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Towards A More Physically Adequate Definition of Randomness: A Topological Approach

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Abstract

Kolmogorov-Martin-Löf definition describes a random sequence as a sequence which satisfies all the laws of probability. This notion formalizes the intuitive physical idea that if an event has probability 0, then this event cannot occur. Physicists, however, also believe that if an event has a very small probability, then it cannot occur. In our previous papers, we proposed a modification of the Kolmogorov-Martin-Löf definition which formalizes this idea as well. It turns out that our original definition is too general: e.g., it includes some clearly non-physical situations when the set of all random elements is a one-point set. In this paper, we propose a new definition which avoids such situations and is, thus, a more physically adequate description of randomness.

1 Formulation of the Problem

Intuitive notion of randomness. In the traditional probability theory, we can talk about probabilities of different events, but we cannot distinguish between "random" and "non-random" sequences. Intuitively, however, some sequences are random and some are not.

For example, if we have a fair coin which produces heads (denoted by 1) and tails (denoted by 0) with equal probability, then we expect the actual sequence of the results of flipping this coin to be random, while the sequence 0101..., in which a sequence 01 is repeated again and again, is clearly not random.

Kolmogorov-Martin-Löf definition of randomness. The most well-known formalization of the notion of randomness was proposed in the 1970s by A. N. Kolmogorov and P. Martin-Löf; see, e.g., [8].

Informally, we call a sequence random if it satisfies all the laws of probability. Laws of probability usually state that some property holds with probability 1, i.e., that this property holds for all sequences (function, objects, etc.) except for sequences from a set of the probability measure 0. For example, the large numbers theorem states that with probability 1, the frequency f of 1's in a sequence tends to $1/2$. A central limit theorem states that with probability 1, the distribution of the difference $f - 1/2$ between the actual frequency f and $1/2$ tends to Gaussian, etc.

It is natural to say that a sequence satisfies the large numbers law if for this sequence, the frequency of 1s tends to $1/2$. Similarly, we say that a sequence satisfies the central limit theorem if for this sequence, the distribution of the deviations $f - 1/2$ tends to the normal distribution, etc.

In other words, we say that a sequence is random if it does not belong to any set of measure 0 which describes exceptions to a law of probability. Of course, we cannot define a random sequence as a one which does not belong to *any* set of measure 0 – because, e.g., in the probability measure corresponding to coin flipping, every infinite sequence has measure 0. However, we can do this if we restrict ourselves to *measurable* sets of measure 0, i.e., sets which can be described by closed formulas.

Every formula is a finite word in a finite alphabet (used to describe the corresponding mathematical theories). There are no more than countably many such words, hence no more than countably many definable sets of measure 0. The union of all such sets still has measure, so by deleting all of them we keep the set of measure 1. Elements of this set are called *random in the sense of Kolmogorov-Martin-Löf.*

From efficient to definable sets of measure 0. In the original definition of a Kolmogorov-Martin-Löf random sequences, the authors only considered *computable* sets of measure 0, i.e., sets corresponding to efficient tests of randomness.

From the viewpoint of computational statistics, when our main objective is to check whether a sequence is indeed random with respect to a given distribution, thus restricted definition is sufficient. However, in physical applications, we may be interested in more general properties which are not necessarily described by computable sets of measure 0. To handle such properties, P. Benioff, in his pioneer paper [1] on the use of randomness in physics, extended the original definition to the general case of definability.

In order to formally describe the corresponding notion, let us recall what definability means; for details, see, e.g., [6]. To make formal definitions, we must fix a formal theory $\mathcal{L}$ that has sufficient expressive power and deductive strength to conduct all the arguments and calculations necessary for working physics. For simplicity, in the arguments presented in this paper, we consider ZF, one of the most widely used formalizations of set theory.

Definition 1 *Let $\mathcal{L}$ be a theory, and let $P(x)$ be a formula from the language of the theory $\mathcal{L}$, with one free variable x for which, in the theory $\mathcal{L}$, there exists a set $\{x \mid P(x)\}$. We will then call the set $\{x \mid P(x)\}$* $\mathcal{L}$-definable.

Crudely speaking, a set is $\mathcal{L}$-definable if we can explicitly *define* it in $\mathcal{L}$. The set of all real numbers, the set of all solutions of a well-defined equation, every set that we can describe in mathematical terms is $\mathcal{L}$-definable.

This does not mean, however, that *every* set is $\mathcal{L}$-definable: indeed, every $\mathcal{L}$-definable set is uniquely determined by formula $P(x)$, i.e., by a text in the language of set theory. We have already mentioned that there are only countably many words and therefore, there are only countably many $\mathcal{L}$-definable sets. Since, e.g., in a standard model of set theory ZF, there are more than countably many sets of integers, some of them are thus not $\mathcal{L}$-definable.

In our definitions, we need to make mathematical statements about $\mathcal{L}$-definable sets. Therefore, in addition to the theory $\mathcal{L}$, we must have a stronger theory $\mathcal{M}$ in which the class of all $\mathcal{L}$-definable sets is a set – and it is a countable set.

Denotation. *For every formula F from the theory $\mathcal{L}$, we denote its Gödel number by $\lfloor F \rfloor$.*

Comment. A Gödel number of a formula is an integer that uniquely determines this formula. For example, we can define a Gödel number by describing what this formula will look like in a computer. Specifically, we write this formula in LaTeX , interpret every LaTeX symbol as its ASCII code (as computers do), add 1 at the beginning of the resulting sequence of 0s and 1s, and interpret the resulting binary sequence as an integer in binary code.

Definition 2 *We say that a theory $\mathcal{M}$ is* stronger *than $\mathcal{L}$ if it contains all formulas, all axioms, and all deduction rules from $\mathcal{L}$, and also contains a special predicate* $\mathrm{def}(n, x)$ *such that for every formula $P(x)$ from $\mathcal{L}$ with one free variable, the formula* $\forall y\,(\mathrm{def}(\lfloor P(x) \rfloor, y) \leftrightarrow P(y))$ *is provable in $\mathcal{M}$.*

The existence of a stronger theory can be easily proven; see, e.g., [6]. Now, we are ready for a formal definition.

Comment. In this paper, we will consider several different definitions of randomness. To distinguish between different versions, Kolmogorov-Martin-Löf randomness will be denoted by an index 0 and its consequent modifications by indices 1, 2, ...

Definition 3 *Let μ be a definable measure on a definable set X. We say that an element $x \in X$ is* random$_0$ *if it does not belong to any definable set of μ-measure 0.*

In the following text, the set of all random$_0$-elements will be denoted by R_0.

Limitations of Kolmogorov-Martin-Löf definition. The above definition, in effect, says that if an event has *probability 0*, then this event cannot happen. Physicists actually believe in a stronger statement: if an event has a *very small* probability, then this event cannot happen.

For example, according to physicists, the result of flipping a fair coin cannot start with 10,000 heads. This argument is used in statistical physics, to explain why processes with a very small probability – e.g., that all the molecules in a gas gather in one half of the bottle – simply cannot occur.

How to overcome these limitations? A challenge. We cannot simply fix a threshold $p_0 \ll 1$ and claim that all events with probability $\le p_0$ cannot happen. Indeed, in a coin flipping example, all sequences of a given length n are equally probable, with probability 2^{-n}. So, if we prohibit a sequence starting with 10,000 1s because its probability is too small $2^{-10000} \le p_0$, then we will prohibit *all* sequences of this length – but this does not make physical sense, because we can easily flip a coin 10,000 times and get some result 0110 ...

Levin's suggestion. A solution to the above challenge was first proposed by L. Levin [7, 8]. Levin's idea is that for a simple easy-to-describe event like "a sequence starts with 10,000 heads", the condition that $\mu(E) \le p_0$ should be sufficient to conclude that E is impossible. However, when we go to more complex, more difficult-to-describe events, e.g., that the sequence starts with the "random" subsequence 0110 ... of the same length, the impossibility threshold should be much smaller – definitely larger than 2^{-10000}.

In other words, the threshold should depend on the *complexity* of an event. Complexity can be described in algorithmic terms – e.g., as Kolmogorov complexity $K(x)$, the shortest length of a program which generates a given object [8], so this definition can be formalized.

Limitations of Levin's definition. Kolmogorov complexity is defined in terms of algorithms. Its *asymptotic* properties do not depend on what programming language we use to describe the corresponding algorithms, but its *numerical* value for a given object strongly depends on the choice of this language. So, strictly speaking, we have different numerical functions describing complexity.

Different functions lead to different definition of randomness. We do not know which definition is physically most adequate. It is therefore desirable, instead of defining a single calls of random sequences, to describe a *collection* of such classes, i.e., to provide an *axiomatic* descriptions of the corresponding sets.

Our previous definition of randomness. Such a definition was proposed and analyzed in our papers [2, 3, 6].

What does it mean that events with small probability cannot happen? For coin flipping, we can consider events A_n meaning that the results of flipping start with n heads. We know that $A_0 \supseteq A_1 \supseteq \ldots \supseteq A_n \supseteq \ldots$, and that the probability $\mu(A_n) = 2^{-n}$ of the event A_n tend to 0 as $n \to \infty$. Tending to 0 means that as n increases, these probabilities become smaller and smaller. Eventually, this probability will become so small that the corresponding event simply cannot happen, i.e., all events from the corresponding set A_N will be non-random.

We do not know for which N this will happen, it depends on physics, a physicist must tell us whether a sequence of 50 heads is already impossible – or it is possible but a sequence of 10,000 heads is not. In any case, we know that for some sufficiently large N, a sequence starting with N heads is impossible.

Similarly, for every other definable sequence $A_n \supseteq A_{n+1}$ for which $\mu(A_n) \to 0$, there must exist a value $N(\{A_n\})$ depending on this sequence for which all elements from the set A_N are not random. We have mentioned that this value $N(\{A_n\})$ depends on the complexity of the sequence $\{A_n\}$ – in some sense, this value can be taken as a measure of this complexity: for simple sequences, N is smaller, for more complex sequences, this N is larger.

For monotonically decreasing sequences of sets A_n, the condition $\mu(A_n) \to 0$ is equivalent to $\mu(\cap A_n) = 0$. Thus, we arrive at the following definition.

Definition 4 *Let μ be a definable measure on a definable set X. We say that a non-empty set S is* a set of all random$_1$ elements *(or R_1-set, for short) if for every definable sequence of sets $\{A_n\}$ for which $A_n \supseteq A_{n+1}$ and $\mu(\cap A_n) = 0$, there exists an integer N for which $S \cap A_N = \emptyset$.*

Comment. It is easy to see that we cannot formulate this property for *all* (not necessarily definable) sequences $\{A_n\}$ with the above properties. Indeed, for coin flipping, for every random sequence $x \in S$, we can consider sets A_n consisting of all sequences which share the first n results with x. For this sequence of sets, $A_n \supseteq A_{n+1}$, and $\mu(A_n) = 2^{-n} \to 0$ – but since $x \in \cap A_n$ and $x \in S$, we cannot require that $S \cap A_N = \emptyset$ for some N.

From randomness to typicalness. In some physical situations, we do not know the corresponding probability measure, but we still want to distinguish between "degenerate" (abnormal, physically impossible) elements and "typical" (physically possible) ones.

For example, the equations of general relativity allow many solutions; some of them are degenerate in the sense that small deviations from the initial conditions would lead to a drastically different dynamics. Physicists believe that such solutions are not physically possible; see, e.g., [9].

How can we formalize this notion of "typicalness"? Here, we do not know the probability measure μ, so we have to restrict ourselves only to sequences for which $\mu\left(\bigcap\limits_{n=0}^{\infty} A_n\right) = 0$. The only set whose measure is always 0 for all measures is an empty set, so we arrive at the following definition:

Definition 5 *Let X be a definable set. We say that a non-empty set S is* a set of all typical$_1$ elements *(or T_1-set, for short) if for every definable sequence of sets $\{A_n\}$ for which $A_n \supseteq A_{n+1}$ and $\bigcap\limits_{n=0}^{\infty} A_n = \emptyset$, there exists an integer N for which $S \cap A_N = \emptyset$.*

Comment. Similarly to the random case, we also believe that we cannot drop the requirement that the sequence $\{A_n\}$ is definable. However, in contrast to the random case, we do not have a proof that such "dropping" will make the corresponding notion inconsistent.

Known properties of these definitions. First, we must prove existence.

Proposition 1 ([6]) *Let X be a set, and let μ be a definable probability measure on the set X in which all $\mathcal{L}$-definable sets are μ-measurable. Then, for every $\varepsilon > 0$, there exists an R_1-set S that is μ-measurable and for which $\mu(S) \geq 1 - \varepsilon$.*

Comment. For example, all arithmetic subsets of the interval $[0, 1]$ are Lebesgue-measurable, so for an arithmetic theory $\mathcal{L}$ and for the Lebesgue measure μ, every definable set is measurable. It is worth mentioning that some other set theories have non-measurable definable subsets of the set $[0, 1]$.

Proposition 2 ([6]) *Let X be a set, and let μ be a definable probability measure on the set X in which all $\mathcal{L}$-definable sets are μ-measurable. Then, for every $\varepsilon > 0$, there exists a T_1-set S that is μ-measurable and for which $\mu(S) \geq 1 - \varepsilon$.*

These definitions have many useful properties. For example, if X is a definably separable metric space, then every R_1-set and every T_1-set is *precompact* (i.e., its closure is compact). This is important for inverse problems, when we want to reconstruct the state (e.g., the density distribution) $x \in X$ from the results $f(x)$ of the measurements, where $f : X \to Y$ is a definable continuous function. In general, even if we can uniquely reconstruct x from $f(x)$, the inverse mapping f^{-1} from Y to X is not necessarily continuous – i.e., small changes in the measurement result can lead to drastic changes in the reconstructed x. Such problems are called *ill-defined.* It is known that if we restrict ourselves to a compact set $X_0 \subseteq X$, then the problem becomes well-defined [11]. Thus, if we require that x is random or typical, all inverse problems become well-defined [6].

Similarly, it is known that it is not possible to have an algorithm which, given a continuous function $f(x)$, returns the point $x_{\max}$ where this function attains its maximum. However, if we restrict ourselves to *typical* functions (e.g., functions from a T_1-subset of the set of all functions), then algorithmic optimization becomes, in some sense, possible [2] – "in some sense" because for this determination, we need to know the values $N(\{A_n\})$ provided by the physicists.

Limitations of our original definition. Stephen G. Simpson noticed if S is an R_1- or a T_1-set, then every non-empty subset $S' \subseteq S$ is also correspondingly, an R_1- or a T_1-set. In particular, if we take any point $s \in S$, then the corresponding 1-element set $S' = \{s\}$ is also a R_1-set.

The situation when the set of all random elements is a one-point set is clearly not very physically adequate. It is therefore desirable to modify our original definition to make it more physically adequate.

What we do in this paper. In this paper, we propose new definitions which provide a more physically adequate description of randomness.

2 Analysis of the Problem: Properties of Our Previous Definitions

In order to describe these definitions, let us first analyze the situation, i.e., investigate the properties of the above definitions. Actually, we had three definitions: of the set R_0 of all Kolmogorov-Martin-Löf sequences, of R_1-sets, and of T_1-sets. Before we start analyzing these properties one by one, let us describe the relation between these three notions.

Proposition 3 *Every R_1-set S is a subset of the set R_0 of all objects which are random in the sense of Kolmogorov-Martin-Löf.*

Comment. In other words, the notion of an R_1-set is a refinement of the notion of Kolmogorov-Martin-Löf randomness.

Proof. We will prove that if $x \notin R_0$, then $x \notin S$. Indeed, by definition of the set R_0, the fact that $x \notin R_0$ means that x belongs to some definable set A of measure 0. We can then take a sequence $A_0 = A_1 = \ldots = A$. For this sequence, $A_n \supseteq A_{n+1}$ and $\mu(\cap A_n) = 0$. So, by the definition of a R_1-set, there exists an N for which $A_N \cap S = \emptyset$. This means that the element $x \in A = A_N$ does not belong to S. The proposition is proven.

Comment. It is easy to see that the two notions of randomness differ even for the simplest probability measures. Indeed, for the coin-flipping measure, as we have mentioned, the measure of the set R_0 is 1. On the other hand, for every R_1-set, for the sequence A_n of the sequences which start with n heads, we have $A_N \cap S = \emptyset$. Thus, a complement to S contains a set A_N of measure 2^{-N} and thus, $\mu(S) \le 1 - 2^{-N} < 1$. So, the set R_0 of measure 1 cannot be a R_1-set.

Proposition 4 *Let X be a set, and let μ be a definable probability measure on the set X in which all $\mathcal{L}$-definable sets are μ-measurable. Then:*

- *every R_1-set is a T_1-set;*
- *if S is a R_1-set, then $S \cap R_0$ is an R_1-set.*

Comment. In other words, a sequence is random$_1$ if and only if it is typical$_1$ and random$_0$.

Proof. Every sequence $\{A_n\}$ with $\cap A_n = \emptyset$ has the property $\mu(\cap A_n) = 0$, hence every R_1-set is indeed a T_1-set.

Vice versa, let S be a T_1-set. Let us show that the intersection $S \cap R_0$ is a R_1-set. Indeed, let $\{A_n\}$ be a sequence of sets for which $A_n \supseteq A_{n+1}$ and $\mu(\cap A_n) = 0$. Since the sequence A_n is definable, its intersection $A \stackrel{\text{def}}{=} \bigcap\limits_{n=0}^{\infty} A_n$ is also definable and has a measure 0. By definition of R_0, we have $R_0 \cap A = \emptyset$. Thus, for the sequence $A'_n \stackrel{\text{def}}{=} A_n - A$, we have $A'_n \supseteq A'_{n+1}$ and $\bigcap\limits_{n=0}^{\infty} A'_n = \emptyset$. Since S is a T_1-set, we conclude that there exists an integer N for which $A'_N \cap S = \emptyset$. Thus, $A_N \cap S \subseteq A$. Due to $A \cap R_0 = \emptyset$, we have $A \subseteq -R_0$, hence $A_N \cap S \subseteq -R_0$ and thus, $A_N \cap (S \cap R_0) = \emptyset$. So, $S \cap R_0$ is indeed a T_1-set. The proposition is proven.

Proposition 5

- *A union $S = S_1 \cup \ldots \cup S_n$ of finitely many R_1-sets S_i is an R_1-set.*
- *A union $S = S_1 \cup \ldots \cup S_n$ of finitely many T_1-sets S_i is a T_1-set.*
- *An arbitrary subset $S' \subseteq S$ of an R_1-set S is an R_1-set.*
- *An arbitrary subset $S' \subseteq S$ of a T_1-set S is an R_1-set.*

The proofs of these results directly follow from the definitions. For T_1-sets, there are two more such easy-to-prove properties:

Proposition 6

- *Every finite set is a T_1-set.*
- *If S is a T_1-set, and $f : X \to Y$ is a definable function, then the image $f(S)$ is also a T_1-set.*

Comment. The result for the image follows from the fact that $A_n \supseteq A_{n+1}$ and $\cap A_n = \emptyset$ imply that $A'_n \supseteq A'_{n+1}$ and $\cap A'_n = \emptyset$, where $A'_n \stackrel{\text{def}}{=} f^{-1}(A_n)$. Thus, for some N, we have $A'_N \cap S = f^{-1}(A_N) \cap S = \emptyset$, hence $f(S) \cap A_n = \emptyset$.

3 A New Definition of Randomness and Typicality: the Corresponding Sets Must Be Maximal

Main idea. As we have mentioned, a physicist must supply us with a mapping $N(\{A_n\})$ which describes what cannot happen. This mapping, in effect, provides a measure of complexity for different sequences of sets. Once this mapping is in place, we can determine the corresponding R_1-set S of random elements.

In principle, we can then arbitrarily dismiss some of the elements from this set S and consider a smaller subset $S' \subset S$. However, this additional narrowing from S to S' is no longer motivated by any physics. It is therefore reasonable to restrict ourselves only to those narrowing which are motivated by physics. In other words, from all R_1-sets which are consistent with a given complexity measure $N(\{A_n\})$, we select the one which is the $\subseteq$-largest.

Thus, we arrive at the following definitions.

Definition 6

- *By a* complexity measure*, we mean a mapping $N(\{A_n\})$ which puts into correspondence, to every definable sequence $\{A_n\}$ for which $A_n \supseteq A_{n+1}$ and $\mu(\cap A_n) = 0$, an integer N.*

- *We say that an R_1-set S is* consistent *with the complexity measure $N(\{A_n\})$ if for every definable sequence $\{A_n\}$ for which $A_n \supseteq A_{n+1}$ and $\mu(\cap A_n) = 0$, we have $S \cap A_{N(\{A_n\})} = \emptyset$.*

- *We say that an R_1-set S is* maximal *with respect to a complexity measure $N(\{A_n\})$ if it is consistent with this measure, but no proper superset S' is consistent with it.*

- *We say that a set S is* a set of all random$_2$ elements *(or R_2-*set*, for short) if it is an R_1-set, and it is maximal with respect to some complexity measure.*

Definition 7

- *By a* complexity measure*, we mean a mapping $N(\{A_n\})$ which puts into correspondence, to every definable sequence $\{A_n\}$ for which $A_n \supseteq A_{n+1}$ and $\cap A_n = \emptyset$, an integer N.*

- *We say that a T_1-set S is* consistent *with the complexity measure $N(\{A_n\})$ if for every definable sequence $\{A_n\}$ for which $A_n \supseteq A_{n+1}$ and $\cap A_n = \emptyset$, we have $S \cap A_{N(\{A_n\})} = \emptyset$.*

- *We say that a T_1-set S is* maximal *with respect to a complexity measure $N(\{A_n\})$ if it is consistent with this measure, but no proper superset S' is consistent with it.*

- *We say that a set S is* a set of all typical$_2$ elements *(or T_2-*set*, for short) if it is a T_1-set, and it is maximal with respect to some complexity measure.*

Let us first prove the existence of such sets.

Proposition 7

- *For every R_1-set S, there exists an R_2-set $S' \supseteq S$.*

- *For every T_1-set S, there exists a T_2-set $S' \supseteq S$.*

Proof. In the previous section, we have described a simple relation between R_1- and T_1-sets. It is easy to show that there is a similar relation between R_2- and T_2-sets. Thus, in all the proofs, it is sufficient to consider only T_2-sets: R_2-sets are simply intersections of these sets with the set R_0 of all Kolmogorov-Martin-Löf random elements.

Let S be a T_1-set. By definition, this means that there exists a complexity measure $N(\{A_n\})$ with which this set S is consistent. Let us now take, as S', a complement to the union U of all the sets $A_{N(\{A_n\})}$ corresponding to all definable sequences $\{A_n\}$. Clearly:

- this complement S' is consistent with the given complexity measure, and
- every T_1-set S'' which is consistent with the complexity measure $N(\{A_n\})$ must have no intersections with all the sets $A_{N(\{A_n\})}$ and thus, with their union U, so it must be a subset of S': $S'' \subseteq S'$.

So, this set S' is indeed maximal, i.e., a T_2-set, and $S \subseteq S'$. The proposition is proven.

Proposition 8 *Let X be a set, and let μ be a definable probability measure on the set X in which all $\mathcal{L}$-definable sets are μ-measurable. Then, for every $\varepsilon > 0$, there exists an R_2-set S that is μ-measurable and for which $\mu(S) \geq 1 - \varepsilon$.*

Proposition 9 *Let X be a set, and let μ be a definable probability measure on the set X in which all $\mathcal{L}$-definable sets are μ-measurable. Then, for every $\varepsilon > 0$, there exists a T_2-set S that is μ-measurable and for which $\mu(S) \geq 1 - \varepsilon$.*

Comment. These results are simple corollaries of Propositions 1, 2, and 7.

Discussion. One can easily see that the new definitions indeed drastically decrease the number of possible random (typical) sets. Indeed, we know that every subset of R_1-set is also R_1. We also know that in many situations, there exists an R_1-set S of cardinality continuum $\aleph_1$ – e.g., a set of measure $\geq 1 - \varepsilon$. In this case, we have at least as many R_1-sets as there are subsets in the set S. So, the number of possible R_1-sets is $2^{\aleph_1}$.

On the other hand, every R_2-set is a complement to a union of a family of definable sets. Since there are at most countable many ($\aleph_0$) definable sets, there are at most $2^{\aleph_0}$ R_2-sets. So, there are $\geq 2^{\aleph_1}$ R_1-sets and $\leq 2^{\aleph_0}$ R_2-sets.

Under the usual assumptions of the Continuum Hypothesis, when $2^{\aleph_0} = \aleph_1$ and $2^{\aleph_1} = \aleph_2$, we conclude that there are $\geq \aleph_2$ R_1-sets and $\leq \aleph_1$ R_2-sets.

Similarly, we can prove that there exist $\geq 2^{\aleph_1}$ (i.e., $\geq \aleph_2$) T_1-sets and $\leq 2^{\aleph_0}$ (i.e., $\leq \aleph_1$) T_2-sets.

4 Topological Interpretation of the New Definition

Corresponding topology. Let us show that the above definitions can be naturally reformulated in topological terms. Indeed, once can easily check that the intersection of two definable sets is definable. Thus, an intersection of finitely many definable sets is also definable. Hence, definable sets form a *basis* of a topology. We will call this topology D-topology (D from "definable"). In general, open sets are defined as unions of sets from a basis; see, e.g., [5].

Definition 8 *A set S is* open *in D-topology (D-*open*, for short) if and only if it is a union of definable sets.*

Proposition 10 *A set S is closed in D-topology (D-closed, for short) if and only if it is an intersection of definable sets.*

Proof. Closed sets are complements to open sets. From Definition 8, it follows that a set is closed if and only if it is an intersection of complements to definable sets. However, a complement to a definable set is also definable – and vice versa. The proposition is proven.

Comment. In the particular case when we have Δ^1_1-definable sets, the corresponding topology is known as the Gandy-Harrington topology. This topology has been actively used to prove deep theorems in descriptive set theory; see, e.g., [4, 10].

Proposition 11 *A T_1-set S is a T_2-set if and only S is D-closed.*

Proof. 1°. In our proof of Proposition 7, we proved, in effect, that every T_2-set is an intersection of definable sets and is, thus, D-closed.

2°. To complete our proof, we must now show that every D-close T_1-set S is indeed a T_2-set, i.e., that S is maximal with respect to some complexity measure.

As such a complexity measure, let us take the following mapping: for every sequence $\{A_n\}$, as $N(\{A_n\})$, we take the smallest N for which $A_N \cap S = \emptyset$.

2.1°. We know, from the proof of Proposition 7, that among all the T_1-sets which are consistent with this complexity measure, there is the maximal one $S' = -U$, where $U = \cup A_{N(\{A_n\})}$. Since this set S' is maximal, we have $S' \supseteq S$.

2.2°. Let us prove that $S' = S$. For that, let us show that if S is contained in a definable set A, then S' is contained in the same set.

Indeed, since $S \subseteq A$, for a definable sequence $A_0 = -A$, $A_1 = \ldots = A_n = \ldots = \emptyset$, the corresponding $N(\{A_n\})$ is equal to 0. Since the set S' is consistent with the same complexity measure, we conclude that $S' \cap A_{N(\{A_n\})} = \emptyset$, i.e., $S' \cap (-A) = \emptyset$ and thus, $S' \subseteq A$.

The set S is an intersection of definable sets A. Since the set S' is contained in each of these sets, it must be also contained in their intersection S: $S' \subseteq S$.

Since we have already proved that $S \subseteq S'$, this implies that $S = S'$, hence S is indeed a T_2-set.

The proposition is proven.

Topological reformulation of the original definition. It turns out that the original definition of a T_1-set can also be naturally reformulated in terms of this topology.

Let us recall that in topology, a set A is called precompact if and only if from every cover $A \subseteq \cup U_\alpha$ of this set by open sets U_α, there exists a finite subcover $A \subseteq U_{\alpha_1} \cup \ldots \cup U_{\alpha_n}$; see, e.g., [5].

Definition 9 *We say that a set S is D-*precompact *if from every definable cover $S \subseteq \cup U_n$ of S by definable sets U_i, there exists a finite subcover $S \subseteq U_{i_1} \cup \ldots \cup U_{i_n}$.*

Proposition 12 *A set S is a T_1-set if and only if it is D-precompact.*

Proof. 1°. Let us first prove that if S is a T_1-set, then S is D-precompact. Indeed, let U_n be a definable open cover of S. Since the family U_n is definable, its union $U = \cup U_n$ is also definable. Let us now take $A_n = U - (U_1 \cup \ldots \cup U_n)$. Due to this definition, this sequence $\{A_n\}$ is a definable sequence of sets for which $A_n \supseteq A_{n+1}$ and $\cap A_n = \emptyset$. Since S is a T_1-set, we conclude that for some N, we have $S \cap A_N = \emptyset$. We assumed that $S \subseteq U$, i.e., every point from S belongs to U. By definition of A_n, the fact that $S \cap A_N = \emptyset$ means that S has no points outside $U_1 \cup \ldots \cup U_N$ – i.e., that $S \subseteq U_1 \cup \ldots \cup U_N$.

2°. Vice versa, let us prove that every D-precompact set S is a T_1-set. Indeed, let S be a D-precompact set, and let $\{A_n\}$ be a definable sequence for which $A_m \supseteq A_{n+1}$ and $\cap A_n = \emptyset$. From the fact that $\cap A_n = \emptyset$, we conclude that $\cup U_n = X$, where $U_n \stackrel{\text{def}}{=} -A_n$. Thus, the definable family U_n forms a definable cover for the set S. Since the set S is D-precompact, there exists a finite subcover $S \subseteq U_{i_1} \cup \ldots \cup U_{i_n}$.

By definition of $U_n = -A_n$, from $A_n \supseteq A_{n+1}$, we conclude that $U_n \subseteq U_{n+1}$. Thus, $U_{i_1} \cup \ldots \cup U_{i_n} = U_N$, where $N \stackrel{\text{def}}{=} \max(i_1, \ldots, i_n)$. From $S \subseteq U_n = -A_n$, we conclude that $S \cap A_n = \emptyset$.

The proposition is proven.

Comment. Combining Propositions 11 and 12, we can conclude that a set S is a T_2-set if and only if it is D-closed and D-precompact. Thus, we get a simple topological reformulation of the notion of a T_2-set.

Remaining open problems. The problem with the new definition is that it still allows one-point sets of random elements. For example, on the interval $[0, 1]$, every point can be described as an interval of (definable) intervals with rational endpoints. Since every one-point set is a T_1-set, we thus conclude that it is also a T_2-set, and thus, that it is an R_2-set.

To avoid such situations, we must explicitly disallow such sets. We must also disallow sets for which in some neighborhood, there is only one random element. In other words, we would like to make sure that if a non-definable sequence belongs to a T-set, then in its every neighborhood, there should be another non-definable element. In topological terms, this means that we would like to require that the set of non-definable typical elements is *perfect*.

For flipping coins and for the uniform measure on the interval $[0, 1]$, such a set is indeed possible.

Proposition 13 *Let $X = [0, 1]$, let μ be a uniform measure on X, and let $\mathcal{L}$ be such that all $\mathcal{L}$-definable sets are μ-measurable. Then, for every $\varepsilon > 0$, there exists a perfect μ-measurable T_2-set S for which $\mu(S) \geq 1 - \varepsilon$.*

Proof. 1°. Let us start with a T_2-set S for which $\mu(S) \geq 1 - \varepsilon$. The existence of such a set follows from Proposition 9.

If this set S is already perfect, we are done. If it is not perfect, this mean that it has isolated points. With each isolated point, comes the entire interval which contains no other points from S; thus, there are no more than countably many such points.

1.1°. If an isolated point x is *definable*, we simply take an intersection of S with the definable set $-\{x\}$.

1.2°. If an isolated point is *not definable*, we pick a small rational-valued interval which contain x and intersect S with the definable complement to this interval.

As a result of all such intersections, we get a new set S' which is S minus all isolated points.

2°. Let us prove that S' is a T_2-set. According to Proposition 11, it is sufficient to prove two statements:

- that S' is a T_1-set and
- that S' is D-closed, i.e., that S' is an intersection of definable sets.

2.1°. Let us first prove that S' is a T_1-set.

Indeed, the set S' is a subset of the previous set S which was a T_1-set. Thus, S' is also a T_1-set.

2.2°. Let us now prove that S' is an intersection of definable sets.

Indeed, the set S was a T_2-set, hence an intersection of definable sets. Our new set S' is an intersection of S and other definable sets – thus, it is also such an intersection. So, S' is indeed a T_2-set.

3°. Since we deleted countably many points, the measure does not change: $\mu(S') = \mu(S) \geq 1 - \varepsilon$.

The proposition is proven.

Comment. It is desirable to extend the above definition and the corresponding result to a more general case.

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