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T-Independence Condition for Fuzzy Random Vector Based on Continuous Triangular Norms*

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Abstract. Min-independence has been proved to be a sufficient condition of a vector of fuzzy random variables to be a fuzzy random vector. The objective of this paper is to study further on the independence condition for fuzzy random vector based on continuous triangular norms. We first discuss measurability criteria for fuzzy random vector, and present two more new equivalent formulations of the measurability criteria. Then, based on the obtained results and T-independence of fuzzy variables, we generalize the independence condition for fuzzy random vector from scenario of minimum triangular norm to that of continuous triangular norms.

Keywords: fuzzy random vector, measurable criteria, triangular norm, independence

1 Introduction

There are two types of uncertainties in the real world — randomness and fuzziness. Two powerful theories, probability theory and possibility theory [2, 9, 19, 20, 21] are used to deal with them, respectively. Based on credibility measure [13], an axiomatic approach, called credibility theory, was proposed by Liu [11], which is an extension of possibility theory, and is theoretical foundation of decision making under possibilistic uncertainty [10]. In a practical decision-making process, we often face a hybrid uncertain environment where fuzzyness and randomness nature coexist. As a combination of credibility theory and probability theory, fuzzy random theory is an appropriate tool to deal with such a twofold uncertainty [11, 17]. In fuzzy random theory, fuzzy random variable is a kernel in this theory, which was introduced by Kwakernaak [6, 7] to depict the phenomena in which fuzziness and randomness appear simultaneously. Since then, its variants as well as extensions were presented by other researchers, aiming at different purposes, e.g., Puri and Ralescu [4], Kruse and Meyer [8], López-Diaz and Gil [18] and Liu and Liu [14].

In a fuzzy random decision making system, we are often required to construct fuzzy random vectors by using various fuzzy random variables. In fuzzy random theory [11], a well known result on fuzzy random vectors is: If $\xi = (\xi_1, \xi_2, \cdots, \xi_n)$ is a fuzzy random vector, then $\xi_i, 1 \leq i \leq n$, are fuzzy random variables. However, the converse proposition may not necessarily hold. That implies, some more conditions should be added to the considered fuzzy random variables so as to construct a fuzzy random vector. Using possibility measure, Feng and Liu [3] established the measurability criteria for fuzzy random vector, and studied the relationship between fuzzy random vectors and fuzzy random variables under the assumption of min-independence. Furthermore, Liu and Wang [16] characterized the measurability of fuzzy random vectors through credibility measure.

Our goal in this paper is to study further on the independence condition for fuzzy random vector based on triangular norms. We aim to generalize the independence condition for fuzzy random vector from scenario of minimum triangular norm to that of continuous triangular norms.

This paper is organized as follows. In Section 2, we recall some basic concepts on fuzzy random vectors. Section 3 introduces T-independence of fuzzy variables. In Section 4, we first discusses the measurability criteria for fuzzy random vector and obtained two more new equivalent formulations of measurability criteria. After that, applying the obtained results, we generalize the min-independence condition for fuzzy random vector to the scenario of continuous triangular norms. Finally, a conclusion is provided in Section 5.

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2 Fuzzy Random Vector

Given a universe Γ, an ample field [19] $\mathcal{A}$ on Γ is a class of subsets of Γ that is closed under arbitrary unions, intersections, and complementation in Γ. Let Pos be a possibility measure defined on the ample field $\mathcal{A}$, a self-dual set function Cr, called *credibility measure*, can be formally defined as follows:

Definition 2.1 ([13]) *Let Γ be a universe, $\mathcal{A}$ an ample field on Γ. The credibility measure, denoted* Cr*, is defined as*

$$\mathrm{Cr}(A) = \frac{1}{2}\left[1 + \mathrm{Pos}(A) - \mathrm{Pos}(A^c)\right], \quad A \in \mathcal{A}, \tag{1}$$

where A^c is the complement of A.

The triplet $(\Gamma, \mathcal{A}, \mathrm{Cr})$ is called a credibility space [12]. A credibility measure has the following properties:

(1) $\mathrm{Cr}(\emptyset)$=0, and $\mathrm{Cr}(\Gamma) = 1$.

(2) Monotonicity: $\mathrm{Cr}(A) \le \mathrm{Cr}(B)$ for all $A, B \subset \Gamma$ with $A \subset B$.

(3) Self-duality: $\mathrm{Cr}(A) + \mathrm{Cr}(A^c) = 1$ for all $A \subset \Gamma$.

(4) Subadditivity: $\mathrm{Cr}(A \cup B) \le \mathrm{Cr}(A) + \mathrm{Cr}(B)$ for all $A, B \subset \Gamma$.

Definition 2.2 *Let $(\Gamma, \mathcal{A}, \mathrm{Cr})$ be a credibility space. A fuzzy vector is a map $X = (X_1, X_2, \cdots, X_n)$ from Γ to $\Re^n$ such that*

$$\{\gamma \in \Gamma \mid X(\gamma) \le t\} \in \mathcal{A} \tag{2}$$

for every $t \in \Re^n$. As $n = 1$, it is called a fuzzy variable.

In credibility theory, possibility distribution of fuzzy vector X is defined as

$$\mu_X(t) = \mathrm{Pos}\{\gamma \mid X(\gamma) = t\} \tag{3}$$

for every $t = (t_1, t_2, \cdots, t_n) \in \Re^n$. A fuzzy vector X is said to be upper semicontinuous (abbreviated by usc) if its possibility distribution $\mu_X(x)$ is usc at every $t \in \Re^n$.

We assume that $(\Omega, \Sigma, \mathrm{Pr})$ is a probability space, and $\mathcal{F}_v^n$ is a collection of fuzzy vectors defined on a credibility space $(\Gamma, \mathcal{A}, \mathrm{Cr})$.

Definition 2.3 ([14]) *A fuzzy random vector is a map $\xi = (\xi_1, \xi_2, \cdots, \xi_n) : \Omega \to \mathcal{F}_v^n$ such that for any closed subset $F \subset \Re^n$, $\mathrm{Pos}\{\gamma \mid \xi_\omega(\gamma) \in F\}$ is Σ-measurable, i.e., for any Borel subset $B \subset [0, 1]$, we have*

$$\{\omega \in \Omega \mid \mathrm{Pos}\{\gamma \mid \xi_\omega(\gamma) \in F\} \in B\} \in \Sigma.$$

As $n = 1$, it is called a fuzzy random variable.

A fuzzy random vector ξ is said to be usc if for each $\omega \in \Omega$, fuzzy vector ξ_ω is usc.

3 T-independence of Fuzzy Variables

A triangular norm (t-norm for short) is a function $T : [0, 1]^2 \to [0, 1]$ such that for any $x, y, z \in [0, 1]$ the following four axioms are satisfied [5]:

($T1$) Commutativity: $T(x, y) = T(y, x)$.

($T2$) Associativity: $T(x, T(y, z)) = T(T(x, y), z)$.

($T3$) Monotonicity: $T(x, y) \le T(x, z)$ whenever $y \le z$.

($T4$) Boundary condition: $T(x, 1) = x$.

The associativity ($T2$) allows us to extend each t-norm T in a unique way to an n-ary operation in the usual way by induction, defining for each n-tuple $(x_1, x_2, \cdots, x_n) \in [0,1]^n$

$$T_{k=1}^n x_k = T(T_{k=1}^{n-1} x_k, x_n) = T(x_1, x_2, \cdots, x_n).$$

Definition 3.1 *Let $X_k, k = 1, 2, \cdots, n$ be n_k-ary fuzzy vectors, T a t-norm. We say $X_1, \cdots, X_n$ are T-independent if*

$$\text{Pos}\{X_1 \in B_1, \cdots, X_n \in B_n\} = T_{k=1}^n \text{Pos}\{X_k \in B_k\} \tag{4}$$

for any sets $B_k \in \Re^{n_k}, k = 1, 2, \cdots, n$.

Furthermore, a sequence $\{X_n\}$ of fuzzy vectors is said to be T-independent if for all $n \geq 2$, fuzzy vectors $X_i, i = 1, 2, \cdots, n$ are T-independent. Particularly, in the above definition, when all $n_k = 1$ for $k = 1, 2, \cdots, n$, the T-independence of fuzzy vectors degenerates to that of fuzzy variables (see Cooman [1]).

Theorem 3.1 *Suppose that T is a right continuous triangular norm, μ_k are the possibility distributions of fuzzy variables $\xi_k, k = 1, 2, \cdots, n$, respectively, and μ is the possibility distribution function of fuzzy vector $(\xi_1, \xi_2, \cdots, \xi_n)$. Then $\xi_1, \xi_2, \cdots, \xi_n$ are T-independent if and only if*

$$\mu(t_1, t_2, \cdots, t_n) = T_{k=1}^n \mu_k \tag{5}$$

for any real numbers $t_1, t_2, \cdots, t_n$.

Proof. *Necessity:* Suppose $\xi_1, \xi_2, \cdots, \xi_n$ are T-independent variables. Then for any real numbers $t_k, k = 1, 2, \cdots, n$, letting $B_k = \{t_k\}, k = 1, 2, \cdots, n$, we have

$$\text{Pos}\{\xi_1 = t_1, \xi_2 = t_2, \cdots, \xi_n = t_n\} = T_{k=1}^n \text{Pos}\{\xi_k = t_k\}$$

which implies (5) is valid. The necessity of the theorem is proved.

Sufficiency: Suppose (5) is valid. Then for any $B_k \subset \Re, k = 1, 2, \cdots, n$, we have

$$\text{Pos}\{\xi_k \in B_k, k = 1, 2, \cdots, n\} = \text{Pos}\left(\bigcap_{k=1}^n \bigcup_{t_k \in B_k} \{\xi_k = t_k\}\right) = \sup_{t_k \in B_k, 1 \leq k \leq n} T_{k=1}^n \text{Pos}\{\xi_k = t_k\}. \tag{6}$$

Therefore, for any $\epsilon > 0$, there is $(\check{t}_1, \check{t}_2, \cdots, \check{t}_n) \in \prod_{k=1}^n B_k$ such that

$$\text{Pos}\{\xi_k \in B_k, k = 1, 2, \cdots, n\} < T_{k=1}^n \text{Pos}\{\xi_k = \check{t}_k\} + \epsilon \leq T_{k=1}^n \text{Pos}\{\xi_k \in B_k\} + \epsilon,$$

By the arbitrary of ϵ, we obtain

$$\text{Pos}\{\xi_k \in B_k, k = 1, 2, \cdots, n\} \leq T_{k=1}^n \text{Pos}\{\xi_k \in B_k\}.$$

On the other hand, for every $\delta > 0$, there is $\check{t}_k \in B_k$ such that $\mu_k(\check{t}_k) + \delta > \text{Pos}\{\xi_k \in B_k\}$. Moreover, by the right continuity of t-norm T, for any $\epsilon > 0$, there exists $\delta > 0$ such that

$$T_{k=1}^n (\mu_k(\check{t}_k) + \delta) \leq T_{k=1}^n \mu_k(\check{t}_k) + \epsilon.$$

As a consequence, we have

$$\begin{aligned} T_{k=1}^n \text{Pos}\{\xi \in B_k\} &< T_{k=1}^n (\mu_k(\check{t}_k) + \delta) \leq T_{k=1}^n \mu_k(\check{t}_k) + \epsilon \\ &= \text{Pos}\{\xi_k = \check{t}_k, k = 1, 2, \cdots, n\} + \epsilon \\ &\leq \text{Pos}\{\xi_k \in B_k, k = 1, 2, \cdots, n\} + \epsilon. \end{aligned}$$

Letting $\epsilon \to 0$, we deduce

$$\text{Pos}\{\xi_k \in B_k, k = 1, 2, \cdots, n\} \geq T_{k=1}^n \text{Pos}\{\xi_k \in B_k\}.$$

The sufficiency of the theorem is proved.

Theorem 3.2 *Let T be a t-norm. The fuzzy variables $X_1, X_2, \cdots, X_n$ are T-independent if and only if*

$$2\mathrm{Cr}\left\{\bigcap_{k=1}^{n}\{X_k \in B_k\}\right\} \wedge 1 = T_{k=1}^{n}[2\mathrm{Cr}\{X_k \in B_k\} \wedge 1] \tag{7}$$

for any subsets $B_1, B_2, \cdots, B_n$ of $\Re$.

Proof. Noting that for any $A \in \mathcal{A}$, we have

$$\mathrm{Pos}(A) = 2\mathrm{Cr}\{A\} \wedge 1.$$

Replacing all $\mathrm{Pos}\{\cdot\}$ in formula (4) by $2\mathrm{Cr}\{\cdot\} \wedge 1$, we proved the theorem.

Remark 3.1 *If the t-norm T in Theorems 3.2 is taken as "min" t-norm, then (7) degenerates to*

$$\mathrm{Cr}\left\{\bigcap_{k=1}^{n}\{X_k \in B_k\}\right\} = \min_{1\le k\le n} \mathrm{Cr}\{\xi_k \in B_k\}$$

for any sets $B_1, B_2, \cdots, B_n$ of $\Re$. That is just the min-independence of fuzzy variables [15, 11].

4 T-independence Condition for Fuzzy Random Vectors

In [3], Feng and Liu established the following measurability criteria for fuzzy random vectors. Based on those criteria they concluded that under min-independence condition, i.e., $\xi_{k,\omega}, k = 1, 2, \cdots, n$ are min-independent fuzzy variables for any $\omega \in \Omega$, if $\xi_k, k = 1, 2, \cdots, n$ are usc fuzzy random variables, then $\xi = (\xi_1, \xi_2, \cdots, \xi_n)$ is a fuzzy random vector. In this section, we will generate the result to the case of T-independence condition, where T is any continuous t-norm.

Theorem 4.1 ([3]) *Let $(\Omega, \Sigma, \mathrm{Pr})$ be a complete probability space, and ξ a map from Ω to usc-$\mathcal{F}_v^n$. Then the following six statements are equivalent:*

(i) ξ is a fuzzy random vector in the sense of Definition 2.3.

(ii) For every open subset $G \subset \Re^n$, $\mathrm{Pos}\{\xi_\omega \in G\}$ is Σ-measurable.

(iii) For every open ball $B(t;r)$ $(t \in \Re^n, r > 0)$, $\mathrm{Pos}\{\xi_\omega \in B(t;r)\}$ is Σ-measurable.

(iv) For every compact set $K \subset \Re^n$, $\mathrm{Pos}\{\xi_\omega \in K\}$ is Σ-measurable.

(v) For each $\alpha \in (0,1]$, ξ^α is a random set from Ω to $\Re^n$.

(vi) For every Borel subset $B \subset \Re^n$, $\mathrm{Pos}\{\gamma \mid \xi_\omega(\gamma) \in B\}$ is Σ-measurable.

In this section, we first derive the following two more measurability criteria for fuzzy random vectors.

Lemma 4.1 *Let $(\Omega, \Sigma, \mathrm{Pr})$ be a complete probability space, and ξ a map from Ω to usc-$\mathcal{F}_v^n$. Then ξ is a fuzzy random vector if and only if for every open-closed interval $I \subset \Re^n$, $\mathrm{Pos}\{\xi_\omega \in I\}$ is Σ- measurable.*

Proof. From Assertion (vi) in Theorem 4.1, the *Necessity* is obviously valid since every interval $I = (c, d] \subset \Re^n$ is a Borel subset of $\Re^n$.

Sufficiency: Since ξ is a map from Ω to usc-$\mathcal{F}_v^n$, for any $\omega \in \Omega$, $\xi_\omega = (\xi_{1,\omega}, \cdots, \xi_{n,\omega})$ is an n-ary fuzzy vector. Note that every open subset $G \subset \Re^n$ can be expressed as the union of at most countable many disjoint open-closed intervals $\{I_k\}$, $G = \cup_{k=1}^{\infty} I_k$, where

$$I_k = \prod_{j=1}^{n}(c_j^k, d_j^k], \quad (c_j^k, d_j^k] \subset \Re.$$

Therefore,

$$\mathrm{Pos}\{\xi_\omega \in G\} = \mathrm{Pos}\left\{\xi_\omega \in \bigcup_{n=1}^{\infty} I_n\right\} = \sup_{n\ge 1} \mathrm{Pos}\left\{\xi_\omega \in I_n\right\}.$$

Since $\mathrm{Pos}\{\xi_\omega \in I\}$ is Σ- measurable, we have $\mathrm{Pos}\{\xi_\omega \in G\}$ is Σ- measurable. Furthermore, by Assertion (ii) in Theorem 4.1, ξ is a fuzzy random vector.

Lemma 4.2 *Let $(\Omega, \Sigma, \Pr)$ be a complete probability space, and ξ a map from Ω to usc-$\mathcal{F}_v^n$. Then ξ is a fuzzy random vector if and only if for every open-closed interval $I \subset \Re^n$, $\mathrm{Cr}\{\xi_\omega \in I\}$ is Σ- measurable.*

Proof. *Necessity*: Suppose that ξ is a fuzzy random vector. For any open-closed interval $I \subset \Re^n$, $\mathrm{Cr}\{\xi_\omega \in I\}$ can be expressed by

$$\mathrm{Cr}\{\xi_\omega \in I\} = \frac{1}{2}[1 + \mathrm{Pos}\{\xi_\omega \in I\} - \mathrm{Pos}\{\xi_\omega \in I^c\}].$$

Noting that I and I^c both are Borel subset of $\Re^n$, by Assertion (vi) in Theorem 4.1, $\mathrm{Cr}\{\xi_\omega \in I\}$ is a Σ-measurable function.

Sufficiency: We note that for any open-closed interval $I \subset \Re^n$, $\mathrm{Pos}\{\xi_\omega \in I\}$ can be written as

$$\mathrm{Pos}\{\xi_\omega \in I\} = 2\mathrm{Cr}\{\xi_\omega \in I\} \wedge 1.$$

Therefore, the measurability of $\mathrm{Cr}\{\xi_\omega \in I\}$ implies that of $\mathrm{Pos}\{\xi_\omega \in I\}$. Furthermore, by Lemma 4.1, ξ is a fuzzy random vector.

Example 4.1 *Assume that Ω is a complete probability space, and C and W are random variables on Ω. Try to testify ξ is a fuzzy random variable, where*

$$\mu_{\xi_\omega}(x) = \exp\left(-\left(\frac{x - C(\omega)}{W(\omega)}\right)^2\right) \quad x \in \Re.$$

We use Lemma 4.1 to testify ξ is a fuzzy random variable. For any open-closed interval $(a, b] \subset \Re$, to testify $\mathrm{Pos}\{\xi_\omega \in (a, b]\}$ is Σ-measurable, it suffices to show the equation

$$\{\omega \in \Omega \mid \mathrm{Pos}\{a < \xi_\omega \leq b\} \geq t\} \in \Sigma$$

holds for any $t \in (0, 1]$. Noting that

$$\begin{aligned}
&\{\omega \in \Omega \mid \mathrm{Pos}\{a < \xi_\omega \leq b\} \geq t\} \\
= &\{\omega \in \Omega \mid [C(\omega) - W(\omega)\sqrt{-\ln t}, C(\omega) + W(\omega)\sqrt{-\ln t}] \cap (a, b] \neq \emptyset\} \\
= &\{\omega \in \Omega \mid C(\omega) - W(\omega)\sqrt{-\ln t} \leq b\} \cap \{\omega \in \Omega \mid C(\omega) - W(\omega)\sqrt{-\ln t} > a\} \in \Sigma,
\end{aligned}$$

we have $\mathrm{Pos}\{\xi_\omega \in (a, b]\}$ is Σ-measurable. Furthermore, since $\mu_{\xi_\omega}(x)$ is continuous, by Lemma 4.1, ξ is a fuzzy random variable.

Theorem 4.2 *Let $\xi_k, k = 1, 2, \cdots, n$ be the fuzzy random variables defined on a complete probability space $(\Omega, \Sigma, \Pr)$. If $\xi_{k,\omega}, k = 1, 2, \cdots, n$ are usc T-independent fuzzy variables for any $\omega \in \Omega$, where T is a continuous t-norm, then $\xi = (\xi_1, \xi_2, \cdots, \xi_n)$ is a fuzzy random vector.*

Proof. From Lemma 4.1, it suffices to prove $\mathrm{Pos}\{\xi_\omega \in I\}$ is Σ- measurable for any open-closed interval $I \subset \Re^n$. Denoting $I = \prod_{k=1}^n J_k = \prod_{k=1}^n (c_k, d_k]$, we have

$$\mathrm{Pos}\{\xi_\omega \in I\} = \mathrm{Pos}\left\{(\xi_{1,\omega}, \xi_{2,\omega}, \cdots, \xi_{n,\omega}) \in \prod_{k=1}^n J_k\right\} = \mathrm{Pos}\left\{\bigcap_{k=1}^n \{\xi_{k,\omega} \in J_k\}\right\}.$$

Note that $\xi_{k,\omega}, k = 1, 2, \cdots, n$ are T-independent fuzzy variables, we have

$$\mathrm{Pos}\{\xi_\omega \in I\} = \mathrm{Pos}\left\{\bigcap_{k=1}^n \{\xi_{k,\omega} \in J_k\}\right\} = T_{k=1}^n \mathrm{Pos}\left\{\xi_{k,\omega} \in J_k\right\}.$$

By Proposition 4.1, we know $\mathrm{Pos}\left\{\xi_{k,\omega} \in J_k\right\}$ is Σ-measurable, this fact together with that every continuous t-norm T is a Borel measurable function deduce that $\mathrm{Pos}\{\xi_\omega \in I\}$ is a Σ-measurable function. This completes the proof of the theorem.

Example 4.2 *Assume that Ω is a complete probability space, and Y, C and W are random variables on Ω. Try to testify (ξ, ζ) is a fuzzy random vector, where*

$$\mu_{\xi_\omega}(x) = \exp\left(-\left(\frac{x - C(\omega)}{W(\omega)}\right)^2\right) \quad x \in \Re,$$

$$\zeta_\omega = (Y(\omega) - 2, Y(\omega), Y(\omega) + 3),$$

and ξ_ω and ζ_ω are mutual T-independent fuzzy variables, T is a continuous t-norm.

By Example 4.1, we know ξ is a fuzzy random variable, similarly, we can testify ζ is a fuzzy random variable. Since ξ_ω and ζ_ω are mutually T-independent under a continuous t-norm, by Theorem 4.2, we can deduce (ξ, ζ) is a fuzzy random vector.

5 Concluding Remarks

In this paper, we studied on the independence condition for fuzzy random vector under continuous triangular norms, and obtained the following new results.

First, we derived two more new measurability criteria for fuzzy random vector.

Second, based on T-independence of fuzzy variables and the obtained results, we discussed the relationship between fuzzy random vectors and fuzzy random variables, and generalized the min-independence condition of fuzzy random vector to T-independence condition in the scenario of continuous triangular norms.

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