field of the space \(E\). Let \(f = \{f(t), \mathcal{A}_t : t \in I\}\) be a \(R^d\)-valued adapted stochastic process. It is said that \(f\) is progressively measurable if for any \(t \in I\), the mapping \((s, \omega) \mapsto f(s, \omega)\) from \([0, t] \times \Omega\) to \(R^d\) is \(\mathcal{B}([0, t]) \times \mathcal{A}_t\)-measurable. If let
\[ \mathcal{C} = \{A \subseteq I \times \Omega : \forall t \in I, A \cap ([0, t] \times \Omega) \in \mathcal{B}([0, t]) \times \mathcal{A}_t\}, \]
then \(f\) is progressively measurable if and only if \(f\) is \(\mathcal{C}\)-measurable. Each right continuous (left continuous) adapted process is progressively measurable.

Assume that \(L^p(R^d)\) \((p \geq 1)\) denotes the set of \(R^d\)-valued stochastic processes \(f = \{f(t), \mathcal{A}_t : t \in I\}\) such that \(f\) satisfying (a) \(f\) is progressively measurable; and (b)
\[ \|f\|_p = \left[ E \left( \int_0^T \|f(t, \omega)\|^p ds \right) \right]^{1/p} < \infty. \tag{6} \]
Let \( f, f' \in L^p(\mathbb{R}^d) \), \( f = f' \) if and only if \( \|f - f'\|_p = 0 \). Then \( (L^p(\mathbb{R}^d), \|\cdot\|_p) \) is complete.

Now we review notation and concepts of set-valued stochastic processes.

Assume that \( \mathbf{K}(R^d) \) is the family of all nonempty, closed subsets of \( R^d \), and \( \mathbf{K}_c(R^d) \) (resp. \( \mathbf{K}_k(R^d) \)) is the family of all nonempty closed convex (resp. compact, compact convex) subsets of \( R^d \). For any \( x \in R^d \), \( A \) is a nonempty subset of \( R^d \), define the distance between \( x \) and \( A \) as \( d(x, A) = \inf_{y \in A} \|x - y\| \).

The Hausdorff metric on \( \mathbf{K}(R^d) \) is defined as

\[
d_H(A, B) = \max\{\sup_{a \in A} d(a, B), \sup_{b \in B} d(b, A)\}
\]

for \( A, B \in \mathbf{K}(R^d) \). For \( B \in \mathbf{K}(R^d) \), define \( \|B\|_H = d_H(\{0\}, B) = \sup_{a \in B} \|a\| \).

For a set-valued random variable \( F \) (cf. \([6], [21]\)), define the set

\[
S^p_F = \{ f \in L^p[\Omega; R^d] : f(\omega) \in F(\omega) \ \text{a.e.} \},
\]

where \( L^p[\Omega; R^d] \) is the set of all \( R^d \)-valued random variables \( f \) such that \( \|f\|_p = [E(\|f\|^p)]^{1/p} < \infty \). The expectation of \( F \) is defined as \( E[F] = \{E[f] : f \in S^p_F\} \). It is called Aumann integrable introduced by Aumann in 1965 (cf. \([2]\)). A set-valued random variable \( F : \Omega \to \mathbf{K}(R^d) \) is called integrable if \( S^p_F \) is non-empty. \( F \) is called \( L^p \)-bounded if \( \int_{\Omega} \|F(\omega)\|^p d\mu < \infty \). Let \( L^p[\Omega; \mathbf{K}(R^d)] \) (resp. \( L^p[\Omega; \mathbf{K}_c(R^d)] \), \( L^p[\Omega; \mathbf{K}_k(R^d)] \)) denote the family of \( \mathbf{K}(R^d) \)-valued (resp. \( \mathbf{K}_c(R^d) \), \( \mathbf{K}_k(R^d) \)-valued) \( L^p \)-bounded random variables. For any two set-valued random variables \( F_1, F_2 \in L^p[\Omega; \mathbf{K}(R^d)] \), define

\[
\Delta_p(F_1, F_2) = \left( \int_{\Omega} \|d_H(F_1(\omega), F_2(\omega))\| d\mu \right)^{1/p},
\]

then \( (L^p[\Omega; \mathbf{K}(R^d)], \Delta_p) \) is a complete space. Concerning more definitions and more results of set-valued random variables, readers could refer to \([6]\) or \([21]\).

**Definition 1** A set-valued stochastic process \( F = \{F(t) : t \in I\} \) is called progressively measurable, if it is \( \mathcal{C} \)-measurable, i.e., for any \( A \in \mathcal{B}(R^d) \), \( \{s, \omega) \in I \times \Omega : F(s, \omega) \cap A \neq \emptyset \} \in \mathcal{C} \). \( F \) is called \( L^p \)-bounded, if the real stochastic process \( \{||F(t)||_K, A_t : t \in I\} \in L^p(R) \).

**Definition 2** A \( R^d \)-valued process \( \{f(t), A_t : t \in I\} \in L^p(R^d) \) is called an \( L^p \)-selection of \( F = \{F(t), A_t : t \in I\} \) if \( f(t, \omega) \in F(t, \omega) \ \text{a.e.} (t, \omega) \in I \times \Omega \).

Let \( S^p(F) \) or \( S^p(F) \) denote the family of all \( L^p \)-selections of \( F = \{F(t), A_t : t \in I\} \), i.e.

\[
S^p(F) = \{ \{f(t)\} \in L^p(R^d) : f(t, \omega) \in F(t, \omega), \ a.e. \ (t, \omega) \in I \times \Omega \}.
\]

Let \( L^p(\mathbf{K}(R^d)) \) denote the set of all \( L^p \)-bounded progressively measurable \( \mathbf{K}(R^d) \)-valued stochastic processes.

Similarly, we have notations \( L^p(\mathbf{K}_c(R^d)) \), \( L^p(\mathbf{K}_k(R^d)) \) and \( L^p(\mathbf{K}_k(R^d)) \). Take \( F_i = \{F_i(t) : t \in I\} \in L^p(\mathbf{K}(R^d)) \), \( i = 1, 2 \), define

\[
\Delta_p(F_1, F_2) = \left( E\left( \int_0^T d_H(F_1(s, \omega), F_2(s, \omega)) ds \right) \right)^{1/p}.
\]

\( F_1 \) and \( F_2 \) are said to be equivalent, if \( \Delta_p(F_1, F_2) = 0 \), denoted by \( F_1 = F_2 \). We have that \( (L^p(\mathbf{K}(R^d)), \Delta_p) \) is complete, \( L^p(\mathbf{K}_c(R^d)) \), \( L^p(\mathbf{K}_k(R^d)) \) and \( L^p(\mathbf{K}_k(R^d)) \) are closed subsets of \( (L^p(\mathbf{K}(R^d)), \Delta_p) \). Denote \( \|F\|_p = \left( E\left( \int_0^T \|F(s)\|_K ds \right) \right)^{1/p} \).

Now we introduce the concept of decomposability.

**Definition 3** A non-empty set \( \Gamma \subseteq L^p(R^d) \) is called decomposable with respect to the progressively measurable \( \sigma \)-field \( \mathcal{C} \), if for any \( f, g \in \Gamma \), any \( U \in \mathcal{C} \), we have \( I_U \cdot f + I_U \cdot g \in \Gamma \).

Firstly, we know that for any set-valued progressively measurable stochastic process \( F \in L^p(\mathbf{K}(R^d)) \), \( S^p(F) \) is decomposable with respect to \( \sigma \)-field \( \mathcal{C} \). Furthermore we have the following Theorem.
There exists a progressively measurable set-valued stochastic process \( \Gamma \) is decomposable with respect to progressively measurable \( \sigma \)-field \( \mathcal{C} \) if and only if there exists a progressively measurable set-valued stochastic process \( F \in \mathcal{L}^p(\mathcal{K}(R^d)) \) such that \( \Gamma = S^p(F) \). Furthermore, \( \Gamma \) is convex if and only if \( F \in \mathcal{L}^p(\mathcal{K}_c(R^d)) \).

Now we consider the integral of set-valued stochastic process. To avoid trouble of dealing with almost every problem, we assume that \( \mathcal{A} \) is \( \mu \)-separable in the following. In this case, for any \( p \geq 1 \), \( \mathcal{L}^p[\mathcal{I} \times \Omega, \mathcal{B}(\mathcal{I}) \times \mathcal{A}, \lambda \times \mu; R^d] \) is a separable Banach space (cf. [31]), \( \mathcal{L}^p(R^d) \) can be considered as its closed subset so that it is separable with respect to \( ||| \cdot |||_p \). Thus, For any \( F = \{ F(t) : t \in I \} \in \mathcal{L}^p(\mathcal{K}(R^d)) \), \( S^p(F) \) is separable. We may ignore almost everywhere problem and assume that the following definition is well-defined for all \((t, \omega) \in I \times \Omega \) rather than for almost everywhere \((t, \omega) \in I \times \Omega \).

**Definition 4** Let a set-valued stochastic process \( F = \{ F(t) : t \in I \} \in \mathcal{L}^p(\mathcal{K}(R^d)) \), \( 1 \leq p < +\infty \). For any \( \omega \in \Omega \), \( t \in I \), define

\[
(A) \int_0^t F(s, \omega)ds := \left\{ \int_0^t f(s, \omega)ds : f \in S^p(F) \right\},
\]

where \( \int_0^t f(s, \omega)ds \) is the Lebesgue integral. \((A) \int_0^t F(s, \omega)ds \) is called the Aumann type Lebesgue integral of set-valued stochastic process \( F \) with respect to time \( t \) introduced in [14]. For any \( 0 \leq u < t < T \),

\[
(A) \int_u^t F(s, \omega)ds := (A) \int_0^t I_{[u,t]}(s)F(s, \omega)ds.
\]

**Remark 1** In the Definition 4, the set of selections is \( S^p(F) \). As a matter of fact, if we only consider the Lebesgue integral, we can use \( S^1(F) \). But we often consider the sum of integral of a set-valued stochastic process with respect to time \( t \) and integral of a set-valued stochastic process with respect to a Brownian motion, where we have to use \( S^p(F) \). Thus we here use \( S^p(F) \) for more general case.

**Remark 2** If a set-valued stochastic process \( F = \{ F(t) : t \in I \} \in \mathcal{L}^p(\mathcal{K}(R^d)) \), then for any \( t \in I \), \( \Gamma(t) := (A) \int_0^t F(s)ds \) is a non-empty subset of \( \mathcal{L}^p[\Omega, \mathcal{A}, \mu; R^d] \). Furthermore, if \( F \in \mathcal{L}^p(\mathcal{K}_c(R^d)) \), then we can prove that \((A) \int_0^t F(s)ds \) is a non-empty convex subset of \( \mathcal{L}^p[\Omega, \mathcal{A}, \mu; R^d] \). However, it is natural to hope that the result of integral is a set-valued stochastic process taking values in \( \mathcal{K}(R^d) \) rather than in \( \mathcal{L}^p[\Omega, \mathcal{A}, \mu; R^d] \). If for any fixed \( t \in I \), let \( \Gamma(t)(\omega) := (A) \int_0^t F(s, \omega)ds \) \( (\omega \in \Omega) \), we also do not know whether \( \Gamma(t)(\omega) \) is a closed subset or not, whether it is measurable or not. So it is necessary to give a new definition so that the integral is still a set-valued stochastic process. Since we can not prove directly that \( \{ \Gamma(t) : t \in I \} \) is decomposable with respect to \( \mathcal{C} \), we firstly give the definition of decomposable closure.

**Definition 5** For any non-empty subset \( \Gamma \subseteq \mathcal{L}^p[\mathcal{I} \times \Omega, \mathcal{C}, \lambda \times \mu; R^d] \), define the decomposable closure \( \overline{\Gamma} \) of \( \Gamma \) with respect to \( \mathcal{C} \) as

\[
\overline{\Gamma} = \left\{ \{ g(t, \omega) : t \in I \} : \text{for any } \varepsilon > 0, \text{there exists a } \mathcal{C}\text{-measurable finite partition } \{ A_1, \ldots, A_n \} \text{ of } I \times \Omega \text{ and } f_1, \ldots, f_n \in \Gamma \text{ such that } \| g - \sum_{i=1}^n I_{A_i} f_i \|_p < \varepsilon \right\}.
\]

**Theorem 2** \((20)\) Assume that \( F = \{ F(t) : t \in I \} \in \mathcal{L}^p(\mathcal{K}(R^d)) \), \( \Gamma(t) := (A) \int_0^t F(s)ds \), then there exists a \( \mathcal{C}\)-measurable set-valued stochastic process \( L(F) = \{ L_t(F) : t \in I \} \in \mathcal{L}^p(\mathcal{K}(R^d)) \) such that \( S^p(L(F)) = \overline{\mathcal{C}\Gamma} \{ \Gamma(t) : t \in I \} \). Furthermore, if \( F \in \mathcal{L}^p(\mathcal{K}_c(R^d)) \), then \( \{ L_t(F) : t \in I \} \in \mathcal{L}^p(\mathcal{K}_c(R^d)) \).

The set-valued stochastic process \( L(F) = \{ L_t(F) : t \in I \} \) defined in Theorem 2 is called the Lebesgue integral of a set-valued stochastic process \( F = \{ F(t) : t \in I \} \in \mathcal{L}^p(\mathcal{K}(R^d)) \) with respect to the time \( t \), and denoted as \( L_t(F) = (L) \int_0^t F(s)ds \).

**Theorem 3** \((20)\) Let \( F = \{ F(t) : t \in I \} \in \mathcal{L}^p(\mathcal{K}(R^d)) \), then there exists a sequence of \( R^d \)-valued stochastic processes \( \{ F^i(t) : t \in I \} : i \geq 1 \} \subseteq \mathcal{S}^p(F) \) such that

\[
F(t, \omega) = \text{cl} \{ F^i(t, \omega) : i \geq 1 \}, \text{ a.e. } (t, \omega) \in I \times \Omega,
\]

and

\[
L_t(F) = \text{cl} \{ \int_0^t F^i(s, \omega)ds : i \geq 1 \} \text{ a.e. } (t, \omega) \in I \times \Omega.
\]
Theorem 4 Let set-valued stochastic process \( \{F(t) : t \in I\} \in \mathcal{L}^2(K(R^d)) \). Then there exists a measurable subset \( A \subseteq I \times \Omega \) with \( (\lambda \times \mu)(A) = 0 \), so that the following holds

\[
L_t(F)(\omega) = \overline{\left\{ L_{t_1}(F)(\omega) + \left( L \int_{t_1}^t F(s, \omega)ds \right) \mid t_1 \leq t \right\}}
\]

for any \((t, \omega), (t_1, \omega) \in I \times \Omega \setminus A, \ t_1 \leq t,\)

where the closure is taken in \( R^d \).

**Proof:** From Theorem 3, there exist a sequence \( \{(f^i(t))_{t \in I} : i = 1, 2, \cdots \} \subseteq S^2(F(\cdot)) \) and a measurable subset \( A \subseteq I \times \Omega \) with \((\lambda \times \mu)(A) = 0\) such that for each \((t, \omega) \in I \times \Omega \setminus A\), we have

\[
F(t, \omega) = \overline{\{ (f^i(t, \omega)) : i = 1, 2, \cdots \}},
\]

and

\[
L_t(F)(\omega) = \overline{\left\{ \int_0^t f^i(s, \omega)ds : i = 1, 2, \cdots \right\}}.
\]

Then for \( 0 \leq t_1 < t \) with \((t_1, \omega) \in I \times \Omega \setminus A\), we have

\[
L_{t_1}(F)(\omega) = \overline{\left\{ \int_0^{t_1} f^i(s, \omega)ds : i = 1, 2, \cdots \right\}},
\]

\[
(L) \int_{t_1}^t F(s, \omega)ds = \overline{\left\{ \int_{t_1}^t f^i(s, \omega)ds : i = 1, 2, \cdots \right\}}.
\]

It is obvious that

\[
L_t(F)(\omega) \subseteq \overline{\left\{ L_{t_1}(F)(\omega) + (L) \int_{t_1}^t F(s, \omega)ds \right\}}.
\]

Conversely, take \( a \in \overline{\{L_{t_1}(F)(\omega) + (L) \int_{t_1}^t F(s, \omega)ds\}}\), by (9) and (10) for any given \( \epsilon > 0 \), we can find \( m(\epsilon), k(\epsilon) \in N \), such that

\[
\left\| a - \left( \int_0^{t_1} f^{m(\epsilon)}(s, \omega)ds + \int_{t_1}^t f^{k(\epsilon)}(s, \omega)ds \right) \right\| < \frac{\epsilon}{2}.
\]

Let \( g(s, \omega) = f^{m(\epsilon)}(s, \omega)I_{[0,t_1]}(s) + f^{k(\epsilon)}(s, \omega)I_{[t_1,t]}(s) \), where \( I_{[0,t_1]}(s) \) and \( I_{[t_1,t]}(s) \) are indicator functions. Then \( \int_0^t g(s, \omega)ds \in L_t(F)(\omega) \). From (8), there exists \( n(\epsilon) \in N \), such that

\[
\left\| \int_0^t f^{n(\epsilon)}(s, \omega)ds - \int_0^t f^{m(\epsilon)}(s, \omega)ds \right\| < \frac{\epsilon}{2}.
\]

By (11) and (12), we obtain

\[
\left\| a - \int_0^t f^{n(\epsilon)}(s, \omega)ds \right\| < \epsilon,
\]

which implies \( a \in L_t(F)(\omega) \). Thus \( L_t(F)(\omega) \subseteq \overline{\{L_{t_1}(F)(\omega) + (L) \int_{t_1}^t F(s, \omega)ds\}} \).

Now we prove an inequality of set-valued Legesgue integrals which will be used in the next section.

Theorem 5 Let set-valued stochastic processes \( F = \{F(t) : t \in I\}, G = \{G(t) : t \in I\} \in \mathcal{L}^2(K(R^d)) \), then there exists a measurable subset \( A \subseteq I \times \Omega \) with \((\lambda \times \mu)(A) = 0\) so that the following holds

\[
d_H^2(L_t(F)(\omega), L_t(G)(\omega)) \leq \int_0^t d_H^2(F(s, \omega), G(s, \omega))ds,
\]

for any \((t, \omega) \in I \times \Omega \setminus A\).

**Proof:** Suppose \( \Phi(t) = \int_0^t F(s)ds, \Psi(t) = \int_0^t G(s)ds \). From Theorem 3, there exist \( \{f^i = \{f^i(t) : t \in I\}, i \geq 1\} \subseteq S^2(F), \{g^j = \{g^j(t) : t \in I\}, j \geq 1\} \subseteq S^2(G) \), and a measurable subset \( A \subseteq I \times \Omega \) with \((\lambda \times \mu)(A) = 0\) such that for each \((t, \omega) \in I \times \Omega \setminus A\),

\[
F(t, \omega) = \overline{\{f^i(t, \omega) : i \geq 1\}}, \quad G(t, \omega) = \overline{\{g^j(t, \omega) : j \geq 1\}}.
\]
and
\[
\Phi(t)(\omega) = \text{cl}\left\{ \int_0^t f^i(s, \omega) ds : i \geq 1 \right\},
\]
\[
\Psi(t)(\omega) = \text{cl}\left\{ \int_0^t g^j(s, \omega) ds : j \geq 1 \right\}.
\]
Hence, we have
\[
\inf_{y \in \mathcal{L}_1(F)(\omega)} \left\| \int_0^t f^i(s, \omega) ds - y \right\|^2 = \inf_{j \geq 1} \left\| \int_0^t f^i(s, \omega) ds - \int_0^t g^j(s, \omega) ds \right\|^2
\]
\[
\leq \inf_{j \geq 1} t \int_0^t \left\| f^i(s, \omega) - g^j(s, \omega) \right\|^2 ds.
\]
Further, we can show along the same arguments as in the proof of [21, Lemma 1.3.12]
\[
\inf_{j \geq 1} \int_0^t \left\| f^i(s, \omega) - g^j(s, \omega) \right\|^2 ds = \int_0^t \inf_{y \in \mathcal{L}_1(G)(\omega)} \left\| f^i(s, \omega) - y \right\|^2 ds
\]
\[
\leq \int_0^t d^2_H(F(s, \omega), G(s, \omega))ds.
\]
Noticing that
\[
\sup_{x \in \mathcal{L}_1(F)(\omega)} \inf_{y \in \mathcal{L}_1(G)(\omega)} \left\| x - y \right\| = \sup_{i \geq 1} \inf_{y \in \mathcal{L}_1(G)(\omega)} \left\| \int_0^t f^i(s, \omega) ds - y \right\|
\]
we obtain
\[
\sup_{x \in \mathcal{L}_1(F)(\omega)} \inf_{y \in \mathcal{L}_1(G)(\omega)} \left\| x - y \right\|^2 \leq t \int_0^t d^2_H(F(s, \omega), G(s, \omega))ds.
\]
Similarly, we have
\[
\sup_{x \in \mathcal{L}_1(G)(\omega)} \inf_{y \in \mathcal{L}_1(F)(\omega)} \left\| x - y \right\|^2 \leq t \int_0^t d^2_H(F(s, \omega), G(s, \omega))ds.
\]
Hence, by the definition of Hausdorff distance, we arrive at the result.

3 The Existence and Uniqueness of the Solution of Itô Type Set-Valued Stochastic Differential Equation

We consider the following Itô type set-valued stochastic differential equation
\[
dF(t) = f(t, F(t))dt + g(t, F(t))dB_t,
\]
where the set-valued stochastic process \( F \in \mathcal{L}^2(K(R^d)) \) with initial condition \( F(0) \) being an \( L^2 \)-bounded set-valued random variable, \( f : I \times K(R^d) \to K(R^d) \) is measurable, \( g : I \times K(R^d) \to R^d \) is measurable, \( B_t \) is an \( m \)-dimensional Brown motion. If \( f \in \mathcal{L}^2(K(R^d)) \) and \( g \in \mathcal{L}^2(R^d \otimes R^m) \), then equation (13) is equivalent to the integral form:
\[
F(t) = F(0) + (L) \int_0^t f(s, F(s)) ds + \int_0^t g(s, F(s)) dB_s.
\]

**Theorem 6 (Existence and uniqueness Theorem)** Assume that \( f(t, F), g(t, F), t \in I, F, F_1, F_2 \in K(R^d) \) satisfy the following conditions:
(i) Linear increasing condition
\[
\| f(t, F) \|_K^2 + \| g(t, F) \|_K^2 \leq K^2(1 + \| F \|_K^2),
\]
where \( K \) is a positive constant.
(ii) Lipschitz continuous condition
\[
d_H(f(t, F_1), f(t, F_2)) + \| g(t, F_1) - g(t, F_2) \| \leq K d_H(F_1, F_2).
\]
Then for any given initial \( L^2 \)-bounded set-valued random variable \( F(0) \), there is a solution to the equation (13), and the solution is unique in the space of \( (\mathcal{L}^2(K(R^d)), \Delta_2) \).
Proof: Without loss of generality, we assume that Theorems 4 and 5 are right for all \(t, t_1\). If \(F \in \mathcal{L}^2(K(R^d))\), then for any \(t \in I\),

\[
E\|f(t, F(t))\|_K^2 + E\|g(t, F(t))\|^2 \leq K^2(1 + E\|F(t)\|_K^2).
\]

We have \(f \in \mathcal{L}^2(K(R^d)), g \in \mathcal{L}^2(R^d \otimes R^n)\).

**Step 1.** We prove the existence by successively approaching. For simplification, we omit the character “(L)” before the symbol of the set-valued Lebesgue integral in the proof of this theorem.

For any \(t \in I\), define

\[
F_0(t) = F(0), \quad F_{n+1}(t) = F(0) + \int_0^t f(s, F_n(s))ds + \int_0^t g(s, F_n(s))dB_s, \quad n \geq 0.
\]

We firstly prove that for any \(n \geq 0\), \(F_n\) is well-defined and satisfies:

(\(\alpha\)) \(F_n \in \mathcal{L}^2(K(R^d))\);

(\(\beta\)) \(\lim_{n \to \infty} Ed_H^2(F_n(t), F_n(s)) = 0\).

For \(n = 0\), it is obviously right. Suppose that \(F_n\) has properties (\(\alpha\)), (\(\beta\)) for any fixed \(n\), we shall prove so does \(F_{n+1}\). Indeed, since \(F_n \in \mathcal{L}^2(K(R^d)), f \in \mathcal{L}^2(K(R^d))\), let

\[
Y(t) := \int_0^t f(s, F_n(s))ds,
\]

we have that \(Y \in \mathcal{L}^2(K(R^d))\) by the definition of set-valued Lebesgue integral. For any \(s, t \in I\), by using triangular inequality and Hölder inequality, we have

\[
\left| E\|F_n(t)\|_K^2 - E\|F_n(s)\|_K^2 \right| \\
= \left| Ed_H^2(F_n(t), 0) - Ed_H^2(F_n(s), 0) \right| \\
\leq E\left| d_H(F_n(t), 0) + d_H(F_n(s), 0) \right| d_H(F_n(t), 0) - d_H(F_n(s), 0) \right| \\
= E[d_H(F_n(t), 0) + d_H(F_n(s), 0)][d_H(F_n(t), 0) - d_H(F_n(s), 0)] \\\n\leq E(d_H(F_n(t), 0) + d_H(F_n(s), 0))d_H(F_n(t), F_n(s)) \\\n\leq \left( E[d_H(F_n(t), 0) + d_H(F_n(s), 0)]^2 Ed_H^2(F_n(t), F_n(s)) \right)^{1/2} \\\n\leq \left( 2 Ed_H^2(F_n(t), 0) + Ed_H^2(F_n(s), 0) Ed_H^2(F_n(t), F_n(s)) \right)^{1/2}. \\
\]

Thus, we know that \(E\|F_n(t)\|_K^2\) is continuous in \(I\) by the assumptions.

By virtue of Theorems 4 and 5 and the assumptions of theorems, we obtain

\[
Ed_H^2(Y(t), Y(s)) = Ed_H^2\left( \int_0^t f(s_1, F_n(s_1))ds_1, \int_0^t f(s_1, F_n(s_1))ds_1 \right) \\
= Ed_H^2\left( \int_0^t f(s_1, F_n(s_1))ds_1 + L_s(f) \right) \\
\leq Ed_H^2\left( \int_0^t f(s_1, F_n(s_1))ds_1, 0 \right) \\
= E\left( \int_s^t \|f(s_1, F_n(s_1))\|_K ds_1 \right)^2 \\
\leq E\left( \int_s^t \|f(s_1, F_n(s_1))\|_K ds_1 \right)^2 \\
\leq (t - s)E\left( \int_s^t \|f(s_1, F_n(s_1))\|_K ds_1 \right)^2 \\
\leq (t - s)E\int_s^t K^2(1 + \|F_n(s_1)\|_K)ds_1.
\]
Thus, \( Y \) satisfies properties (\( \alpha \)) and (\( \beta \)).

Now we investigate the Itô integral part. Since \( g(t, F_n(t)) \) is square integrable, the Itô integral

\[
Z(t) := \int_0^t g(s, F_n(s)) dB_s
\]

exists and \( Z(t) \) is square integrable. By using the properties of the classical Itô integral, we have

\[
E\|Z(t) - Z(s)\|^2 = E\left\| \int_s^t g(s, F_n(s)) dB_s - \int_s^t g(s, F_n(s)) dB_s \right\|^2
\]

\[
= E\| \int_s^t g(s, F_n(s)) dB_s \|^2
\]

\[
= E \int_s^t \| g(s, F_n(s)) \|^2 ds
\]

\[
\leq \int_s^t K^2 (1 + E\|F_n(s)\|^2_\mathcal{K}) ds.
\]

Hence, \( Z \) satisfies (\( \alpha \)), (\( \beta \)). Since

\[
F_{n+1}(t) = F(0) + Y(t) + Z(t),
\]

\( F_{n+1} \) exists and satisfies (\( \alpha \)) and (\( \beta \)).

Next we prove that \( F_n \) converges to \( F \) in \( L^2(\mathcal{K}(\mathbb{R}^d)) \). Let

\[
F_0(t) = F(0),
\]

\[
F_{n+1}(t) = F(t_0) + \int_0^t f(s, F_n(s)) ds + \int_0^t g(s, F_n(s)) dB_s, \quad n \geq 0.
\]

Then due to the property of \( d_H \) and triangular inequality, we have

\[
d_H(F_1(t), F_0(t)) = d_H \left( F_0(t) + \int_0^t f(s, F_0(s)) ds + \int_0^t g(s, F_0(s)) dB_s, F_0(t) \right)
\]

\[
\leq d_H \left( \int_0^t f(s, F_0(s)) ds, 0 \right) + d_H \left( \int_0^t g(s, F_0(s)) dB_s, 0 \right)
\]

\[
= \left\| \int_0^t f(s, F_0(s)) ds \right\|_\mathcal{K} + \left\| \int_0^t g(s, F_0(s)) dB_s \right\|_\mathcal{K},
\]

and for the first part of (15), by the Hölder inequality and the assumptions of theorem, we obtain

\[
E \left\| \int_0^t f(s, F_0(s)) ds \right\|^2_\mathcal{K} \leq E \left( \int_0^t \| f(s, F_0(s)) \|_\mathcal{K} ds \right)^2
\]

\[
\leq t E \left( \int_0^t \| f(s, F_0(s)) \|_\mathcal{K} ds \right)^2
\]

\[
= t E \left( \int_0^t \| f(s, F_0(s)) \|_\mathcal{K} ds \right)^2
\]

\[
\leq t K^2 \int_0^t (1 + E\|F_0(s)\|^2_\mathcal{K}) ds
\]

\[
\leq A^2 t,
\]

where \( A^2 = K^2 (1 + E\|F(0)\|^2_\mathcal{K}) T \). For the second part of (15), from classical Itô isometric property and the assumptions of theorem, we have

\[
E \left\| \int_0^t g(s, F_0(s)) dB_s \right\|^2 \leq E \int_0^t \| g(s, F_0(s)) \|^2 ds
\]

\[
\leq E \int_0^t K^2 (1 + \| F_0(s) \|^2_\mathcal{K}) ds
\]

\[
= t K^2 (1 + E\|F(0)\|^2_\mathcal{K})
\]

\[
\leq B^2 t,
\]

where \( B^2 = K^2 (1 + E\|F(0)\|^2_\mathcal{K}) \). Put (16) (17) into (15), we get

\[
Ed_H^2(F_1(t), F_0(t)) \leq 2E \left( \int_0^t f(s, F_0(s)) ds \right)^2_\mathcal{K} + 2E \left( \int_0^t g(s, F_0(s)) dB_s \right)^2_\mathcal{K}
\]

\[
\leq 2(A + B)^2 t.
\]
By the same way, we have

\[ E\tilde{d}_H^2(F_{n+1}(t), F_n(t)) \]
\[ = E\left( d_H(F(0) + \int_0^t f(s, F_n(s))ds + \int_0^t g(s, F_n(s))dB_s, F(0) \right. \]
\[ + \left. \int_0^t f(s, F_{n-1}(s))ds + \int_0^t g(s, F_{n-1}(s))dB_s \right)^2 \]
\[ \leq E\left[ d_H(F(0), F(0)) + d_H\left( \int_0^t f(s, F_n(s))ds, \int_0^t f(s, F_{n-1}(s))ds \right) \right. \]
\[ + \left. \left\| \int_0^t g(s, F_n(s))dB_s - \int_0^t g(s, F_{n-1}(s))dB_s \right\|^2 \right] \]
\[ \leq 2Ed_H^2\left( \int_0^t f(s, F_n(s))ds, \int_0^t f(s, F_{n-1}(s))ds \right) \]
\[ + 2E\left\| \int_0^t g(s, F_n(s))dB_s - \int_0^t g(s, F_{n-1}(s))dB_s \right\|^2 \]
\[ \leq 2tEd_H^2(f(s, F_n(s)), f(s, F_{n-1}(s)))ds \]
\[ + 2E\int_0^t \| g(s, F_n(s)) - g(s, F_{n-1}(s)) \|^2 ds \]
\[ \leq 2tE\int_0^t K^2d_H^2(F_n(s), F_{n-1}(s))ds + 2E\int_0^t K^2d_H^2(F_n(s), F_{n-1}(s))ds \]
\[ = 2(t + 1)K^2E\int_0^t d_H^2(F_n(s), F_{n-1}(s))ds. \]

Iterating the above process, we obtain

\[ E\tilde{d}_H^2(F_{n+1}(t), F_n(t)) \leq K^{2n}2^{n+1}(A + B)^2(t + 1)T!(n + 1)! \]

then

\[ (E\tilde{d}_H^2(F_{n+1}(t), F_n(t)))^{1/2} \leq [K^{2n}2^{n+1}(A + B)^2(T + 1)^{n+1}(n + 1)!]^{1/2}. \quad (18) \]

Since the sum of the right of (18) is a series which is convergent and not dependent on t, we have that for any \( t \in I, \)

\[ \Delta_2(F_n(t), F_m(t)) = \left( E\tilde{d}_H^2(F_n(t), F_m(t)) \right)^{1/2} \]
\[ \leq \sum_{k=m}^{n-1} \Delta_2(F_{k+1}(t), F_k(t)) \]
\[ = \sum_{k=m}^{n-1} \left( E\tilde{d}_H^2(F_{k+1}(t), F_k(t)) \right)^{1/2} \to 0 \quad (m, n \to \infty), \]

by noticing the triangular inequality with respect to the metric \( \Delta_2. \) Noting for any \( m < n, \Delta_2(F_n(t), F_m(t)) \) is also bounded in \( I. \) Thus, by using Fubini Theorem and bounded dominated theorem, we have

\[ \Delta_2(F_n, F_m) = \left[ E \left( \int_0^T d_H^2(F_n(s, \omega), F_m(s, \omega))ds \right) \right]^{1/2} \to 0, \quad as \ m, n \to \infty, \]

that is, \( F_n \) is convergent to \( F \) in \( L^2(K(R^d)). \) By triangular inequality, we have

\[ E\tilde{d}_H^2(F(s), F(t)) \]


By property (β) and the convergence of $F_n$, we have $\lim_{n \to \infty} Ed_H(F(s), F(t)) = 0$. Thus the solution of the set-valued stochastic integral equation is continuous.

**Step 2.** We prove the uniqueness. Let $F$ and $G$ are two solutions of the equation (14). Similar to the proof of the existence, we have

$$Ed_H^2(F(t), G(t)) \leq 2(t + 1)K^2E \int_0^t d_H^2(F(s), G(s))ds. \quad (19)$$

Since the solutions $F, G \in L^2(K, (\Omega, \mathcal{F}, P))$, we have

$$E \int_0^t d_H^2(F(s), G(s))ds \leq E \int_0^T d_H^2(F(s), G(s))ds \leq 2E \int_0^T (\|F(s)\|_K + \|G(s)\|_K)ds = C^2 < \infty.$$

Together with (19) once, we have

$$Ed_H^2(F(t), G(t)) \leq 2(t + 1)K^2C^2.$$

Together with (19) twice, we have

$$Ed_H^2(F(t), G(t)) \leq 2(t + 1)K^2E \int_0^t d_H^2(F(s), G(s))ds = 2(t + 1)K^2 \int_0^t 2(s + 1)K^2C^2 ds \leq 2^2(t + 1)^2K^2 \int_0^t C^2 ds = (2K^2)^2(t + 1)^2C^2t.$$

Iterating the above process, we get

$$Ed_H^2(F(t), G(t)) \leq K^{2(n+1)}(2(t + 1))^{n+1}C^2T^n \frac{n!}{n!}, \quad (20)$$

Let $n \to \infty$, the right of (20) converges to 0. By using Fubini theorem and classical bounded dominated theorem, we have

$$\Delta_2(F, G) = \left[ E \left( \int_0^T d_H^2(F(s, \omega), F(s, \omega))ds \right) \right]^{1/2} \leq 0,$$

the uniqueness is proved.

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References


