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Bivariate Credibility-Copulas

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Abstract

In this paper, based on the review of the conditional distribution of fuzzy variable under Liu's [4] non-classical credibility measure theory (i.e., $(\vee,\cdot)$-credibility measure theory), we propose the concept of fuzzy copula, called as credibility-copula, and explore its mathematical properties and data-assimilation of the copula parameter.

1 Introduction

Since Liu (2004) and his colleagues proposed an axiomatic foundation for modeling fuzzy phenomena, fuzzy credibility measure theory has entered a new phase. The credibility measure possesses self-duality property. Fuzzy variable concept and its distribution, which are parallel to those in probability theory, can be developed naturally on the foundation of fuzzy credibility measure theory.

On the one hand, we note that the concept of covariance is established based on fuzzy credibility measure theory for describing the association between two fuzzy variables in a way similar to that in probability theory, which is merely a measure of linear association between fuzzy variables and has limited applications. On the other hand, we note that in probability theory, copula theory has obtained great attention and enjoyed very wide applications, particularly in option pricing, portfolio selection, risk management, and in geo-statistical modeling, etc. Therefore, we believe that it is necessary to explore the counterpart in fuzzy theory of (probabilistic) copula, named as credibility copula, on the credibility measure theoretical platform.

(Probabilistic) copula function is, in a nutshell, a joint distribution function with uniform marginals. This allows us to apply any form of distributions to the marginals since they can be transformed into uniform distributions. Furthermore, a complete association structure can then be specified by choice of a copula. In other words, the study of the association between two random variables and their marginal cumulative distributions can be completely separated. This means that it is free to choose any parametric family of distributions for the marginals, or even choose to derive an empirical probability distribution function.

In this paper, based on Liu's [4] non-classical credibility measure theory, i.e., $(\vee,\cdot)$-credibility measure theory, we explore the basic property of uniform-distributed fuzzy variable and explore joint uniform distribution for the fuzzy bivariate variables. Then, we propose the concept of credibility copula on the ground of $(\vee,\cdot)$-credibility measure theory, for the characterization of the full association between fuzzy variables. Finally, we explore a decomposition of credibility copula function into product copula and an adjusted association function.

2 A Review of $(\vee,\cdot)$ -Credibility Measure Theory

Let Θ be a nonempty set, and 2^{Θ} the power set on Θ. Each element, say, $A\subset\Theta$, $A\in 2^{\Theta}$ is called an event. A number denoted as $\mathrm{Cr}\{A\}$, $0\le \mathrm{Cr}\{A\}\le 1$, is assigned to event $A\in 2^{\Theta}$, which indicates the credibility grade that event A occurs. $\mathrm{Cr}\{A\}$ satisfies following axioms (Liu, [4]):

Axiom 1. $\mathrm{Cr}\{\Theta\}=1$.

Axiom 2. Cr *is non-decreasing, i.e.,* $\mathrm{Cr}\{A\}\le \mathrm{Cr}\{B\}$ *whenever* $A\subset B$.

Axiom 3. Cr *is self-dual, i.e.,* $\mathrm{Cr}\{A\}+\mathrm{Cr}\{A^c\}=1$ *for any* $A\in 2^{\Theta}$.

Axiom 4. $\mathrm{Cr}\{\bigcup_i A_i\}\wedge 0.5=\sup_i\left[\mathrm{Cr}\{A_i\}\right]$ *for any* $\{A_i\}$ *with* $\mathrm{Cr}\{A_i\}\ \le\ 0.5$.

Axiom 5. *Let set functions* $\mathrm{Cr}_k:2^{\Theta_k}\to[0,1]$ *satisfy Axioms 1-4 and let* $\Theta=\Theta_1\times\Theta_2\times\cdots\times\Theta_p$ *.Then*

$$\mathrm{Cr}\left(\theta_1,\theta_2,\cdots,\theta_p\right)=\begin{cases}\frac{1}{2}\prod_{k=1}^{p}\left(2\mathrm{Cr}_k\left(\theta_k\right)\wedge 1\right) & \text{if } \min_{1\le k\le p}\left\{\mathrm{Cr}_k\left(\theta_k\right)\right\}<0.5\\ \min_{1\le k\le p}\left\{\mathrm{Cr}_k\left(\theta_k\right)\right\} & \text{if } \min_{1\le k\le p}\left\{\mathrm{Cr}_k\left(\theta_k\right)\right\}\ge 0.5\end{cases} \tag{1}$$

for each $\left(\theta_1,\theta_2,\cdots,\theta_p\right)\in\Theta$. *In this case, we write* $\mathrm{Cr}=\mathrm{Cr}_1\times\mathrm{Cr}_2\times\cdots\times\mathrm{Cr}_p$.

Definition 2.1. (Liu [3]) *Any set function* $\mathrm{Cr}:2^{\Theta}\to[0,1]$ *satisfies Axioms 1-4 is called a* $(\vee,\cdot)$*-credibility measure. The triple* $\left(\Theta,2^{\Theta},\mathrm{Cr}\right)$ *is called the* $(\vee,\cdot)$*-credibility measure space.*

Definition 2.2. (Liu [3]) *The (induced) membership function of a fuzzy variable* ξ *on* $\left(\Theta,2^{\Theta},\mathrm{Cr}\right)$ *is*

.
$$\mu(x)=\left(2\mathrm{Cr}\{\xi=x\}\right)\wedge 1,\quad x\in\mathbb{R}. \tag{2}$$

Conversely, for given membership function the credibility measure is determined by the credibility inversion theorem.

Theorem 2.3 (Liu [3]) *Let* ξ *be a fuzzy variable with a membership function* μ. *Then for* $\forall B\subset\mathbb{R}$,

$$\mathrm{Cr}\{\xi\in B\}=\frac{1}{2}\left(\sup_{x\in B}\mu(x)+1-\sup_{x\in B^c}\mu(x)\right). \tag{3}$$

Corollary 2.4 (Liu [3]) *Let* ξ *be a fuzzy variable on* $\left(\Theta,2^{\Theta},\mathrm{Cr}\right)$ *with a membership function* μ *.Then the credibility measure of event* $\{\theta:\xi(\theta)=x\}$ *is*

$$\mathrm{Cr}\{\xi=x\}=\frac{1}{2}\left(\mu(x)+1-\sup_{x\ne x}\mu(x)\right),\ \forall x\in\mathbb{R}. \tag{4}$$

That is, the credibility distribution $\Phi(x)$ is the accumulated credibility grade that the fuzzy variable ξ takes a value less than or equal to a real-valued number $x\in\mathbb{R}$. Generally speaking, a credibility distribution Φ is neither left-continuous nor right-continuous.

Theorem 2.5. (Liu [3]) *Let ξ be a fuzzy variable on $(\Theta, 2^{\Theta}, \mathrm{Cr})$ with membership function μ. Then its credibility distribution is defined by*

$$\Phi(x) = \frac{1}{2}\left(\sup_{y \le x} \mu(y) + 1 - \sup_{y > x} \mu(y)\right), \ \forall x \in \mathbb{R} . \tag{5}$$

Definition 2.6 *Let Φ be the credibility distribution of the fuzzy variable ξ. A function $\phi : \mathbb{R} \to [0, +\infty)$ is called a credibility density function of a fuzzy variable if*

$$\Phi(x) = \int_{-\infty}^{x} \phi(y) \mathrm{d}y, \quad \forall x \in \mathbb{R} . \tag{6}$$

We now consider the credibility of an event A after it has been learned that some other event B has occurred. This new credibility of A is called the conditional credibility of the event A given B, denoted by $\mathrm{Cr}\{A \mid B\}$.

Definition 2.7 (Liu [4]) Let $(\Theta, 2^{\Theta}, \mathrm{Cr})$ *be a credibility measure space and let* $\forall A, B \in 2^{\Theta}$. *Then the conditional credibility measure of the event A given B, $\mathrm{Cr}\{A \mid B\}$ is*

$$\mathrm{Cr}\{A \mid B\} = \frac{1}{2}\left(\frac{(2\mathrm{Cr}\{A \cap B\}) \wedge 1}{(2\mathrm{Cr}\{B\}) \wedge 1} + 1 - \frac{(2\mathrm{Cr}\{A^c \cap B\}) \wedge 1}{(2\mathrm{Cr}\{B\}) \wedge 1}\right) \tag{7}$$

provided $\mathrm{Cr}\{B\} > 0$.

3 Concepts and Properties of Bivariate Credibility Copula

In this section, we will explore the basic property of uniform distributed fuzzy variable and then the credibility distribution transformation on the ground of $(\vee, \cdot)$-credibility measure.

3.1 Uniform Distributed Fuzzy Variable

In probabilistic context, uniform distribution and the distribution transformation play very important roles. Let us explore whether similar developments exist on the ground of the credibility measure theory. We say a fuzzy variable υ to be (standard) uniform distributed if its credibility density takes the form

$$\phi_{\upsilon}(x) = \begin{cases} 1, & 0 \le x \le 1 \\ 0, & \text{otherwise.} \end{cases} \tag{8}$$

The credibility distribution of fuzzy standard uniform variable υ is

$$\Phi_{\upsilon}(x) = \begin{cases} 0, & x < 0 \\ x, & 0 \le x < 1 \\ 1, & x \ge 1 . \end{cases} \tag{9}$$

And thus the induced membership function for the fuzzy standard uniform variable υ is

$$\mu_{\upsilon}(x)=\begin{cases}2x, & 0\le x<0.5\\ 2-2x, & 0.5\le x\le 1\\ 0, & \text{otherwise.}\end{cases} \tag{10}$$

Conversely, the isosceles triangle membership located at (0,0), (0.5,1), and (1,0) characterizes a fuzzy standard uniform-distributed variable.

3.2 Credibility Distribution Transformation

Let ξ be a fuzzy variable with credibility distribution function $\Phi_{\xi}(\cdot)$. We are interested in the distribution for fuzzy variable $\Phi_{\xi}(\xi)$. Recall that the *L-R* membership function for fuzzy variable ξ can be expressed by the credibility distribution in a form as

$$\mu_{\xi}(\xi)=\begin{cases}2\Phi_{\xi}(\xi), & \text{if } \Phi_{\xi}(\xi)<0.5\\ 1, & \text{if } \lim_{y\uparrow\xi}\Phi_{\xi}(y)<0.5\le\Phi_{\xi}(\xi)\\ 2-2\Phi_{\xi}(\xi), & \text{if } 0.5\le\lim_{y\uparrow\xi}\Phi_{\xi}(y).\end{cases} \tag{11}$$

Then for any given value $x=\Phi_{\xi}(\xi)$, $\xi=\Phi^{-1}(x)$, thus we obtain the membership function for $X=\Phi_{\xi}(\xi)$

$$\mu_{\upsilon}(x)=\begin{cases}2x, & 0\le x<0.5\\ 2-2x, & 0.5\le x\le 1\\ 0, & \text{otherwise.}\end{cases} \tag{12}$$

In other words, the transformation of a fuzzy variable by its credibility distribution will result in a fuzzy standard (i.e., unit) uniform distributed variable.

3.3 Definition of Bivariate Credibility-Copula

Similar to (probabilistic) copula theory [5], copula is a dependence index measuring a couple of two fuzzy variables. Let X and Y be two continuous fuzzy variable, with credibility distributions Φ_X and Φ_Y, respectively, and $H_{XY}(\cdot,\cdot)$ be their joint credibility distribution. Let $\mathrm{I}=[0,1]$.

Definition 3.1 *A bivariate credibility copula is a function* $C: I\times I\rightarrow I$ *such that*
(i) $C(0,x)=C(x,0)=0$ *and* $C(1,x)=C(x,1)=x$ *for all* $x\in \mathrm{I}$ *;*
(ii) $C(\cdot,\cdot)$ *is 2-increasing: for* $\forall a,b,c,d\in\mathbb{I}$, $a<b$, *and* $c<d$ *,*

$$\nu_C([a,b]\times[c,d])=C(b,d)-C(a,d)-C(b,c)+C(a,c)\ge 0 \tag{13}$$

The function ν_C is called the C-volume of the rectangle $[a,b]\times[c,d]$.

In other words, a credibility-copula is the restriction to the unit square $I\times I=[0,1]\times[0,1]$ of a bivariate credibility distribution function whose margins are standard uniforms.

More formally, a copula C induces a credibility measure on $I\times I$ in term of ν_C, i.e.,

$$\nu_C\left([0,u]\times[0,v]\right)=C(u,v)\,. \tag{14}$$

Sklar's theorem plays a fundamental role in probabilistic copula theory, see Nelsen [5]. Now let us state the version of Sklar's theorem on the credibility measure theoretical platform.

Theorem 3.2: *Let H be an absolutely continuous bivariate-credibility distribution function with absolutely continuous marginal credibility distributions Φ_X and Φ_Y respectively. Then there exists a credibility copula C such that $H(x,y)=C(\Phi_X(x),\Phi_Y(y))$. Conversely, for any absolutely continuous credibility distribution functions Φ_X and Φ_Y and any credibility-copula C, the function H defined above is a two-dimensional credibility distribution with margins Φ_X and Φ_Y,respectively. Furthermore, if Φ_X and Φ_Y are absolutely continuous, then the credibility-copula is unique.*

Based on the fuzzy credibility measure based Sklar's theorem, given a absolutely continuous two-dimensional credibility distribution $H(x,y)$ and absolutely continuous marginal credibility distributions Φ_X and Φ_Y, the credibility-copula is

$$C(u,v)=H\left(\Phi_X^{-1}(u),\Phi_Y^{-1}(v)\right), \tag{15}$$

where the inverse of the credibility distribution denoted by $F^{-1}(\cdot)$ is defined as

$$F^{-1}(u)=\sup\left\{x \mid F(x)\le u\right\}. \tag{16}$$

By noticing the credibility distribution transformation arguments in Subsection *B*, $U=\Phi_X(X)$ and $V=\Phi_Y(Y)$ are fuzzy standard uniform-distributed variables on $[0,1]$, the credibility-copula is the joint uniform credibility distribution of two uniform-distributed fuzzy variables on $[0,1]\times[0,1]$. Therefore, credibility copula fully characterizes the functional association between two fuzzy variables.

3.4 Properties of Bivariate Credibility-Copulas

If $C(u,v)$, in nature a joint credibility uniform distribution, possesses a joint credibility density, denoted by $\partial C(u,v)/\partial u\partial v$, then we say $C(u,v)$ is absolutely continuous. Given a two-dimensional credibility distribution $H(x,y)$ and marginal credibility distributions Φ_X and Φ_Y, then it can be shown that

$$\max\left\{\Phi_X(x)+\Phi_Y(y)-1,0\right\}\le H(x,y)\le \min\left\{\Phi_X(x),\Phi_Y(y)\right\} \tag{17}$$

or, equivalently

$$\varpi(u,v)=\max\left\{u+v-1,0\right\}\le C(u,v)\le \min\left\{u,v\right\}=\omega(u,v)\,. \tag{18}$$

It can also be shown that $\varpi(u,v)$, $\omega(u,v)$, and $\chi(u,v)=uv$ are all credibility copulas. We also note that for continuous fuzzy variables X and Y (i) if and only if any one of X and Y is an increasing function of the other, then $C(u,v)=\omega(u,v)$; (ii) if and only if any one of X and Y is a decreasing function of the other, then $C(u,v)=\varpi(u,v)$; and (iii) if and only if X and Y are independent, then $C(u,v)=\chi(u,v)=uv$ (which is referred to as *product copula*).

3.5 Archimedean Family of Bivariate Credibility-Copulas

Archimedean copula family [5] plays an important role in (probabilistic) copula theory and applications. Paralleling to (probabilistic) copula theory, the Archimedean credibility copula is defined as a function C: $I\times I\to I$ defined by the generator ϕ such that

$$C(u,v)=\phi^{[-1]}\left(\phi(u)+\phi(v)\right), \tag{19}$$

where $\phi^{[-1]}$ is called as the pseudo-inverse of ϕ :

$$\phi^{[-1]}(t)=\begin{cases}\phi^{-1}(t), & t\in[0,\phi(0)]\\ 0, & t>\phi(0).\end{cases} \tag{20}$$

The generator $\phi: \mathrm{I}\to[0,\infty]$, is a strictly monotone decreasing and continuous convex function. An important property of Archimedean copula family is the associativity: i.e., $C(C(u,v),w)=C(u,C(v,w))$. A simple example of Archimedean copula is the generator with a form $\ln\left((1-\theta(1-t))/t\right)$ and the generating copula is

$$C(u,v)=uv\big/\left(1-\theta(1-u)(1-v)\right). \tag{21}$$

4 A Decomposition Of Bivariate Credibility-Copulas

Recently, Anjos [1] pointed out that in probabilistic context there exists a local association measure, which provides explicit and precise information of the underlying association structure and helps to reformulate bivariate distribution and associated copula. Now let us explore the parallel developments on the ground of credibility theory.

Definition 4.1 *(Expected value and variance) Let* X *be a fuzzy variable with credibility distribution* Φ_X *. If*

$$\lim_{x\to-\infty}\Phi_X(x)=0,\quad \lim_{x\to\infty}\Phi_X(x)=1 \tag{22}$$

and the Lebesgue-Stieltjes integral $\int_{-\infty}^{\infty} x\mathrm{d}\Phi_X(x)$ *is finite, then we define* $\int_{-\infty}^{\infty} x\mathrm{d}\Phi_X(x)$ *as the expected value of fuzzy variable* X *and denoted as* $E[X]$. *Furthermore, we define* $E\left[(X-E[X])^2\right]$ *as the variance of fuzzy variable and denoted by* $V[X]$.

Definition 4.2 *(Covariance) Let two fuzzy variables* X *and* Y *have a bivariate credibility distribution* $H(x,y)$, *and the two marginal credibility distributions* Φ_X *and* Φ_Y *with finite expected values* $E[X]$ *and* $E[Y]$ *,respectively. Then*

$$E\left[(X-E[X])(Y-E[Y])\right] \text{ and } E\left[(X-E[X])(Y-E[Y])\right]\Big/\left(\sqrt{V[X]}\sqrt{V[Y]}\right)$$

is called the covariance and correlation of fuzzy variables X *and* Y *,respectively.*

Remark 4.3 *Under certain conditions, it can be shown that*

$$E\left[\left(X-E[X]\right)\left(Y-E[Y]\right)\right]=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(x-E[X]\right)\left(y-E[Y]\right)dH(x,y) \tag{23}$$

Definition 4.4 *Let two fuzzy variables X and Y have a bivariate credibility distribution $H(x,y)$, and the two marginal credibility distributions Φ_X and Φ_Y. The underlying Spearman function ρ_H is the correlation coefficient between indicator $\vartheta\{\Phi_X(X)\le\Phi_X(x)\}$ of fuzzy event $\{\Phi_X(X)\le\Phi_X(x)\}$ and indicator $\vartheta\{\Phi_Y(Y)\le\Phi_Y(y)\}$ of fuzzy event $\{\Phi_Y(Y)\le\Phi_Y(y)\}$ for each $(x,y)\in[-\infty,\infty]\times[-\infty,\infty]$,*

$$\rho_H(x,y)=\frac{H(x,y)-\Phi_X(x)\Phi_Y(y)}{\sqrt{\Phi_X(x)\Phi_Y(y)\left(1-\Phi_X(x)\right)\left(1-\Phi_Y(y)\right)}}, \tag{24}$$

where the indicator function is defined as usually

$$\vartheta_A(x)=\begin{cases}1, & \text{if } x\in A\\ 0, & \text{otherwise.}\end{cases} \tag{25}$$

Similarly, for copulas

$$\rho_C(u,v)=\frac{C(u,v)-uv}{\sqrt{uv(1-u)(1-v)}}. \tag{26}$$

Theorem 4.5 *Let two fuzzy variables X and Y have a bivariate credibility distribution $H(x,y)$, and the two marginal credibility distributions Φ_X and Φ_Y with associated copula $C(u,v)$.Then*

$$C(u,v)=uv+\rho_C(u,v)\sqrt{uv(1-u)(1-v)}, \tag{27}$$

and

$$H(x,y)=\Phi_X(x)\Phi_Y(y)+\rho_H(x,y)\sqrt{\Phi_X(x)\Phi_Y(y)\left(1-\Phi_X(x)\right)\left(1-\Phi_Y(y)\right)}. \tag{28}$$

Furthermore,

$$\rho_H\left(\Phi_X(x),\Phi_Y(y)\right)=\rho_C(u,v) \tag{29}$$

for all $(x,y)\in[-\infty,\infty]\times[-\infty,\infty]$ such that $u=\Phi_X(x)$ and $v=\Phi_Y(y)$.

Remark 4.6 *As a common sense, the product copula, $\chi(u,v)$, represents non-association or independence. The value of product copula, $\chi(u,v)$, measures the degree of non-association. The alternative representation of copula reveals a fundamental fact that the degree of association between two fuzzy variables is the sum of the product copula factor $\chi(u,v)=uv$ and an adjusted local association factor, $D(u,v)$. In this sense, copula is a quantity composed of two factors: non-association measure and* *association measure. Therefore, the difference between copula,* $C(u,v)$ *and the product copula,* $\chi(u,v)$, $D(u,v)=C(u,v)-\chi(u,v)$ *, measures the true degree of association, i.e., the degree apart from independence. While* ρ_C *is the standardized true degree of association in a local sense, i.e., at about given point* $(u,v)\in I\times I$. *The decomposition of a copula help us a refined understanding of the measure of dependence. For example, let* $C(u,v)=uv/(1-\theta(1-u)(1-v))$, *then* $D(u,v)=uv\left(1-1/(1-\theta(1-u)(1-v))\right)$ *measures the pure bivariate association. Another example is let* $C(u,v)=uv+\theta uv(1-u)(1-v)$, $|\theta|\le 1$, *then* $D(u,v)=uv\theta(1-u)(1-v)$ *and* $\rho_C(u,v)=\theta\sqrt{uv(1-u)(1-v)}$.

5 A Kernel Estimation of 1-Dimensinal Credibility Distribution

Finding an estimated fuzzy credibility distribution based on observed data on a fuzzy variable is a very critical task in practices because a serious researcher has to defend an important principle – *objectiveness*. In other words, we should establish the fuzzy credibility distribution in terms of data information collected objectively from the fuzzy variable itself. We will explore a nonparametric approach – kernel estimation under maximum entropy principle in this section.

5.1 Review on Kernel Estimation in Statistics

A kernel is a function $K(x)=c\kappa\left(\|x\|^2\right)$ mapping from $\mathbb{R}^d$ to $[0,\infty)$, where $\kappa(\cdot)$ is a piecewise nonnegative monotone decreasing function such that $\int_0^\infty \kappa(r)dr<\infty$ and c is a constant.

Two common kernels used in statistics are Gaussian kernel, $\left(1/\sqrt{2\pi}\right)e^{-\|x\|^2/2}$, and Epanechikov kernel, $(3/4)\left(1-\|x\|^2\right)$. For finite support, both kernel reduced to $K(x/h)=0$, if $\|x\|>h$, where parameter $h>0$ is called the bandwidth for the kernel function K. For bivariate case, the Epanechikov product kernel takes the form

$$K(x,y;h)=\begin{cases}\dfrac{9}{16}\left(1-\left(\dfrac{x}{h}\right)^2\right)\left(1-\left(\dfrac{y}{h}\right)^2\right), & \text{if } |x|<h,\ |y|<h\\ 0, & \text{otherwise,}\end{cases} \tag{30}$$

and the Epanechikov radial kernel takes the form

$$K(x,y;h)=\frac{3}{4}\left(1-\frac{x^2+y^2}{h^2}\right). \tag{31}$$

For data assimilation purpose, given a one-dimensional data sample $\{x_1,x_2,\cdots,x_n\}$, the credibility kernel density takes the form

$$f_h(x)=\frac{1}{nh}\sum_{i=1}^{n}K\left(\frac{x_i-x}{h}\right). \tag{32}$$

While for the two-dimensional case, the data sample $\{(x_1,y_1),(x_2,y_2),\cdots,(x_n,y_n)\}$, the credibility kernel density takes the form

$$f_h(x,y)=\frac{1}{nh}\sum_{i=1}^{n}K(x,y;h). \tag{33}$$

Different from inference in the probabilistic sense, the inference on the ground of credibility theory must develop its own criterion. One of the criteria is the maximum entropy principle.

5.2 Maximum Entropy Principle

Entropy is a measure of uncertainty. It is expected that the degree of uncertainty is 0 when the fuzzy variable degenerates to a crisp number, and is maximum when the fuzzy variable is an equipossible one, i.e., all values have the same possibility. In order to meet such a requirement, Liu [3] provided a new definition based on credibility measure.

Definition 3.2: *(Fuzzy Entropy) Let* ξ *be a continuous fuzzy variable defined on a credibility space* $(\Theta,2^{\Theta},\mathrm{Cr})$. *Then*

$$H(\xi)=\int_{-\infty}^{\infty}S\left(\mathrm{Cr}\left(\{\theta:\xi(\theta)=u\}\right)\right)du\ , \tag{34}$$

where

$$S(t)=-t\ln t-(1-t)\ln(1-t). \tag{35}$$

5.3 Maximum Entropy Kernel Estimation of 1- Dimensional Credibility Distribution

The arguments of this approach in general will follow the route, which starts from credibility kernel density to credibility distribution, then to the induced membership function and finally reaches the fuzzy entropy.

Given the sample $\{x_1,x_2,\cdots,x_n\}$, the credibility kernel density is

$$f_h(x)=(1/nh)\sum_{i=1}^{n}K((x_i-x)/h),$$

then the credibility distribution takes the form

$$\hat{\Phi}_\xi(x)=\frac{1}{nh}\sum_{i=1}^{n}\int_{-\infty}^{x}K\left(\frac{x_i-u}{h}\right)du. \tag{36}$$

Thus the membership function can be determined by

$$\hat{\mu}_\xi(x)=\begin{cases}\dfrac{2}{nh}\displaystyle\sum_{i=1}^{n}\int_{-\infty}^{x}K\left(\frac{x_i-u}{h}\right)du, & \text{if } \hat{\Phi}_\xi(x)<0.5\\ 1, & \text{if } \lim_{y\uparrow x}\hat{\Phi}_\xi(y)<0.5\le\hat{\Phi}_\xi(x)\\ 2-\dfrac{2}{nh}\displaystyle\sum_{i=1}^{n}\int_{-\infty}^{x}K\left(\frac{x_i-u}{h}\right)du, & \text{if } 0.5\le\lim_{y\uparrow x}\hat{\Phi}_\xi(y).\end{cases} \tag{37}$$

Note that

$$\mathrm{Cr}\left\{\theta:\xi(\theta)=x\right\}=\frac{1}{2}\left(\mu(x)+1-\sup_{y\neq x}\mu(y)\right). \quad (38)$$

Accordingly, we suggest an empirical object function for parameter searching since the optimal value of the data-dependent object function has to reflect the constraints specified by system performance data implicitly. The data constrained object function is the average of entropies evaluated at $\left\{x_1,x_2,\cdots,x_n\right\}$, i.e.,

$$J\left(h\mid x_1,x_2,\cdots,x_n\right)=\frac{1}{n}\sum_{i=1}^{N}S\left(\mathrm{Cr}\left\{\xi(\theta)=x_i\right\}\right). \quad (39)$$

Then it is expected that $J(h)\to H$, as $n\to\infty$, if the membership function is defined in a finite interval $\left[-L_1,L_2\right],\ L_1>0,\ L_2>0$ by which the theoretical entropy $H[\xi]$ exists and finite in general.

Once the optimal bandwidth, $\hat{h}$, is obtained, the maximum entropy kernel credibility distribution based on sample data $\left\{x_1,x_2,\cdots,x_n\right\}$ will be obtained, denoted by $\hat{\Phi}_{\xi,h}(x)$. Accordingly, the values of the kernel credibility distribution at the points $\left\{x_1,x_2,\cdots,x_N\right\}$ will be denoted by $\left\{\hat{u}_1,\hat{u}_2,\cdots,\hat{u}_n\right\}$, where $\hat{u}_i=\hat{\Phi}_{X,h}(x_i)$. Similarly, the kernel credibility distribution $\hat{\Phi}_{Y,h}(y)$ can be obtained and values of the kernel credibility distribution at the points $\left\{y_1,y_2,\cdots,y_n\right\}$ will be denoted by $\left\{\hat{v}_1,\hat{v}_2,\cdots,\hat{v}_n\right\}$, where $\hat{v}_i=\hat{\Phi}_{Y,h}(y_i),\ i=1,2,\cdots,n$.

5.3 Maximum Entropy Kernel Estimation of Bivariate Credibility Copula

Bivariate credibility-copula is in nature a bivariate joint uniform distribution. However, the copula representation has an advantage of its simple form, particularly, the one-parameter copula family. This feature may make the data assimilation process easier than that of the kernel estimation of the credibility distribution directly.

In terms of the arguments of Section 5, for bivariate sample data $\left\{(x_1,y_1),(x_2,y_2),\cdots,(x_n,y_n)\right\}$, the entropy kernel estimated bivariate uniform values are $\left\{(\hat{u}_1,\hat{v}_1),(\hat{u}_2,\hat{v}_2),\cdots,(\hat{u}_n,\hat{v}_n)\right\}$.

Recall that a copula, $C(u,v)$, in nature a bivariate joint credibility uniform distribution, possesses a joint credibility density, denoted by $\partial^2C(u,v)/\partial u\partial v$, we say $C(u,v)$ is absolutely continuous. Given a two-dimensional credibility distribution $H(x,y)$ and marginal credibility distributions Φ_X and Φ_Y, then its associated copula is $C(u,v)$. Conversely, given a copula $C(u,v)$ and marginal credibility distributions Φ_X and Φ_Y, then the bivariate joint distribution $H(x,y)$ can be found.

The joint bivariate membership function is defined by

$$\mu(x,y)=\left(2\mathrm{Cr}\left\{\theta:(X,Y)(\theta)=(x,y)\right\}\right)\wedge 1. \quad (40)$$

In terms of the equality $H(x,y)=C\left(\Phi_X(x),\Phi_Y(y)\right)$, we have

$$\mu(x,y)=\begin{cases} C\left(\Phi_X(x),\Phi_Y(y)\right), & \text{if } C\left(\Phi_X(x),\Phi_Y(y)\right)<0.5 \\ 1, & \text{if } \lim\limits_{(s,t)\uparrow(x,y)} C\left(\Phi_X(s),\Phi_Y(t)\right)<0.5\le C\left(\Phi_X(x),\Phi_Y(y)\right) \\ 2-C\left(\Phi_X(x),\Phi_Y(y)\right), & \text{if } 0.5\le \lim\limits_{(s,t)\uparrow(x,y)} C\left(\Phi_X(s),\Phi_Y(t)\right). \end{cases} \tag{41}$$

Also,

$$\mathrm{Cr}\left(\left\{\theta:(X,Y)(\theta)=(x,y)\right\}\right)=\frac{1}{2}\left(\mu(x,y)+1-\sup_{(s,t)\uparrow(x,y)}\mu(s,t)\right). \tag{42}$$

Accordingly, the entropy is

$$H\left[(X,Y)\right]=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} S\left(\mathrm{Cr}\left\{\theta:(X,Y)(\theta)=(s,t)\right\}\right)dsdt\,, \tag{43}$$

where $S(\cdot)$ is defined by Equation (35).

Recall that the kernel estimated bivariate joint uniform distribution has estimated sample data $\left\{(\hat{u}_1,\hat{v}_1),(\hat{u}_2,\hat{v}_2),L,(\hat{u}_n,\hat{v}_n)\right\}$, thus the observed copula are $\left\{C(\hat{u}_1,\hat{v}_1),C(\hat{u}_2,\hat{v}_2),\cdots,C(\hat{u}_n,\hat{v}_n)\right\}$. Accordingly, the observed entropy can be calculated in the form

$$J\left(\alpha \mid C(\hat{u}_i,\hat{v}_i),i=1,2,\cdots,n\right)=\frac{1}{n}\sum_{i=1}^{n} S\left(\mathrm{Cr}\left\{\theta:(X,Y)(\theta)=(x_i,y_i)\right\}\right). \tag{44}$$

Solving equation $dJ/d\alpha=0$ will give the maximum entropy estimation of the copula.

8 Conclusions

In this paper, we heavily reviewed the related work on $(\vee,\cdot)$-credibility measure theory. Based on the review, we propose the concept of copula for bivariate fuzzy variables, called as credibility-copula, which is a bivariate uniform credibility distribution in the sense of the credibility measure theory. Parallel to the basic understanding in probabilistic copula, we regard the credibility-copula to be a measure of the full bivariate dependence. Furthermore, we establish a decomposition of credibility-copula. The decomposition explicitly reveals that copula is a total measure of independence as well as dependence and only the removal of the product copula component from the copula could accurately describe the true dependence between the two fuzzy variables. We must point out that the mathematical formulation developments for credibility-copula and the decomposition are similar to those in the probabilistic copula theory. However, the underlying mechanism on the ground of credibility theory is different from that on the ground of probability theory because credibility measure theory and probability measure theory describe different phenomena: one is fuzzy uncertainty and the other is random uncertainty although we do not explore the bivariate uniform credibility distribution in details. The significance of the introduction of credibility-copula concept is to let fuzzy research community be aware that the traditional correlation modeling of fuzzy dependence is not the only approach. The multi-variate copula concept can be similarly defined in system reliability modeling. Furthermore, we establish a two-stage procedure for credibility-copula based on data-assimilation under the fuzzy maximum entropy principle.

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