Journal of Uncertain Systems
Vol.2, No.1, pp.17-21, 2008
Online at: www.jus.org.uk

Option Pricing Formula for Fuzzy Financial Market

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Received 1 August 2007; Accepted 1 December 2007

Abstract

The option pricing problem is one of central contents in modern finance. In this paper, European option pricing formula is formulated for fuzzy financial market and some mathematical properties of them are discussed. This formula may be regarded as the fuzzy counterpart of Black-Scholes option pricing formula. In addition, some illustrative examples are also documented with MATLAB codes.

Keywords: finance, fuzzy process, option pricing, Liu process

1 Introduction

In the early 1970s, Black and Scholes [2] and, independently, Metron [13] used the geometric Brownian motion to construct a theory for determining the stock options price. The Black-Scholes formula has become an indispensable tool in today's daily financial market practice.

Different from randomness, fuzziness is another type of uncertainty in real world. Since the concept of fuzzy set was initiated by Zadeh [17] via membership function in 1965, fuzzy set theory has been widely applied in practice. In order to measure a fuzzy event, Liu and Liu [12] presented the concept of credibility measure in 2002. Afterward, a sufficient and necessary condition for credibility measure was given by Li and Liu [7]. Credibility theory, founded by Liu [8] in 2004 and refined by Liu [10] in 2007, is a branch of mathematics for studying the behavior of fuzzy phenomena.

In order to deal with the evolution of fuzzy phenomena with time, Liu [11] proposed a fuzzy process, a differential formula and a fuzzy integral. Later, the community renamed them Liu process, Liu formula and Liu integral due to their importance and usefulness, just like Brownian motion, Ito formula and Ito integral. Some researches surrounding the subject have been made. You [16] studied differential and integral of multi-dimensional Liu process. Qin [15] considered some properties of analytic functions of complex Liu process. Dai [3] gave a reflection principle related to Liu process. Peng [14] studied credibilistic stopping problems for fuzzy stock market.

Stochastic financial mathematics was founded based on the assumption that stock price follows geometric Brownian motion. As a different doctrine, Liu [11] presented an alternative assumption that stock price follows geometric Liu process. Moreover, a basic stock model for fuzzy financial market was also proposed in Liu [11]. In this paper, we will call it Liu's stock model in order to differentiate it from Black-Scholes stock model.

Option pricing problem is a fundamental problem in financial market. European options are the most classical and useful options. European option pricing formulas for Liu's stock model are considered in this paper. Some mathematical properties of them are proved and demonstrated to be consistent with the reality.

The rest of this paper is organized as follows. Some basic concepts and properties about Liu process are recalled and the reasonableness of Liu's stock model is interpreted in Section 2. European call and put option price formulas are derived and some properties of them are studied in Sections 3 and 4. Finally, some conclusions are listed.

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2 Liu's Stock Model

In Black-Scholes stock model, there is a cash bond and a risky stock whose price is assumed to follow geometric Brownian motion. Thus, stochastic financial mathematics was founded and has considerably developed in real life. In fuzzy environment, Liu proposed a counterpart of Brownian motion.

Definition 1 *(Liu [11]) A fuzzy process C_t is said to be a Liu process if*
(i) $C_0 = 0$,
(ii) C_t has stationary and independent increments,
(iii) every increment $C_{t+s} - C_s$ is a normally distributed fuzzy variable with expected value et and variance $\sigma^2 t^2$ whose membership function is

$$\mu(x) = 2\left(1 + \exp\left(\frac{\pi|x - et|}{\sqrt{6}\sigma t}\right)\right)^{-1}, \quad -\infty < x < +\infty.$$

Liu process is said to be standard if $e = 0$ and $\sigma = 1$. If C_t is a Liu process, then the fuzzy process $X_t = \exp(C_t)$ is called a geometric Liu process.

An assumption that stock price follows geometric Liu process was presented by Liu [11] for fuzzy financial market. Based on this assumption, Liu's stock model is formulated. This is just a fuzzy counterpart of Black-Scholes stock model [2]. In Liu's stock model, the bond price X_t and the stock price Y_t are assumed to be governed by

$$\begin{cases} \mathrm{d}X_t = rX_t\mathrm{d}t \\ \mathrm{d}Y_t = eY_t\mathrm{d}t + \sigma Y_t \mathrm{d}C_t \end{cases} \tag{1}$$

where r is the riskless interest rate, e is the stock drift, σ is the stock diffusion, and C_t is a standard Liu process. In this model, the market is comprised of a riskless cash bond and a risky tradable stock.

It is reasonable to assume that the stock price follows geometric Liu process. To see this, for any given positive integer n, suppose that Y_{t_k} is the price of some stock at time t_k where $t_k = k/n$ for $k = 0, 1, 2, \cdots, n^2$. Generally speaking, the percentage changes of stock price are independent and identically distributed. Let $Z_{t_k} = Y_{t_k}/Y_{t_{k-1}}$. Then Z_{t_k} are independent and identically distributed for $k = 1, 2, \cdots, n^2$. Obviously,

$$Y_{t_k} = Z_{t_k} Y_{t_{k-1}}.$$

Iterating this equality gives

$$Y_{t_k} = Z_{t_k} Z_{t_{k-1}} \cdots Z_{t_1} Y_{t_0}.$$

Thus, we have

$$\ln(Y_{t_k}) = \sum_{i=1}^{k} \ln(Z_{t_i}) + \ln(Y_{t_0}).$$

Assume that $\ln(Z_{t_k})$ is a normal fuzzy variable for each k. Since $\ln(Z_{t_k})$ are independent and identically distributed for $k = 1, 2, \cdots, n^2$, letting $n \to \infty$, $\ln(Y_{t_k})$ will be approximately a Liu process, and Y_{t_k} will be approximately a geometric Liu process.

3 European Call Option Pricing Formula

A European call option gives the holder the right, but not the obligation, to buy a stock at a specified time for a specified price. Considering Liu's stock model, we assume that a European call option has strike price K and expiration time T. If Y_T is the final price of the underlying stock, then the payoff from buying a European call option is $(Y_T - K)^+$. Considering the time value of money, the present value of this payoff is $\exp(-rT)(Y_T - K)^+$. Therefore, the below definition is reasonable.

Definition 2 *European call option price f for Liu's stock model is defined as*

$$f(Y_0, K, e, \sigma, r) = \exp(-rT)E[(Y_0 \exp(eT + \sigma C_T) - K)^+] \tag{2}$$

where K is the strike price at expiration time T.

In order to calculate this European call option price, we solve Equation (2) and give an integral form as follows:

Theorem 1 *European call option price formula for Liu's stock model is*

$$f(Y_0, K, e, \sigma, r) = Y_0 \exp(-rT) \int_{K/Y_0}^{+\infty} \frac{1}{1 + \exp\left(\frac{\pi}{\sqrt{6}\sigma T}(\ln x - eT)\right)} \mathrm{d}x. \tag{3}$$

Proof: By the definition of expected value of fuzzy variable, we have

$$\begin{aligned} f(Y_0, K, e, \sigma, r) &= \exp(-rT) E[(Y_0 \exp(eT + \sigma C_T) - K)^+] \\ &= \exp(-rT) \int_0^{+\infty} \mathrm{Cr}\{(Y_0 \exp(eT + \sigma C_T) - K)^+ \geq x\} \mathrm{d}x \\ &= \exp(-rT) \int_0^{+\infty} \mathrm{Cr}\{Y_0 \exp(eT + \sigma C_T) - K \geq x\} \mathrm{d}x \\ &= Y_0 \exp(-rT) \int_{K/Y_0}^{+\infty} \mathrm{Cr}\{\exp(eT + \sigma C_T) \geq u\} \mathrm{d}u \\ &= Y_0 \exp(-rT) \int_{K/Y_0}^{+\infty} \frac{\exp\left(\frac{\pi e}{\sqrt{6}\sigma}\right)}{\exp\left(\frac{\pi e}{\sqrt{6}\sigma}\right) + \exp\left(\frac{\pi \ln x}{\sqrt{6}\sigma T}\right)} \mathrm{d}x \\ &= Y_0 \exp(-rT) \int_{K/Y_0}^{+\infty} \frac{1}{1 + \exp\left(\frac{\pi}{\sqrt{6}\sigma T}(\ln x - eT)\right)} \mathrm{d}x. \end{aligned}$$

Theorem 2 *European call option formula $f = f(Y_0, K, e, \sigma, r)$ has the following properties:*
(a). f is an increasing and convex function of Y_0;
(b). f is a decreasing and convex function of K;
(c). f is an increasing function of e;
(d). f is an increasing function of σ;
(e). f is a decreasing function of r.

Proof: (a). This property means that if the other four variables remain unchanged, then the option price is an increasing and convex function of the stock's initial price. To prove it, first note that for any positive constant a, the function $\exp(-rT)(Y_0 a - K)^+$ is an increasing and convex function of Y_0. Consequently, the quantity $\exp(-rT)(Y_0 \exp(et + \sigma C_T) - K)^+$ is increasing and convex in Y_0. Since the credibility distribution of $\exp(eT + \sigma C_T)$ does not depend on Y_0, the desired result is verified.

(b). This follows from the fact that $\exp(-rT)(Y_0 \exp(eT + \sigma C_T) - K)^+$ is decreasing and convex in K. It means that European call option price is a decreasing and convex function of the stock's strike price when the other four variables remain unchanged.

(c). At first, it is obvious that $\exp(\pi(\ln x - eT)/(\sqrt{6}\sigma T))$ is a decreasing function of e. Therefore, the integrand $1/(1 + \exp(\pi(\ln x - eT)/(\sqrt{6}\sigma T)))$ is an increasing function of e. According to the properties of integral, the result is verified. This means that European call option price will increase with the stock drift.

(d). This follows from the fact that the integrand $1/(1 + \exp(\pi(\ln x - eT)/(\sqrt{6}\sigma T)))$ is an increasing function of σ immediately. This property means that European call option price will increase with the stock diffusion.

(e). Since $\exp(-rT)$ is a decreasing function of r and the expected value is independent of r, the result is verified. This means that European call option price will decrease with the riskless interest rate.

In essential, European call option price is a generalized integral. Considering the complexity of the integrand, we can employ numerical integral techniques to calculate it in real life.

Example 1: Suppose that a stock is presently selling for a price of $Y_0 = 30$, the riskless interest rate r is 8% per annum, the stock drift e is 0.06 and the stock diffusion σ is 0.25. We would like to find a European call option price that expires in three months and has a strike price of $K = 34$.

To calculate this European call option price, the following MATLAB codes may be employed in a personal computer:

```
syms x;
y='30*exp(-0.08*0.25)./(1+exp((log(x)-0.06*0.25)*pi/(sqrt(6)*0.25*0.25)))';
f=quad(y,34/30,100)
```

The calculation result shows that $f = 0.1696$. This means the appropriate call option price in the example is about 17 cents.

4 European Put Option Pricing Formula

A European put option gives the holder the right, but not the obligation, to sell a stock at a specified time for a specified price. Suppose that there is a European put option with strike price K and expiration time K in Liu's stock model. If Y_T is the final price of the underlying stock, then the payoff from buying a European put option is $(K - Y_T)^+$. Considering the time value of money, the definition is given as follows:

Definition 3 *European put option price f for Liu's stock model is defined as*

$$f(Y_0, K, e, \sigma, r) = \exp(-rT)E[(K - Y_0 \exp(eT + \sigma C_T))^+] \tag{4}$$

where K is the strike price at expiration time T.

Theorem 3 *European put option price formula for Liu's stock model is*

$$f(Y_0, K, e, \sigma, r) = Y_0 \exp(-rT) \int_0^{K/Y_0} \frac{1}{1 + \exp\left(\frac{\pi}{\sqrt{6}\sigma T}(eT - \ln x)\right)} \mathrm{d}x. \tag{5}$$

Proof: According to the definition of expected value of fuzzy variable, we have

$$\begin{aligned}
f(Y_0, K, e, \sigma, r) &= \exp(-rT)E[(K - Y_0 \exp(eT + \sigma C_T))^+] \\
&= \exp(-rT) \int_0^{+\infty} \mathrm{Cr}\{(K - Y_0 \exp(eT + \sigma C_T))^+ \geq x\} \mathrm{d}x \\
&= \exp(-rT) \int_0^{+\infty} \mathrm{Cr}\{Y_0 \exp(eT + \sigma C_T) \leq K - x\} \mathrm{d}x \\
&= Y_0 \exp(-rT) \int_{-\infty}^{K/Y_0} \mathrm{Cr}\{\exp(eT + \sigma C_T) \leq u\} \mathrm{d}u \\
&= Y_0 \exp(-rT) \int_0^{K/Y_0} \mathrm{Cr}\{\exp(eT + \sigma C_T) \leq u\} \mathrm{d}u \\
&= Y_0 \exp(-rT) \int_0^{K/Y_0} \frac{1}{1 + \exp\left(\frac{\pi}{\sqrt{6}\sigma T}(eT - \ln x)\right)} \mathrm{d}x.
\end{aligned}$$

Theorem 4 *European put option formula $f = f(Y_0, K, e, \sigma, r)$ has the following properties:*
(a). f is a decreasing and convex function of Y_0;
(b). f is an increasing and convex function of K;
(c). f is an increasing function of e;
(d). f is an increasing function of σ;
(e). f is a decreasing function of r.

Proof:(a). It is obvious that the function $\exp(-rT)(K - Y_0 a)^+$ is a decreasing and convex function of Y_0 for any fixed positive constant a. Consequently, the quantity $\exp(-rT)(K - Y_0 \exp(eT + \sigma C_T))^+$ is decreasing and convex in Y_0. Since the credibility distribution of $\exp(eT + \sigma C_T)$ does not depend on Y_0, the desired result is verified. This property means that if the other four variables remain unchanged, European put option price is a decreasing and convex function of the stock's initial price.

(b). This property follows from the fact that $\exp(-rT)(K - Y_0 \exp(eT + \sigma C_T))^+$ is increasing and convex in K. It means that European put option price is an increasing and convex function of the stock's strike price when the other four variables remain unchanged.

(c). It is easy to see that the integrand $1/(1+\exp(\pi(\ln x - eT)/(\sqrt{6}\sigma T)))$ of equation (5) is a decreasing function of e. Thus, the result is verified. This means that European put option price will increase with the stock drift.

(d). This result follows from the fact that the integrand $1/(1+\exp(\pi(\ln x - eT)/(\sqrt{6}\sigma T)))$ is an increasing function of σ. This property means that European put option price will increase with the stock diffusion.

(e). Note that $\exp(-rT)$ is a decreasing function of r and the expected value is dependent of r. The result is verified. This means that European put option price will decrease with the riskless interest rate.

Example 2: Suppose that a stock is presently selling for an initial price $Y_0 = 30$, the riskless interest rate r is 8% per annum, the stock drift e is 0.06 and the stock diffusion σ is 0.25. Find a European put option price that expires in three months and has a strike price of $K = 29$.

The following MATLAB codes may be employed to calculate European put option price:

```
syms x;
y='30*exp(-0.08*0.25)./(1+exp((0.06*0.25-log(x))*pi/(sqrt(6)*0.25*0.25)))';
f=quad(y,0,29/30)
```

The result shows that $f = 0.4109$. This means the appropriate put option price is about 41 cents.

5 Conclusions

In this paper, we investigated the option pricing problems for fuzzy financial market. European call and put option price formulas were defined and computed for Liu's stock model. Some mathematical properties of them were also proved.

Acknowledgments

This work was supported by National Natural Science Foundation of China Grant No.60425309.

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