Fuzzy Process, Hybrid Process and Uncertain Process

Baoding Liu*

Department of Mathematical Sciences

Tsinghua University, Beijing 100084, China

liu@tsinghua.edu.cn   http://orsc.edu.cn/liu

Abstract

This paper first reviews different types of uncertainty. In order to construct fuzzy counterparts of Brownian motion and stochastic calculus, this paper proposes some basic concepts of fuzzy process, including fuzzy calculus and fuzzy differential equation. Those new concepts are also extended to hybrid process and uncertain process. A basic stock model is presented, thus opening up a way to fuzzy financial mathematics.

©2008 World Academic Press, UK. All rights reserved.

Keywords: fuzzy process, fuzzy calculus, fuzzy differential equation, financial mathematics

1 Introduction

Randomness is a basic type of objective uncertainty, and probability theory is a branch of mathematics for studying the behavior of random phenomena. The study of probability theory was started by Pascal and Fermat (1654), and an axiomatic foundation of probability theory was given by Kolmogoroff (1933) in his *Foundations of Probability Theory.*

The concept of fuzzy set was initiated by Zadeh [13] via membership function in 1965. In order to measure a fuzzy event, Liu and Liu [8] introduced the concept of credibility measure in 2002. Li and Liu [5] gave a sufficient and necessary condition for credibility measure in 2006. Credibility theory was founded by Liu [9] in 2004 and refined by Liu [11] in 2007 as a branch of mathematics for studying the behavior of fuzzy phenomena. Credibility theory is deduced from the normality, monotonicity, self-duality, and maximality axioms.

Fuzziness and randomness are two basic types of uncertainty. In many cases, fuzziness and randomness simultaneously appear in a system. In order to describe this phenomena, a fuzzy random variable was introduced by Kwakernaak [3][4] as a random element taking "fuzzy variable" values. In addition, a random fuzzy variable was proposed by Liu [7] as a fuzzy element taking "random variable" values. More generally, a hybrid variable was introduced by Liu [10] as a tool to describe the quantities with fuzziness and randomness. Fuzzy random variable and random fuzzy variable are instances of hybrid variable. In order to measure hybrid events, a concept of chance measure was introduced by Li and Liu [6].

A classical measure is essentially a set function (i.e., a function whose argument is a set) satisfying nonnegativity and countable additivity axioms. Classical measure theory, developed by Borel and Lebesgue around 1900, has been widely applied in both theory and practice. However, the additivity axiom of classical measure theory has been challenged by many mathematicians. The earliest challenge was from the *theory of capacities* by Choquet [2] in which monotonicity and continuity axioms were assumed. Sugeno [12] generalized classical measure theory to *fuzzy measure theory* by replacing additivity axiom with monotonicity and semicontinuity axioms. In order to deal with general uncertainty, "self-duality" plus "countable subadditivity" is much more important than "continuity" and "semicontinuity". For this reason, Liu [11] founded an *uncertainty theory* that is a branch of mathematics based on normality, monotonicity, self-duality, and countable subadditivity axioms.

This paper will review different types of uncertainty, including randomness and fuzziness. This paper also provides some basic concepts of fuzzy process, hybrid process and uncertain process. We develop a fuzzy calculus and propose a fuzzy differential equation. A basic stock model is also presented, thus opening up a way to fuzzy financial mathematics.

---

*Corresponding author. http://orsc.edu.cn/liu. Email: liu@tsinghua.edu.cn (B. Liu)

2 Preliminaries

In this section, we will introduce some useful definitions and properties about random variable, fuzzy variable, hybrid variable and uncertain variable.

2.1 Random Variable

Let Ω be a nonempty set, and $\mathcal{A}$ a σ-algebra over Ω. Each element in $\mathcal{A}$ is called an event. In order to present an axiomatic definition of probability, it is necessary to assign to each event A a number $\Pr\{A\}$ which indicates the probability that A will occur. In order to ensure that the number $\Pr\{A\}$ has certain mathematical properties which we intuitively expect a probability to have, the following three axioms must be satisfied:

Axiom 1. *(Normality)* $\Pr\{\Omega\} = 1$.

Axiom 2. *(Nonnegativity)* $\Pr\{A\} \geq 0$ *for any* $A \in \mathcal{A}$.

Axiom 3. *(Countable Additivity) For every countable sequence of mutually disjoint events* $\{A_i\}$, *we have*

$$\Pr\left\{\bigcup_{i=1}^{\infty} A_i\right\} = \sum_{i=1}^{\infty} \Pr\{A_i\}. \tag{1}$$

Definition 1 *The set function* $\Pr$ *is called a probability measure if it satisfies the normality, nonnegativity, and countable additivity axioms.*

Definition 2 *A random variable is a measurable function from a probability space* $(\Omega, \mathcal{A}, \Pr)$ *to the set of real numbers, i.e., for any Borel set* B *of real numbers, the set* $\{\omega \in \Omega \mid \xi(\omega) \in B\}$ *is an event.*

2.2 Fuzzy Variable

Let Θ be a nonempty set, and let $\mathcal{P}$ be the power set of Θ (i.e., all subsets of Θ). Each element in $\mathcal{P}$ is called an event. In order to present an axiomatic definition of credibility, we accept the following four axioms:

Axiom 1. *(Normality)* $\mathrm{Cr}\{\Theta\} = 1$.

Axiom 2. *(Monotonicity)* $\mathrm{Cr}\{A\} \leq \mathrm{Cr}\{B\}$ *whenever* $A \subset B$.

Axiom 3. *(Self-Duality)* $\mathrm{Cr}\{A\} + \mathrm{Cr}\{A^c\} = 1$ *for any* $A \in \mathcal{P}$.

Axiom 4. *(Maximality)* $\mathrm{Cr}\{\cup_i A_i\} = \sup_i \mathrm{Cr}\{A_i\}$ *for any events* $\{A_i\}$ *with* $\sup_i \mathrm{Cr}\{A_i\} < 0.5$.

Definition 3 *(Liu and Liu [8]) The set function* Cr *is called a credibility measure if it satisfies the normality, monotonicity, self-duality and maximality axioms.*

Now we define a fuzzy variable as a function on a credibility space just as a random variable is defined as a measurable function on a probability space.

Definition 4 *A fuzzy variable is a function from a credibility space* $(\Theta, \mathcal{P}, \mathrm{Cr})$ *to the set of real numbers.*

If a fuzzy variable ξ is defined as a function on a credibility space, then we may get its membership function via

$$\mu(x) = (2\mathrm{Cr}\{\xi = x\}) \wedge 1, \quad x \in \Re. \tag{2}$$

Conversely, if a fuzzy variable ξ is given by a membership function μ, then we may get the credibility value via

$$\mathrm{Cr}\{\xi \in B\} = \frac{1}{2}\left(\sup_{x \in B} \mu(x) + 1 - \sup_{x \in B^c} \mu(x)\right) \tag{3}$$

where B is a set of real numbers.

2.3   Hybrid Variable

Definition 5 *(Liu [10]) Suppose that* $(\Theta, \mathcal{P}, \mathrm{Cr})$ *is a credibility space and* $(\Omega, \mathcal{A}, \mathrm{Pr})$ *is a probability space. The product* $(\Theta, \mathcal{P}, \mathrm{Cr}) \times (\Omega, \mathcal{A}, \mathrm{Pr})$ *is called a chance space.*

The universal set $\Theta \times \Omega$ is clearly the set of all ordered pairs of the form (θ, ω), where $\theta \in \Theta$ and $\omega \in \Omega$. What is the product σ-algebra $\mathcal{P} \times \mathcal{A}$? What is the product measure $\mathrm{Cr} \times \mathrm{Pr}$? Let us discuss these two basic problems.

Definition 6 *(Liu [11]) Let* $(\Theta, \mathcal{P}, \mathrm{Cr}) \times (\Omega, \mathcal{A}, \mathrm{Pr})$ *be a chance space. A subset* $\Lambda \subset \Theta \times \Omega$ *is called an event if* $\Lambda(\theta) = \{\omega \in \Omega \mid (\theta, \omega) \in \Lambda\} \in \mathcal{A}$ *for each* $\theta \in \Theta$.

It has been proved by Liu [11] that the class of all events is a σ-algebra over $\Theta \times \Omega$, and denoted by $\mathcal{P} \times \mathcal{A}$.

Definition 7 *(Li and Liu [6]) Let* $(\Theta, \mathcal{P}, \mathrm{Cr}) \times (\Omega, \mathcal{A}, \mathrm{Pr})$ *be a chance space. Then a chance measure of an event* Λ *is defined as*

$$\mathrm{Ch}\{\Lambda\} = \begin{cases} \sup_{\theta \in \Theta}(\mathrm{Cr}\{\theta\} \wedge \mathrm{Pr}\{\Lambda(\theta)\}), & \text{if } \sup_{\theta \in \Theta}(\mathrm{Cr}\{\theta\} \wedge \mathrm{Pr}\{\Lambda(\theta)\}) < 0.5 \\ 1 - \sup_{\theta \in \Theta}(\mathrm{Cr}\{\theta\} \wedge \mathrm{Pr}\{\Lambda^c(\theta)\}), & \text{if } \sup_{\theta \in \Theta}(\mathrm{Cr}\{\theta\} \wedge \mathrm{Pr}\{\Lambda(\theta)\}) \geq 0.5. \end{cases} \tag{4}$$

It is proved that a chance measure is normal, increasing, and countably subadditive.

Definition 8 *(Liu [10]) A hybrid variable is a measurable function from a chance space* $(\Theta, \mathcal{P}, \mathrm{Cr}) \times (\Omega, \mathcal{A}, \mathrm{Pr})$ *to the set of real numbers, i.e., for any Borel set* B *of real numbers, the set* $\{(\theta, \omega) \in \Theta \times \Omega \mid \xi(\theta, \omega) \in B\}$ *is an event.*

2.4   Uncertain Variable

Let Γ be a nonempty set, and let $\mathcal{L}$ be a σ-algebra over Γ. Each element $\Lambda \in \mathcal{L}$ is called an *event*. In order to present an axiomatic definition of uncertain measure, it is necessary to assign to each event Λ a number $\mathcal{M}\{\Lambda\}$ which indicates the level that Λ will occur. In order to ensure that the number $\mathcal{M}\{\Lambda\}$ has certain mathematical properties, Liu [11] proposed the following four axioms:

Axiom 1. *(Normality)* $\mathcal{M}\{\Gamma\} = 1$.

Axiom 2. *(Monotonicity)* $\mathcal{M}\{\Lambda_1\} \leq \mathcal{M}\{\Lambda_2\}$ *whenever* $\Lambda_1 \subset \Lambda_2$.

Axiom 3. *(Self-Duality)* $\mathcal{M}\{\Lambda\} + \mathcal{M}\{\Lambda^c\} = 1$ *for any event* Λ.

Axiom 4. *(Countable Subadditivity) For every countable sequence of events* $\{\Lambda_i\}$, *we have*

$$\mathcal{M}\left\{\bigcup_{i=1}^{\infty} \Lambda_i\right\} \leq \sum_{i=1}^{\infty} \mathcal{M}\{\Lambda_i\}. \tag{5}$$

.

Definition 9 *(Liu [11]) The set function* $\mathcal{M}$ *is called an uncertain measure if it satisfies the normality, monotonicity, self-duality, and countable subadditivity axioms.*

Definition 10 *(Liu [11]) An uncertain variable is a measurable function from an uncertainty space* $(\Gamma, \mathcal{L}, \mathcal{M})$ *to the set of real numbers, i.e., for any Borel set* B *of real numbers, the set* $\{\gamma \in \Gamma \mid \xi(\gamma) \in B\}$ *is an event.*

2.5   Relations

Probability theory is a branch of mathematics based on the normality, nonnegativity, and countable additivity axioms. In fact, those three axioms may be replaced with four axioms: normality, monotonicity, self-duality, and countable additivity. Thus all of probability, credibility, chance, and uncertain measures meet the normality, monotonicity and self-duality axioms. The essential difference among those measures is how to determine

the measure of union. For any mutually disjoint events $\{A_i\}$ with $\sup_i \pi\{A_i\} < 0.5$, if π satisfies the countable additivity axiom, i.e.,

$$\pi\left\{\bigcup_{i=1}^{\infty} A_i\right\} = \sum_{i=1}^{\infty} \pi\{A_i\}, \tag{6}$$

then π is a probability measure; if π satisfies the maximality axiom, i.e.,

$$\pi\left\{\bigcup_{i=1}^{\infty} A_i\right\} = \sup_{1 \le i < \infty} \pi\{A_i\}, \tag{7}$$

then π is a credibility measure; if π satisfies the countable subadditivity axiom, i.e.,

$$\pi\left\{\bigcup_{i=1}^{\infty} A_i\right\} \le \sum_{i=1}^{\infty} \pi\{A_i\}, \tag{8}$$

then π is an uncertain measure.

Since additivity and maximality are special cases of subadditivity, probability and credibility are special cases of chance measure, and three of them are in the category of uncertain measure. This fact also implies that random variable and fuzzy variable are special cases of hybrid variables, and three of them are instances of uncertain variables.
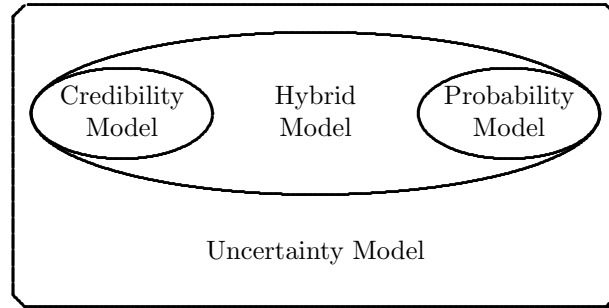

Figure 1: Relations among Uncertainties

3 Fuzzy Process

Definition 11 *Let T be an index set and let $(\Theta, \mathcal{P}, \mathrm{Cr})$ be a credibility space. A fuzzy process is a function from $T \times (\Theta, \mathcal{P}, \mathrm{Cr})$ to the set of real numbers.*

That is, a fuzzy process $X(t, \theta)$ is a function of two variables such that the function $X(t^*, \theta)$ is a fuzzy variable for each t^*. For each fixed θ^*, the function $X(t, \theta^*)$ is called a *sample path* of the fuzzy process. A fuzzy process $X(t, \theta)$ is said to be *sample-continuous* if the sample path is continuous for almost all θ. Instead of longer notation $X(t, \theta)$, sometimes we use the symbol X_t.

Definition 12 *A fuzzy process X_t is said to have independent increments if $X_{t_1} - X_{t_0}$, $X_{t_2} - X_{t_1}$, $\cdots$, $X_{t_k} - X_{t_{k-1}}$ are independent fuzzy variables for any times $t_0 < t_1 < \cdots < t_k$. A fuzzy process X_t is said to have stationary increments if, for any given $t > 0$, the $X_{s+t} - X_s$ are identically distributed fuzzy variables for all $s > 0$.*

Example 1: Assume (i) $X_0 = 0$, (ii) X_t has stationary and independent increments, and (iii) every increment $X_{s+t} - X_s$ is a triangular fuzzy variable (at, bt, ct). Then X_t is a fuzzy process.

Example 2: Assume (i) $X_0 = 0$, (ii) X_t has stationary and independent increments, and (iii) every increment $X_{s+t} - X_s$ is a trapezoidal fuzzy variable (at, bt, ct, dt). Then X_t is a fuzzy process.

Example 3: Assume (i) $X_0 = 0$, (ii) X_t has stationary and independent increments, and (iii) every increment $X_{s+t} - X_s$ is an exponentially distributed fuzzy variable with second moment $m^2 t^2$ whose membership function is

$$\mu(x) = 2 \left(1 + \exp\left(\frac{\pi x}{\sqrt{6}mt} \right) \right)^{-1}, \quad x \geq 0.$$

Then X_t is a fuzzy process.

Example 4: (Fuzzy Renewal Process) Let iid positive fuzzy variables $\xi_1, \xi_2, \cdots$ denote the interarrival times of successive events. Define $S_0 = 0$ and $S_n = \xi_1 + \xi_2 + \cdots + \xi_n$ for $n \geq 1$. Then S_n can be regarded as the waiting time until the occurrence of the nth event. For any $t > 0$, let N_t be the number of renewals in $(0, t]$, i.e.,

$$N_t = \max_{n \geq 0} \left\{ n \mid S_n \leq t \right\}. \tag{9}$$

It is clear that N_t is a fuzzy process, and we call it a *fuzzy renewal process*. Each sample path of N_t is a right-continuous and increasing step function taking only integer values. Furthermore, the size of each jump of N_t is always 1. In other words, N_t has at most one renewal at each time. In particular, N_t does not jump at time 0. Since $N_t \geq n$ if and only if $S_n \leq t$, we have

$$\mathrm{Cr}\{N_t \geq n\} = \mathrm{Cr}\{S_n \leq t\} = \mathrm{Cr}\left\{ \xi_1 \leq \frac{t}{n} \right\}. \tag{10}$$

It is easy to verify that

$$E[N_t] = \sum_{n=1}^{\infty} \mathrm{Cr}\left\{ \xi_1 \leq \frac{t}{n} \right\}. \tag{11}$$

Zhao and Liu [14] proved the following formula,

$$\lim_{t \to \infty} \frac{E[N_t]}{t} = E\left[\frac{1}{\xi_1} \right]. \tag{12}$$

3.1 C Process

Definition 13 *A fuzzy process C_t is said to be a C process if*
(i) $C_0 = 0$,
(ii) C_t has stationary and independent increments,
(iii) every increment $C_{s+t} - C_s$ is a normally distributed fuzzy variable with expected value et and variance $\sigma^2 t^2$, whose membership function is

$$\mu(x) = 2 \left(1 + \exp\left(\frac{\pi|x - et|}{\sqrt{6}\sigma t} \right) \right)^{-1}, \quad x \in \Re.$$

The parameters e and σ are called the *drift* and *diffusion* coefficients, respectively. The C process is said to be *standard* if $e = 0$ and $\sigma = 1$. The C process plays the role of Brownian motion.

Perhaps the readers would like to know why the increment is a normally distributed fuzzy variable. The reason is that a normally distributed fuzzy variable has maximum entropy when its expected value and variance are given, just like a normally distributed random variable.

Theorem 1 *(Existence Theorem) There is a C process that is sample-continuous.*

Sketch of Proof: Without loss of generality, we only prove that there is a standard C process on the range of $t \in [0, 1]$. Let $\left\{ \xi(r) \mid r \text{ represents rational numbers in } [0, 1] \right\}$ be a countable sequence of independently and normally distributed fuzzy variables with expected value zero and variance one. For each integer n, we define a fuzzy process

$$X_n(t) = \begin{cases} \dfrac{1}{n} \displaystyle\sum_{i=1}^{k} \xi\left(\dfrac{i}{n} \right), & \text{if } t = \dfrac{k}{n} \quad (k = 0, 1, \cdots, n) \\[2ex] \text{linear}, & \text{otherwise.} \end{cases}$$

Since the limit $\lim_{n \to \infty} X_n(t)$ exists almost surely, we may verify that the limit meets the conditions of C process and is sample-continuous. Hence there is a standard C process.

3.2 Fuzzy Calculus

Let C_t be a standard C process, and $\mathrm{d}t$ an infinitesimal time interval. Then $\mathrm{d}C_t = C_{t+\mathrm{d}t} - C_t$ is a fuzzy process such that, for each t, the $\mathrm{d}C_t$ is a normally distributed fuzzy variable with

$$E[\mathrm{d}C_t] = 0, \quad V[\mathrm{d}C_t] = \mathrm{d}t^2, \quad E[\mathrm{d}C_t^2] = \mathrm{d}t^2, \quad V[\mathrm{d}C_t^2] \approx 7\mathrm{d}t^4.$$

Definition 14 *Let X_t be a fuzzy process and let C_t be a standard C process. For any partition of closed interval $[a, b]$ with $a = t_1 < t_2 < \cdots < t_{k+1} = b$, the mesh is written as*

$$\Delta = \max_{1 \leq i \leq k} |t_{i+1} - t_i|.$$

Then the fuzzy integral of X_t with respect to C_t is

$$\int_a^b X_t \mathrm{d}C_t = \lim_{\Delta \to 0} \sum_{i=1}^k X_{t_i} \cdot (C_{t_{i+1}} - C_{t_i}) \tag{13}$$

provided that the limit exists almost surely and is a fuzzy variable.

Remark 1: Note that the subscript of X_{t_i} is the left end point of interval $[t_i, t_{i+1}]$. The integral does not remain unchanged if the subscript takes other points, say, the right end point t_{i+1} or the middle point $(t_i + t_{i+1})/2$.

Remark 2: Here the fuzzy integral is defined in the sense of convergence almost surely. In fact, it may also be defined in sense of convergence in mean or in mean square.

Example 5: If C_t is a standard C process, then

$$\int_0^s \mathrm{d}C_t = C_s, \quad \int_0^s t\mathrm{d}C_t = sC_s - \int_0^s C_t \mathrm{d}t, \quad \int_0^s C_t \mathrm{d}C_t = \frac{1}{2}C_s^2, \quad \int_0^s C_t^2 \mathrm{d}C_t = \frac{1}{3}C_s^3.$$

Theorem 2 *Let C_t be a standard C process, and let $h(t, c)$ be a continuously differentiable function. Define $X_t = h(t, C_t)$. Then we have the following chain rule*

$$\mathrm{d}X_t = \frac{\partial h}{\partial t}(t, C_t)\mathrm{d}t + \frac{\partial h}{\partial c}(t, C_t)\mathrm{d}C_t. \tag{14}$$

Proof: Since the function h is continuously differentiable, by using Taylor series expansion, the infinitesimal increment of X_t has a first-order approximation

$$\Delta X_t = \frac{\partial h}{\partial t}(t, C_t)\Delta t + \frac{\partial h}{\partial c}(t, C_t)\Delta C_t.$$

Hence we obtain the chain rule because it makes

$$X_s = X_0 + \int_0^s \frac{\partial h}{\partial t}(t, C_t)\mathrm{d}t + \int_0^s \frac{\partial h}{\partial c}(t, C_t)\mathrm{d}C_t$$

for any $s \geq 0$.

Remark 3: The infinitesimal increment $\mathrm{d}C_t$ in (14) may be replaced with the *derived C process*

$$\mathrm{d}Y_t = u_t \mathrm{d}t + v_t \mathrm{d}C_t \tag{15}$$

where u_t and v_t are absolutely integrable fuzzy processes, thus producing

$$\mathrm{d}h(t, Y_t) = \frac{\partial h}{\partial t}(t, Y_t)\mathrm{d}t + \frac{\partial h}{\partial c}(t, Y_t)\mathrm{d}Y_t. \tag{16}$$

Theorem 3 *(Integration by Parts) Suppose that C_t is a standard C process and $F(t)$ is an absolutely continuous function. Then*

$$\int_0^s F(t)\mathrm{d}C_t = F(s)C_s - \int_0^s C_t\mathrm{d}F(t). \tag{17}$$

Proof: By defining $h(t, C_t) = F(t)C_t$ and using the chain rule, we get $\mathrm{d}(F(t)C_t) = C_t\mathrm{d}F(t) + F(t)\mathrm{d}C_t$. Thus

$$F(s)C_s = \int_0^s \mathrm{d}(F(t)C_t) = \int_0^s C_t\mathrm{d}F(t) + \int_0^s F(t)\mathrm{d}C_t$$

which is just (17).

3.3 Fuzzy Differential Equation

Definition 15 *Suppose C_t is a standard C process, and f and g are some given functions. Then*

$$\mathrm{d}X_t = f(t, X_t)\mathrm{d}t + g(t, X_t)\mathrm{d}C_t \tag{18}$$

is called a fuzzy differential equation. A solution is a fuzzy process X_t that satisfies (18) identically in t.

Example 6: Let C_t be a standard C process. Then the fuzzy differential equation

$$\mathrm{d}X_t = a\mathrm{d}t + b\mathrm{d}C_t$$

has a solution $X_t = at + bC_t$ which is just a C process with drift coefficient a and diffusion coefficient b.

Example 7: Let C_t be a standard C process. Then the fuzzy differential equation

$$\mathrm{d}X_t = aX_t\mathrm{d}t + bX_t\mathrm{d}C_t$$

has a solution $X_t = \exp(at + bC_t)$ which is just a geometric C process.

Example 8: Let C_t be a standard C process. Then the fuzzy differential equations

$$\begin{cases} \mathrm{d}X_t = -Y_t\mathrm{d}C_t \\ \mathrm{d}Y_t = X_t\mathrm{d}C_t \end{cases}$$

have a solution $(X_t, Y_t) = (\cos C_t, \sin C_t)$ which is called a *C process on unit circle* since $X_t^2 + Y_t^2 \equiv 1$.

3.4 A Basic Stock Model

It was assumed that stock price follows geometric Brownian motion, and stochastic financial mathematics was then founded based on this assumption. This paper presents an alternative assumption that stock price follows geometric C process. Based on this assumption, it is expected to reconsider option pricing, optimal stopping, portfolio selection and so on, thus producing a totally new fuzzy financial mathematics.

This paper proposes a basic stock model for fuzzy financial market in which the bond price X_t and the stock price Y_t follow

$$\begin{cases} X_t = X_0\exp(rt) \\ Y_t = Y_0\exp(et + \sigma C_t) \end{cases} \tag{19}$$

or equivalently

$$\begin{cases} \mathrm{d}X_t = rX_t\mathrm{d}t \\ \mathrm{d}Y_t = eY_t\mathrm{d}t + \sigma Y_t\mathrm{d}C_t \end{cases} \tag{20}$$

where r is the riskless interest rate, e is the stock drift, σ is the stock diffusion, and C_t is a standard C process. It is just a fuzzy counterpart of Black-Scholes stock model [1]. This model may also be extended to the cases of multifactor and multi-stock.

3.5   Fuzzy Differential Equation with Jumps

In many cases the stock price is not continuous because of economic crisis or war. In order to incorporate those into stock model, we should develop a fuzzy calculus with jump process. For many applications, a fuzzy renewal process N_t is sufficient. The fuzzy integral of fuzzy process X_t with respect to N_t is

$$\int_a^b X_t \mathrm{d}N_t = \lim_{\Delta \to 0} \sum_{i=1}^k X_{t_i} \cdot (N_{t_{i+1}} - N_{t_i}) = \sum_{a \le t \le b} X_t \cdot (N_t - N_{t-}). \tag{21}$$

Definition 16 *Suppose C_t is a standard C process, N_t is a fuzzy renewal process, and f, g, λ are some given functions. Then*

$$\mathrm{d}X_t = f(t, X_t)\mathrm{d}t + g(t, X_t)\mathrm{d}C_t + \lambda(t, X_t)\mathrm{d}N_t \tag{22}$$

is called a fuzzy differential equation with jumps. A solution is a fuzzy process X_t that satisfies (22) identically in t.

Example 9: Let C_t be a standard C process and N_t a fuzzy renewal process. Then the fuzzy differential equation with jumps

$$\mathrm{d}X_t = a\mathrm{d}t + b\mathrm{d}C_t + c\mathrm{d}N_t$$

has a solution $X_t = at + bC_t + cN_t$ which is just a jump process.

Example 10: Let C_t be a standard C process and N_t a fuzzy renewal process. Then the fuzzy differential equation with jumps

$$\mathrm{d}X_t = aX_t\mathrm{d}t + bX_t\mathrm{d}C_t + cX_t\mathrm{d}N_t$$

has a solution $X_t = \exp(at + bC_t + cN_t)$ which may be employed to model stock price with jumps.

4   Hybrid Process

Definition 17 *Let T be an index set, and $(\Theta, \mathcal{P}, \mathrm{Cr}) \times (\Omega, \mathcal{A}, \mathrm{Pr})$ a chance space. A hybrid process is a measurable function from $T \times (\Theta, \mathcal{P}, \mathrm{Cr}) \times (\Omega, \mathcal{A}, \mathrm{Pr})$ to the set of real numbers, i.e., for each $t \in T$ and any Borel set B of real numbers, the set $\{(\theta, \omega) \in \Theta \times \Omega \mid X(t, \theta, \omega) \in B\}$ is an event.*

That is, a hybrid process $X(t, \theta, \omega)$ is a function of three variables such that the function $X(t^*, \theta, \omega)$ is a hybrid variable for each t^*. For each fixed (θ^*, ω^*), the function $X(t, \theta^*, \omega^*)$ is called a *sample path* of the hybrid process. A hybrid process $X(t, \theta, \omega)$ is said to be *sample-continuous* if the sample path is continuous for almost all (θ, ω). Instead of longer notation $X(t, \theta, \omega)$, sometimes we use the symbol X_t.

Definition 18 *A hybrid process X_t is said to have independent increments if $X_{t_1} - X_{t_0}, X_{t_2} - X_{t_1}, \cdots, X_{t_k} - X_{t_{k-1}}$ are independent hybrid variables for any times $t_0 < t_1 < \cdots < t_k$. A hybrid process X_t is said to have stationary increments if, for any given $t > 0$, the $X_{s+t} - X_s$ are identically distributed hybrid variables for all $s > 0$.*

Example 11: Let X_t be a fuzzy process and let Y_t be a stochastic process. Then $X_t + Y_t$ is a hybrid process.

Example 12: (Hybrid Renewal Process) Let iid positive hybrid variables $\xi_1, \xi_2, \cdots$ denote the interarrival times of successive events. Define $S_0 = 0$ and $S_n = \xi_1 + \xi_2 + \cdots + \xi_n$ for $n \ge 1$. Then S_n can be regarded as the waiting time until the occurrence of the nth event. For any $t > 0$, let N_t be the number of renewals in $(0, t]$, i.e.,

$$N_t = \max_{n \ge 0} \{n \mid S_n \le t\}. \tag{23}$$

It is clear that N_t is a hybrid process, and we call it a *hybrid renewal process*. Each sample path of N_t is a right-continuous and increasing step function taking only integer values. Since $N_t \ge n$ if and only if $S_n \le t$, we have

$$\mathrm{Ch}\{N_t \ge n\} = \mathrm{Ch}\{S_n \le t\}, \quad E[N_t] = \sum_{n=1}^{\infty} \mathrm{Ch}\{S_n \le t\}. \tag{24}$$

4.1 D Process

Definition 19 *Let B_t be a Brownian motion, and let C_t be a C process. Then $D_t = (B_t, C_t)$ is called a D process. The D process is said to be standard if both B_t and C_t are standard.*

Definition 20 *Let B_t be a standard Brownian motion, and let C_t be a standard C process. Then the hybrid process*

$$X_t = et + \sigma_1 B_t + \sigma_2 C_t \tag{25}$$

is called a scalar D process. The parameter e is called the drift coefficient, σ_1 is called the random diffusion coefficient, and σ_2 is called the fuzzy diffusion coefficient.

Definition 21 *Let B_t be a standard Brownian motion, and let C_t be a standard C process. Then the hybrid process $X_t = \exp(et + \sigma_1 B_t + \sigma_2 C_t)$ is called a geometric D process.*

4.2 Hybrid Calculus

Let D_t be a standard D process, and $\mathrm{d}t$ an infinitesimal time interval. Then $\mathrm{d}D_t = D_{t+\mathrm{d}t} - D_t = (\mathrm{d}B_t, \mathrm{d}C_t)$ is a hybrid process.

Definition 22 *Let $X_t = (Y_t, Z_t)$ where Y_t and Z_t are scalar hybrid processes, and let $D_t = (B_t, C_t)$ be a standard D process. For any partition of closed interval $[a, b]$ with $a = t_1 < t_2 < \cdots < t_{k+1} = b$, the mesh is written as*

$$\Delta = \max_{1 \le i \le k} |t_{i+1} - t_i|.$$

Then the hybrid integral of X_t with respect to D_t is

$$\int_a^b X_t \mathrm{d}D_t = \lim_{\Delta \to 0} \sum_{i=1}^k \left(Y_{t_i}(B_{t_{i+1}} - B_{t_i}) + Z_{t_i}(C_{t_{i+1}} - C_{t_i}) \right) \tag{26}$$

provided that the limit exists in mean square and is a hybrid variable.

Remark 4: The hybrid integral may also be written as follows,

$$\int_a^b X_t \mathrm{d}D_t = \int_a^b \left(Y_t \mathrm{d}B_t + Z_t \mathrm{d}C_t \right). \tag{27}$$

Example 13: Let B_t be a standard Brownian motion, and C_t a standard C process. Then

$$\int_0^s \left(\sigma_1 \mathrm{d}B_t + \sigma_2 \mathrm{d}C_t \right) = \sigma_1 B_s + \sigma_2 C_s$$

where σ_1 and σ_2 are constants, random variables, fuzzy variables, or hybrid variables.

Example 14: Let B_t be a standard Brownian motion, and C_t a standard C process. Then

$$\int_0^s (B_t \mathrm{d}B_t + C_t \mathrm{d}C_t) = \frac{1}{2}(B_s^2 - s + C_s^2), \quad \int_0^s (C_t \mathrm{d}B_t + B_t \mathrm{d}C_t) = B_s C_s.$$

Theorem 4 *Let B_t be a standard Brownian motion, C_t a standard C process, and $h(t, b, c)$ a twice continuously differentiable function. Define $X_t = h(t, B_t, C_t)$. Then we have the following chain rule*

$$\mathrm{d}X_t = \frac{\partial h}{\partial t}(t, B_t, C_t)\mathrm{d}t + \frac{\partial h}{\partial b}(t, B_t, C_t)\mathrm{d}B_t + \frac{\partial h}{\partial c}(t, B_t, C_t)\mathrm{d}C_t + \frac{1}{2}\frac{\partial^2 h}{\partial b^2}(t, B_t, C_t)\mathrm{d}t. \tag{28}$$

Proof: Since the function h is twice continuously differentiable, by using Taylor series expansion, the infinitesimal increment of X_t has a second-order approximation

$$\Delta X_t = \frac{\partial h}{\partial t}(t, B_t, C_t)\Delta t + \frac{\partial h}{\partial b}(t, B_t, C_t)\Delta B_t + \frac{\partial h}{\partial c}(t, B_t, C_t)\Delta C_t$$

$$+ \frac{1}{2}\frac{\partial^2 h}{\partial t^2}(t, B_t, C_t)(\Delta t)^2 + \frac{1}{2}\frac{\partial^2 h}{\partial b^2}(t, B_t, C_t)(\Delta B_t)^2 + \frac{1}{2}\frac{\partial^2 h}{\partial c^2}(t, B_t, C_t)(\Delta C_t)^2$$

$$+ \frac{\partial^2 h}{\partial t \partial b}(t, B_t, C_t)\Delta t \Delta B_t + \frac{\partial^2 h}{\partial t \partial c}(t, B_t, C_t)\Delta t \Delta C_t + \frac{\partial^2 h}{\partial b \partial c}(t, B_t, C_t)\Delta B_t \Delta C_t.$$

Since we can ignore the terms $(\Delta t)^2$, $(\Delta C_t)^2$, $\Delta t \Delta B_t$, $\Delta t \Delta C_t$, $\Delta B_t \Delta C_t$ and replace $(\Delta B_t)^2$ with Δt, the chain rule is obtained because it makes

$$X_s = X_0 + \int_0^s \frac{\partial h}{\partial t}\mathrm{d}t + \int_0^s \frac{\partial h}{\partial b}\mathrm{d}B_t + \int_0^s \frac{\partial h}{\partial c}\mathrm{d}C_t + \frac{1}{2}\int_0^s \frac{\partial^2 h}{\partial b^2}\mathrm{d}t$$

for any $s \geq 0$.

Remark 5: The infinitesimal increments $\mathrm{d}B_t$ and $\mathrm{d}C_t$ in (28) may be replaced with the *derived D process*

$$\mathrm{d}Y_t = u_t \mathrm{d}t + v_{1t}\mathrm{d}B_t + v_{2t}\mathrm{d}C_t \tag{29}$$

where u_t and v_{2t} are absolutely integrable hybrid processes, and v_{1t} is a square integrable hybrid process, thus producing

$$\mathrm{d}h(t, Y_t) = \frac{\partial h}{\partial t}(t, Y_t)\mathrm{d}t + \frac{\partial h}{\partial b}(t, Y_t)\mathrm{d}Y_t + \frac{1}{2}\frac{\partial^2 h}{\partial b^2}(t, Y_t)v_{1t}^2\mathrm{d}t. \tag{30}$$

4.3 Hybrid Differential Equation

Definition 23 *Suppose B_t is a standard Brownian motion, C_t is a standard C process, and f, g_1, g_2 are some given functions. Then*

$$\mathrm{d}X_t = f(t, X_t)\mathrm{d}t + g_1(t, X_t)\mathrm{d}B_t + g_2(t, X_t)\mathrm{d}C_t \tag{31}$$

is called a hybrid differential equation. A solution is a hybrid process X_t that satisfies (31) identically in t.

Example 15: Let B_t be a standard Brownian motion, and let $\tilde{a}$ and $\tilde{b}$ be two fuzzy variables. Then the hybrid differential equation $\mathrm{d}X_t = \tilde{a}\mathrm{d}t + \tilde{b}\mathrm{d}B_t$ has a solution $X_t = \tilde{a}t + \tilde{b}B_t$. The hybrid differential equation $\mathrm{d}X_t = \tilde{a}X_t\mathrm{d}t + \tilde{b}X_t\mathrm{d}B_t$ has a solution

$$X_t = \exp\left(\left(\tilde{a} - \frac{\tilde{b}^2}{2}\right)t + \tilde{b}B_t\right).$$

Example 16: Let C_t be a standard C process, and let ξ and η be two random variables. Then the hybrid differential equation $\mathrm{d}X_t = \xi\mathrm{d}t + \eta\mathrm{d}C_t$ has a solution $X_t = \xi t + \eta C_t$. The hybrid differential equation $\mathrm{d}X_t = \xi X_t\mathrm{d}t + \eta X_t\mathrm{d}C_t$ has a solution $X_t = \exp(\xi t + \eta C_t)$.

Example 17: Let B_t be a standard Brownian motion, and C_t a standard C process. Then the hybrid differential equation

$$\mathrm{d}X_t = a\mathrm{d}t + b\mathrm{d}B_t + c\mathrm{d}C_t$$

has a solution $X_t = at + bB_t + cC_t$ which is just a scalar D process.

Example 18: Let B_t be a standard Brownian motion, and C_t a standard C process. Then the hybrid differential equation

$$\mathrm{d}X_t = aX_t\mathrm{d}t + bX_t\mathrm{d}B_t + cX_t\mathrm{d}C_t$$

has a solution

$$X_t = \exp\left(\left(a - \frac{b^2}{2}\right)t + bB_t + cC_t\right)$$

which is just a geometric D process.

4.4 A Basic Stock Model

This paper assumes that stock price follows geometric D process, and presents a basic stock model in which the bond price X_t and the stock price Y_t are determined by

$$\begin{cases} \mathrm{d}X_t = rX_t\mathrm{d}t \\ \mathrm{d}Y_t = eY_t\mathrm{d}t + \sigma_1 Y_t\mathrm{d}B_t + \sigma_2 Y_t\mathrm{d}C_t \end{cases} \tag{32}$$

where r is the riskless interest rate, e is the stock drift, σ_1 is the random stock diffusion, σ_2 is the fuzzy stock diffusion, B_t is a standard Brownian motion, and C_t is a standard C process. This model may also be extended to the cases of multifactor and multi-stock.

4.5 Hybrid Differential Equation with Jumps

Let N_t be a hybrid renewal process. Then the hybrid integral of hybrid process X_t with respect to N_t is

$$\int_a^b X_t\mathrm{d}N_t = \lim_{\Delta \to 0} \sum_{i=1}^k X_{t_i} \cdot (N_{t_{i+1}} - N_{t_i}) = \sum_{a \le t \le b} X_t \cdot (N_t - N_{t-}). \tag{33}$$

Definition 24 *Suppose B_t is a standard Brownian motion, C_t is a standard C process, N_t is a hybrid renewal process, and f, g_1, g_2, λ are some given functions. Then*

$$\mathrm{d}X_t = f(t, X_t)\mathrm{d}t + g_1(t, X_t)\mathrm{d}B_t + g_2(t, X_t)\mathrm{d}C_t + \lambda(t, X_t)\mathrm{d}N_t \tag{34}$$

is called a hybrid differential equation with jumps. A solution is a hybrid process X_t that satisfies (34) identically in t.

Example 19: Let B_t be a standard Brownian motion, C_t a standard C process, and N_t a hybrid renewal process. Then the hybrid differential equation

$$\mathrm{d}X_t = a\mathrm{d}t + b\mathrm{d}B_t + c\mathrm{d}C_t + \lambda\mathrm{d}N_t$$

has a solution $X_t = at + bB_t + cC_t + \lambda N_t$ which is just a jump process.

Example 20: Let B_t be a standard Brownian motion, C_t a standard C process, and N_t a hybrid renewal process. Then the hybrid differential equation

$$\mathrm{d}X_t = aX_t\mathrm{d}t + bX_t\mathrm{d}B_t + cX_t\mathrm{d}C_t + \lambda N_t$$

has a solution $X_t = \exp\left(\left(a - \frac{b^2}{2}\right)t + bB_t + cC_t + \lambda N_t\right)$ which may be employed to model stock price with jumps.

5 Uncertain Process

Definition 25 *Let T be an index set and let $(\Gamma, \mathcal{L}, \mathcal{M})$ be an uncertainty space. An uncertain process is a measurable function from $T \times (\Gamma, \mathcal{L}, \mathcal{M})$ to the set of real numbers, i.e., for each $t \in T$ and any Borel set B of real numbers, the set $\{\gamma \in \Gamma \mid X(t, \gamma) \in B\}$ is an event.*

That is, an uncertain process $X(t, \gamma)$ is a function of two variables such that the function $X(t^*, \gamma)$ is an uncertain variable for each t^*. For each fixed γ^*, the function $X(t, \gamma^*)$ is called a *sample path* of the uncertain process. An uncertain process $X(t, \gamma)$ is said to be *sample-continuous* if the sample path is continuous for almost all γ. Instead of longer notation $X(t, \gamma)$, sometimes we use the symbol X_t.

Definition 26 *An uncertain process X_t is said to have independent increments if $X_{t_1} - X_{t_0}, X_{t_2} - X_{t_1}, \cdots, X_{t_k} - X_{t_{k-1}}$ are independent uncertain variables for any times $t_0 < t_1 < \cdots < t_k$. An uncertain process X_t is said to have stationary increments if, for any given $t > 0$, the increments $X_{s+t} - X_s$ are identically distributed uncertain variables for all $s > 0$.*

Example 21: (Uncertain Renewal Process) Let iid positive uncertain variables $\xi_1, \xi_2, \cdots$ denote the interarrival times of successive events. Define $S_0 = 0$ and $S_n = \xi_1 + \xi_2 + \cdots + \xi_n$ for $n \geq 1$. Then S_n can be regarded as the waiting time until the occurrence of the nth event. For any $t > 0$, let N_t be the number of renewals in $(0, t]$, i.e.,

$$N_t = \max_{n \geq 0} \left\{ n \mid S_n \leq t \right\}. \tag{35}$$

It is clear that N_t is an uncertain process, and we call it an *uncertain renewal process*. Each sample path of N_t is a right-continuous and increasing step function taking only integer values. It is easy to verify that

$$\mathcal{M}\{N_t \geq n\} = \mathcal{M}\{S_n \leq t\}, \quad E[N_t] = \sum_{n=1}^{\infty} \mathcal{M}\{S_n \leq t\}. \tag{36}$$

5.1   Canonical Process

Definition 27 *An uncertain process W_t is said to be a canonical process if*
(i) $W_0 = 0$ and W_t is sample-continuous,
(ii) W_t has stationary and independent increments,
(iii) W_1 is an uncertain variable with expected value 0 and variance 1.

Theorem 5 *(Existence Theorem) There is a canonical process.*

Proof: In fact, standard Brownian motion and standard C process are instances of canonical process.

Theorem 6 *Let W_t be a canonical process. Then $E[W_t] \equiv 0$.*

Proof: Let $f(t) = E[W_t]$. Then for any times t_1 and t_2, by using the property of stationary and independent increments, we have

$$f(t_1 + t_2) = E[W_{t_1+t_2}] = E[W_{t_1+t_2} - W_{t_2} + W_{t_2} - W_0] = E[W_{t_1}] + E[W_{t_2}] = f(t_1) + f(t_2)$$

which implies that there is a constant e such that $f(t) = et$. The theorem is proved via $f(1) = 0$.

Definition 28 *Let W_t be a canonical process. Then $et + \sigma W_t$ is called a derived canonical process, and the uncertain process $X_t = \exp(et + \sigma W_t)$ is called a geometric canonical process.*

5.2   Uncertain Calculus

Let W_t be a canonical process, and $\mathrm{d}t$ an infinitesimal time interval. Then

$$\mathrm{d}W_t = W_{t+\mathrm{d}t} - W_t$$

is an uncertain process with $E[\mathrm{d}W_t] = 0$ and $\mathrm{d}t^2 \leq E[\mathrm{d}W_t^2] \leq \mathrm{d}t$.

Definition 29 *Let X_t be an uncertain process and let W_t be a canonical process. For any partition of closed interval $[a, b]$ with $a = t_1 < t_2 < \cdots < t_{k+1} = b$, the mesh is written as*

$$\Delta = \max_{1 \leq i \leq k} |t_{i+1} - t_i|.$$

Then the uncertain integral of uncertain process X_t with respect to W_t is

$$\int_a^b X_t \mathrm{d}W_t = \lim_{\Delta \to 0} \sum_{i=1}^{k} X_{t_i} \cdot (W_{t_{i+1}} - W_{t_i}) \tag{37}$$

provided that the limit exists in mean square and is an uncertain variable.

Remark 6: Note that the subscript of X_{t_i} is the left end point of interval $[t_i, t_{i+1}]$. The integral does not remain unchanged if the subscript takes other points, say, the right end point t_{i+1} or the middle point $(t_i + t_{i+1})/2$.

Theorem 7 *Let W_t be a canonical process, and let $h(t, w)$ be a twice continuously differentiable function. Define $X_t = h(t, W_t)$. Then we have the following chain rule*

$$\mathrm{d}X_t = \frac{\partial h}{\partial t}(t, W_t)\mathrm{d}t + \frac{\partial h}{\partial w}(t, W_t)\mathrm{d}W_t + \frac{1}{2}\frac{\partial^2 h}{\partial w^2}(t, W_t)\mathrm{d}W_t^2. \tag{38}$$

Proof: Since the function h is twice continuously differentiable, by using Taylor series expansion, the infinitesimal increment of X_t has a second-order approximation

$$\Delta X_t = \frac{\partial h}{\partial t}(t, W_t)\Delta t + \frac{\partial h}{\partial w}(t, W_t)\Delta W_t + \frac{1}{2}\frac{\partial^2 h}{\partial w^2}(t, W_t)(\Delta W_t)^2 + \frac{1}{2}\frac{\partial^2 h}{\partial t^2}(t, W_t)(\Delta t)^2 + \frac{\partial^2 h}{\partial t \partial w}(t, W_t)\Delta t \Delta W_t.$$

Since we can ignore the terms $(\Delta t)^2$ and $\Delta t \Delta W_t$, the chain rule is proved because it makes

$$X_s = X_0 + \int_0^s \frac{\partial h}{\partial t}(t, W_t)\mathrm{d}t + \int_0^s \frac{\partial h}{\partial w}(t, W_t)\mathrm{d}W_t + \frac{1}{2}\int_0^s \frac{\partial^2 h}{\partial w^2}(t, W_t)\mathrm{d}W_t^2$$

for any $s \geq 0$.

Remark 7: The infinitesimal increment $\mathrm{d}W_t$ in (38) may be replaced with the derived canonical process

$$\mathrm{d}Y_t = u_t\mathrm{d}t + v_t\mathrm{d}W_t \tag{39}$$

where u_t is an absolutely integrable uncertain process, and v_t is a square integrable uncertain process, thus producing

$$\mathrm{d}h(t, Y_t) = \frac{\partial h}{\partial t}(t, Y_t)\mathrm{d}t + \frac{\partial h}{\partial w}(t, Y_t)\mathrm{d}Y_t + \frac{1}{2}\frac{\partial^2 h}{\partial w^2}(t, Y_t)v_t^2\mathrm{d}W_t^2. \tag{40}$$

Theorem 8 *(Integration by Parts) Suppose that W_t is a canonical process and $F(t)$ is an absolutely continuous function. Then*

$$\int_0^s F(t)\mathrm{d}W_t = F(s)W_s - \int_0^s W_t\mathrm{d}F(t). \tag{41}$$

Proof: By defining $h(t, W_t) = F(t)W_t$ and using the chain rule, we get

$$\mathrm{d}(F(t)W_t) = W_t\mathrm{d}F(t) + F(t)\mathrm{d}W_t.$$

Thus

$$F(s)W_s = \int_0^s \mathrm{d}(F(t)W_t) = \int_0^s W_t\mathrm{d}F(t) + \int_0^s F(t)\mathrm{d}W_t$$

which is just (41).

5.3 Uncertain Differential Equation

Definition 30 *Suppose W_t is a canonical process, and f and g are some given functions. Then*

$$\mathrm{d}X_t = f(t, X_t)\mathrm{d}t + g(t, X_t)\mathrm{d}W_t \tag{42}$$

is called an uncertain differential equation. A solution is an uncertain process X_t that satisfies (42) identically in t.

Example 22: Let W_t be a sample-continuous uncertain process. Then the uncertain differential equation $\mathrm{d}X_t = a\mathrm{d}t + b\mathrm{d}W_t$ has a solution $X_t = at + bW_t$.

5.4 Uncertain Differential Equation with Jumps

Let N_t be an uncertain renewal process. Then the uncertain integral of uncertain process X_t with respect to N_t is

$$\int_a^b X_t \mathrm{d}N_t = \lim_{\Delta \to 0} \sum_{i=1}^k X_{t_i} \cdot (N_{t_{i+1}} - N_{t_i}) = \sum_{a \le t \le b} X_t \cdot (N_t - N_{t-}). \tag{43}$$

Definition 31 *Suppose W_t is a canonical process, N_t is an uncertain renewal process, and f, g, λ are some given functions. Then*

$$\mathrm{d}X_t = f(t, X_t)\mathrm{d}t + g(t, X_t)\mathrm{d}W_t + \lambda(t, X_t)\mathrm{d}N_t \tag{44}$$

is called an uncertain differential equation with jumps. A solution is an uncertain process X_t that satisfies (44) identically in t.

We may assume that stock price follows geometric canonical process with jumps, and obtain the following stock model in which the bond price X_t and the stock price Y_t are determined by

$$\begin{cases} \mathrm{d}X_t = rX_t \mathrm{d}t \\ \mathrm{d}Y_t = eY_t \mathrm{d}t + \sigma Y_t \mathrm{d}W_t + \lambda Y_t \mathrm{d}N_t \end{cases} \tag{45}$$

where r is the riskless interest rate, e is the stock drift, σ is the stock diffusion, and λ is the jump coefficient.

Acknowledgments

References

[1] Black, F. and M. Scholes, The pricing of option and corporate liabilities, *Journal of Political Economy*, vol.81, pp.637-654, 1973.

[2] Choquet, G., Theory of capacities, *Annals de l'Institute Fourier*, vol.5, pp.131-295, 1954.

[3] Kwakernaak, H., Fuzzy random variables–I: Definitions and theorems, *Information Sciences*, vol.15, pp.1-29, 1978.

[4] Kwakernaak, H., Fuzzy random variables–II: Algorithms and examples for the discrete case, *Information Sciences*, vol.17, pp.253-278, 1979.

[5] Li, X. and B. Liu, A sufficient and necessary condition for credibility measures, *International Journal of Uncertainty, Fuzziness & Knowledge-Based Systems*, vol.14, no.5, pp.527-535, 2006.

[6] Li, X. and B. Liu, Chance measure for hybrid events with fuzziness and randomness, *Soft Computing*, to be published.

[7] Liu, B., *Theory and Practice of Uncertain Programming*, Physica-Verlag, Heidelberg, 2002.

[8] Liu, B. and Y. K. Liu, Expected value of fuzzy variable and fuzzy expected value models, *IEEE Transactions on Fuzzy Systems*, vol.10, no.4, pp.445-450, 2002.

[9] Liu, B., *Uncertainty Theory*, Springer-Verlag, Berlin, 2004.

[10] Liu, B., A survey of credibility theory, *Fuzzy Optimization and Decision Making*, vol.5, no.4, pp.387-408, 2006.

[11] Liu, B., *Uncertainty Theory*, 2nd ed., Springer-Verlag, Berlin, 2007.

[12] Sugeno, M., *Theory of Fuzzy Integrals and its Applications*, Ph.D. Dissertation, Tokyo Institute of Technology, 1974.

[13] Zadeh, L. A., Fuzzy sets, *Information and Control*, vol.8, pp.338-353, 1965.

[14] Zhao, R. and B. Liu, Renewal process with fuzzy interarrival times and rewards, *International Journal of Uncertainty, Fuzziness & Knowledge-Based Systems*, vol.11, no.5, pp.573-586, 2003.