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On One Inverse Problem in Financial Mathematics*

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Abstract

The Black-Scholes formula has been derived under the assumption of constant volatility in stocks. In spite of evidence that this parameter is not constant, this formula is widely used by the markets. This paper addresses the question whether a model for stock price exists such that the Black-Scholes formula holds while the volatility is nonconstant. We give new as well as recent results concerning this question providing as complete as possible an answer at this stage. It is remarkable that while in general for the Black-Scholes formula the answer is 'no', it is 'yes' for a similar question concerning the Bachelier formula.

Keywords: option pricing, volatility, stock price, Black-Scholes model, Bachelier model

1 Introduction

A most spectacular application of mathematics to financial markets has occurred in the past thirty years after the discovery the Black-Scholes formula [1]. This formula gives the price of an option when the stock price is modelled by the Black-Scholes-Merton model. The reason this formula has made such an impact is because it allowed for a new way of looking at risk with wide ranging implications. This formula not only gives the price for an agreement to enter into business transaction in the future, but it also shows how can a riskless profit be made if the price is different to the one given by this formula, eg. [15], [12]. Although the Black-Scholes formula is a product of a complex theory of pricing of assets by no free lunch approach, one would be pressed to find another mathematical formula used so much in practice. Its widespread use motivates mathematicians to examine its assumptions and extend its domain of validity.

Generally speaking all the assumptions used in this formula are widely accepted but one. This is the assumption on the volatility parameter appearing in the model for the price of stock. It is widely believed and experimentally verified that stocks do not have a constant volatility, rather this parameter varies with time, see e.g. [10], [5], [6], [16]. In this paper we discuss the question of existence of a model in which options prices are given by the Black-Scholes formula yet volatility is not constant. This question is of great interest in financial mathematics as well as in practical applications, see [2], [14], [4] and [3]. We show here that, under broad assumptions the answer to this question is negative. That is to say that if the Black-Scholes formula holds with *some* "volatility" parameter not necessarily related to the model of stock, then the model must be the Black-Scholes-Merton model. This paper gives new as well as published results concerning this question providing as complete as possible answer at this stage. The main ideas of proofs are given but the proofs themselves are omitted due to their technical nature, and the interested reader can find them in the

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quoted papers, or they will be forthcoming in a more specialized outlet. It is structured as follows. In the next section we describe the Black-Scholes-Merton model and give the Black-Scholes formula. Then we give results showing why there is no other model with the same option pricing formula, and finally we comment that the same question can be answered in the affirmative if instead of the Black-Scholes model the Bachelier model is used.

2 The Black-Scholes and Other Option Pricing Formulae

We start with the usual set-up of a stochastic basis $(\Omega, \mathcal{F}, \mathcal{F}_t)$, on which we define various probability measures and Brownian motions generically denoted W_t (regardless of the probability measure being used).

The Black-Scholes-Merton model, herein denoted S_t, is described by a randomly perturbed exponential growth. Its evolution is given by the stochastic differential equation

$$dS_t = \mu S_t dt + \sigma S_t dW_t. \tag{1}$$

The strength of the random perturbation is determined by the positive parameter σ, which is known as the volatility of the stock. The above model was used by Merton, Black and Scholes to find the price of an option on stock, such as an agreement to buy the stock at some future time T for the specified at time $t < T$ price K. Their formula states that the price of such an option at time t is given by

$$C_S(T, t, K, \sigma, S_t) = S_t \Phi(h) - K e^{-r(T-t)} \Phi\left(h - \sigma\sqrt{T-t}\right), \tag{2}$$

where Φ denotes the standard normal distribution function and

$$h = \frac{\log \frac{S_t}{K} + (r + \frac{\sigma^2}{2})(T-t))}{\sigma\sqrt{T-t}}.$$

Remarkably, the parameter μ does not enter the formula, but σ does, as well as r, the riskless rate available in a savings account.

As stated in the introduction, a main feature and weakness of the Black-Scholes model (1) is its constant volatility assumption. In this paper, we are concerned with processes, herein denoted $\mathcal{X}_t$, that display non-constant volatilities. $\mathcal{X}_t$ will simply denote a process that is adapted to $\mathcal{F}_t$. It may, for example, take the form

$$d\mathcal{X}_t = \mu \mathcal{X}_t dt + \theta_t \mathcal{X}_t dW_t, \tag{3}$$

where θ_t is a function of time that may or may not be random. For example, θ_t can be a function of stock $\mathcal{X}_t$, as well as include other independent sources of randomness. There is a large literature on non-constant volatility models, both deterministic and stochastic (see e.g. [9]).

In the next paragraph we recall the basic facts on option pricing see e.g. [15], [12]. We assume without loss of generality that the riskless interest rate $r = 0$, otherwise we work with discounted prices, $\mathcal{X}_t e^{-rt}$.

The First Fundamental theorem of asset pricing states that a model does not admit arbitrage if and only if there exists an equivalent probability measure Q such that $\mathcal{X}_t$ is a Q-martingale. The price at time t of a call option that pays $(\mathcal{X}_T - K)^+$ at time T is given by

$$C_t = \mathbb{E}_Q[(\mathcal{X}_T - K)^+ | \mathcal{F}_t], \tag{4}$$

where $\mathbb{E}_Q$ is the expectation under Q.

In particular, (2) is obtained by applying (4) to the Black-Scholes model (1), which under the martingale measure Q, satisfies the reduced stochastic differential equation

$$dS_t = \sigma S_t dW_t. \tag{5}$$

The aim of this paper is to solve the inverse problem of finding another model $(\mathcal{X}_t, \theta_t)$ such that

$$\mathbb{E}_Q[(\mathcal{X}_T - K)^+ | \mathcal{F}_t] = C_S(T, t, K, \theta_t, \mathcal{X}_t).$$

A precise formulation of this problem is given in the Results Section.

Second to the Black-Scholes model is the Bachelier model, which under the martingale measure Q, is simply described as a scaled Brownian motion

$$dB_t = \sigma dW_t. \tag{6}$$

The Bachelier formula is then obtained by applying (4) to B_t,

$$C_B(T, t, K, \sigma, B_t) = \sigma\sqrt{T-t}(\Phi'(h) + h\Phi(h)) \tag{7}$$

where $h = (B_t - K)/\sigma\sqrt{T-t}$.

More generally, one may consider a model of the form

$$dZ_t = \sigma b(Z_t) dW_t, \tag{8}$$

for some function b (with sufficient smoothness and boundedness conditions to guarantee the existence and weak uniqueness of the process Z_t). As in (2) and (7, one can in theory obtain an option pricing formula

$$\mathbb{E}_Q[(Z_T - K)^+ | \mathcal{F}_t] = \mathbb{E}_Q[(Z_T - K)^+ | Z_t] = C_Z(T, t, K, \sigma, Z_t). \tag{9}$$

Similarly, we seek a solution to the inverse problem of finding a model $(\mathcal{X}_t, \theta_t)$ such that

$$\mathbb{E}_Q[(\mathcal{X}_T - K)^+ | \mathcal{F}_t] = C_Z(T, t, K, \theta_t, \mathcal{X}_t).$$

3 Results

In this section we investigate the inverse problems previously eluded to. In fact we look at five situations, four of which yield a non-existence result, while the fifth leads us to the construction of a new family of processes with interesting properties. We begin with a general non-existence result.

3.1 General Non-existence Result – The Case of A Continuum of Strikes

Assume that under the no-arbitrage measure Q, the stock price process satisfies (8), $dZ_t = \sigma b(Z_t) dW_t$, where σ is the volatility parameter.

Theorem 1 *Let $\mathcal{X}_t$ and θ_t be two adapted processes. Assume that $\mathcal{X}_t$ is positive (or bounded from below) or a martingale and that there exist three terminal times $T_1 < T_2 < T_3$ such that for all K and all $t \le T_i$,*

$$\mathbb{E}\left[(\mathcal{X}_{T_i} - K)^+ | \mathcal{F}_t\right] = C_Z(T_i, t, K, \theta_t, \mathcal{X}_t) \tag{10}$$

Then for all $t \le T_1$, $\theta_t^2 = \theta_0^2$. Furthermore, if $\mathcal{F}_t$ is generated by W_t, then $(\mathcal{X}_t)_{t \le T_1} \stackrel{d}{=} (Z_t)_{t \le T_1}$.

We stress that in this result no assumptions are made on the dynamics of $\mathcal{X}_t$ or its relationship with θ_t other than identity (10).

The main tool in establishing this result is to show by using (11) below that $M_t = e^{\theta_t^2(T_i - t)}\phi(\mathcal{X}_t)$ are martingales. Then we use the fact that if M_t, $M_t Y_t$ and $M_t Y_t^\alpha$, for some $\alpha > 1$, are local martingales, then Y_t must be almost surely constant. In this case $Y_t = e^{\theta_t^2(T_2 - T_1)}$.

3.2 Finitely Many Strikes

Theorem 1 is in fact an immediate consequence of the following proposition and the fact that, for any $\mathcal{C}^2$ function ϕ,

$$\phi(x) = \phi(0) + \phi'(0)x + \int_0^{+\infty} (x-K)^+ \phi''(K) dK. \tag{11}$$

Let A denote the infinitesimal generator of the general stock price process given by (8),

$$Af(x) = \frac{1}{2} b(x)^2 f''(x).$$

Proposition 2 *Let $\mathcal{X}_t$ and θ_t be two adapted processes. Assume that $\mathcal{X}_t$ is a martingale and that there exist three terminal times $T_1 < T_2 < T_3$ such that for all $t \le T_i$,*

$$\mathbb{E}\left[\phi(\mathcal{X}_{T_i})|\mathcal{F}_t\right] = \mathbb{E}\left[\phi(Z_{T_i})|Z_t = z\right]_{\sigma=\theta_t, z=\mathcal{X}_t}, \tag{12}$$

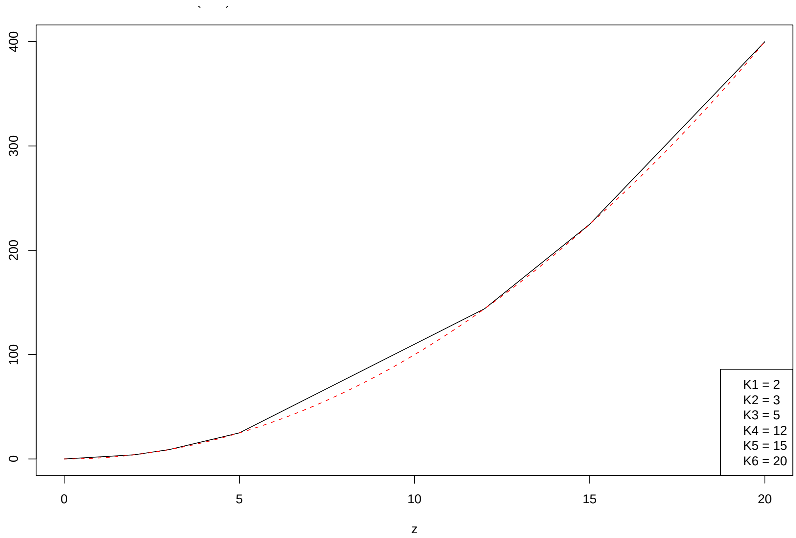
where ϕ is $\mathcal{C}^2$ and satisfies $A\phi = \phi$. Then for all $t \le T_1$, $\theta_t^2 = \theta_0^2$. Furthermore, if $\mathcal{F}_t$ is generated by W_t, then $(\mathcal{X}_t)_{t \le T_1} \overset{d}{=} (Z_t)_{t \le T_1}$.

Next we argue how this result could be used to lessen the unrealistic requirement that (10) be satisfied for all strikes. Instead, we assume that it is satisfied for finitely many strikes, $0 < K_1 < K_2 < \ldots < K_n$.

Using (11), one obtains the following approximation,

$$\phi(x) \simeq \phi(0) + \phi'(0)x + \frac{1}{2}\sum_{i=1}^{n} (x - K_{i-1})^+ \phi''(K_{i-1})(K_i - K_{i-2}),$$

where $K_0 = K_{-1} = 0$. A graphical representation of this approximation, in the case $\phi(x) = x^2$, is given below.



For a martingale $\mathcal{X}_t$, this approximation in turn yields

$$\mathbb{E}\left[\phi(\mathcal{X}_{T_i})|\mathcal{F}_t\right] \simeq \mathbb{E}\left[\phi(Z_{T_i})|Z_t = z\right]_{\sigma=\theta_t, z=\mathcal{X}_t}.$$

It is therefore natural to replace (approximate) the requirement that (10) be satisfied for finitely many strikes by (12).

3.3 Local Black-Scholes – Maturity-independent Volatility

The special case of the Black-Scholes model is, of course, of great importance. It is discussed at length in [7]. We reproduce here the main findings.

Proposition 3 *Let $\mathcal{X}_t$ and θ_t be two adapted processes. Assume that $\mathcal{X}_t$ is a martingale and that there exist three terminal times $T_1 < T_2 < T_3$ such that for all $t \le T_i$,*

$$\mathbb{E}\left[\mathcal{X}_{T_i}^2 | \mathcal{F}_t\right] = \mathbb{E}\left[S_{T_i}^2 | S_t = s\right]_{\sigma=\theta_t, s=\mathcal{X}_t}. \tag{13}$$

Then for all $t \le T_1$, $\theta_t^2 = \theta_0^2$. Furthermore, if $\mathcal{F}_t$ is generated by W_t, then $(\mathcal{X}_t)_{t \le T_1} \stackrel{d}{=} (S_t)_{t \le T_1}$.

Observe that in this case $Af(x) = \frac{1}{2}x^2 f''(x)$ and $\phi(x) = x^2$ satisfies $A\phi = \phi$.

Again using the approximation,

$$x^2 \simeq \sum_{i=1}^{n} (x - K_{i-1})^+ (K_i - K_{i-2}),$$

where $K_0 = K_{-1} = 0$, one can argue that no other martingale than Z_t satisfies (10) for finitely many strikes and three terminal times.

3.4 Local Black-Scholes – Maturity-dependent volatility

In this section we consider the case when the process θ_t is allowed to depend on the maturity T. The Black-Scholes model with time varying but non-random spot volatility, $V(t)$, is given by

$$d\mathcal{S}_t = V(t)\mathcal{S}_t dW_t.$$

Then the random variable $\mathcal{S}_T$ is Lognormal, and the Black-Scholes formula holds with the averaged future volatility

$$\vartheta^2(t,T) = \frac{1}{T-t}\int_t^T V^2(u)du.$$

That is

$$\mathbb{E}\left[(\mathcal{S}_T - K)^+ | \mathcal{F}_t\right] = C_S(T, t, K, \vartheta^2(t,T), \mathcal{S}_t).$$

The next result shows that if the Black-Scholes formula holds locally, this is the only model.

Theorem 4 *Assume that for some T^*, $\mathcal{X}_{T^*} > 0$, that $\mathcal{X}$ is a continuous martingale, and that the Black-Scholes formula holds locally for $t \le T^*$, with a predictable process $\theta(t, T^*)$,*

$$\mathbb{E}[\mathcal{X}_{T^*}^2 | \mathcal{F}_t] = \mathbb{E}[S_{T^*}^2 | S_t = s]_{\sigma=\theta(t,T^*), s=\mathcal{X}_t} \tag{14}$$

Then

$$\theta^2(t, T^*) = \frac{\langle \mathcal{L}(\mathcal{X}), \mathcal{L}(\mathcal{X})\rangle_{T^*} - \langle \mathcal{L}(\mathcal{X}), \mathcal{L}(\mathcal{X})\rangle_t}{T^* - t},$$

where $\mathcal{L}(\mathcal{X})$ is a stochastic logarithm of $\mathcal{X}$.

In particular, if $\mathcal{F}_0$ is trivial then $\langle \mathcal{L}(\mathcal{X}), \mathcal{L}(\mathcal{X})\rangle_{T^*}$ is non-random.

Further, suppose that Black-Scholes formula holds locally for $t \le T$ and all $T \le T^*$. Then there exists a non-random function $V(t)$, such that $d\mathcal{X}_t = V(t)\mathcal{X}_t dW_t$, moreover, for all $T \le T^*$, and all $t \le T$,

$$\theta^2(t,T) = \frac{1}{T-t}\int_t^T V^2(u)du. \tag{15}$$

Recall that we call stochastic logarithm of $\mathcal{X}$, a process X such that $\mathcal{X} = \mathcal{X}_0 \mathcal{E}(X)$, see for example [11] and [12], p. 236.

3.5 Non-Gaussian Martingales with Gaussian Marginals

In this section we address the inverse problem of finding a model such that the Bachelier formula holds for all strikes, all maturities but a single point in time taken for simplicity to be 0. That is, we seek a martingale $\mathcal{X}_t$ for which, for all K and all T,

$$\mathbb{E}_Q[(\mathcal{X}_T - K)^+] = C_B(T, 0, K, \sigma, \mathcal{X}_0).$$

Since the function $H_T(K) = \mathbb{E}_Q[(\mathcal{X}_T - K)^+]$ defines the distribution of $\mathcal{X}_T$, this problem is equivalent to that of finding a non-Gaussian martingale the one-dimensional marginals of which are Gaussian coincide with the marginal one-dimensional distributions of the Bachelier model (mean zero and variance t).

In [13], the authors use Azèma-Yor's solution to the Skorokhod embedding problem to obtain such a process. In [8], we construct an entire family of processes with the desired property. Here we give the main results and refer to [8] for details.

3.5.1 A Two-step Process

The main idea in our construction starts with a two-step process (Y_1, Y_2) that has the desired property of being a non-Gaussian martingale for which both Y_1 and Y_2 are Gaussian. This initial step relies on the following observation.

Proposition 5 *For any triple (R, Y_1, ξ) of independent random variables, such that R takes values in $[0, 1]$ and, ξ is standard normal and Y_1 is normal with mean zero and variance q^2, the random variable $Y_2 = p(\sqrt{R}Y_1 + q\sqrt{1-R}\xi)$ is normal with mean zero and variance p^2q^2. However, (Y_1, Y_2) is a bivariate normal pair if and only if R is non-random.*

This is easily seen as the conditional joint distribution of (Y_1, Y_2) given R is bivariate normal with zero means and covariance matrix

$$\begin{pmatrix} q^2 & pq^2\sqrt{R} \\ pq^2\sqrt{R} & p^2q^2 \end{pmatrix}.$$

Since the marginals of a bivariate normal pair have distributions that do not depend on their correlation, the unconditional distribution of Y_2 is shown to be normal. However, (Y_1, Y_2) is clearly not a bivariate normal pair unless R is non-random.

Now, the two-step process (Y_1, Y_2) is a martingale if and only if

$$Y_1 = \mathbb{E}[Y_2|Y_1] = \mathbb{E}[p(\sqrt{R}Y_1 + q\sqrt{1-R}\xi)|Y_1] = p\mathbb{E}[\sqrt{R}]Y_1,$$

in other words R must satisfy the condition $\mathbb{E}\left[\sqrt{R}\right] = 1/p$.

The processes we are about to define reproduce the above at all times $s < t$. Indeed, we look for a process $\mathcal{X}$ such that for any $0 < s < t$, the random variables $\mathcal{X}_s$ and $\mathcal{X}_t$ satisfy the representation:

$$\mathcal{X}_t = \sqrt{\frac{t}{s}}\left(\sqrt{R_{s,t}}\mathcal{X}_s + \sqrt{s}\sqrt{1-R_{s,t}}\xi_{s,t}\right). \tag{16}$$

It turns out that this is possible whenever the distributions of $R_{s,t}$ form what we call a log-convolution semi-group (plus one extra requirement that ensures that the process is indeed a martingale). The precise definition is given in the next section. By construction, these processes will be Markovian, albeit time non-homogeneous.

3.5.2 The Transition Densities

As Markov processes, our constructs can be defined by their transition density functions. We use (16) to obtain these. First, we introduce the concept of a log-convolution semi-group.

Definition 6 *The family of distributions on* $(0, +\infty)$ $(G_p)_{p\geq 1}$ *is a log-convolution semi-group if* G_1 *is the Dirac mass at 1 and the the distribution of the product of any two independent random variables with distributions* G_p *and* G_q*, is* G_{pq}*.*

Proposition 7 *Define, for* $x \in \mathbb{R}$*,* $s > 0$ *and* $t = p^2 s \geq s$*,* $P_{s,t}(x, dy)$ *as,*

$$P_{0,t}(x, dy) = \frac{1}{\sqrt{2\pi}\sqrt{t}} \exp\left(-\frac{(y-x)^2}{2t}\right) dy \tag{17}$$

and

$$P_{s,t}(x, dy) = \gamma(p)\varepsilon_{px}(dy) + \left[\int_{(0,1)} \frac{1}{\sqrt{2\pi}\sqrt{t}\sqrt{1-r}} \exp\left(-\frac{(y-p\sqrt{r}x)^2}{2t(1-r)}\right) G_p(dr)\right] dy, \tag{18}$$

where $\gamma(p) = G_p(\{1\})$*.*

If $(G_p)_{p\geq 1}$ *is a log-convolution semi-group on* $(0, 1]$ *then the Chapman-Kolmogorov equations hold. That is, for any* $u > t > s > 0$ *and any* x

$$\int P_{s,t}(x, dy)P_{t,u}(y, dz) = P_{s,u}(x, dz) \tag{19}$$

and, for any $u > t > 0$*,*

$$\int P_{0,t}(0, dy)P_{t,u}(y, dz) = P_{0,u}(0, dz). \tag{20}$$

It is then natural to seek a representation of log-convolution semi-groups. This is easily done by relating them to (classical) convolution semi-groups and use the well-known Lévy-Khinchin representation. This yields to the main theorem of this section.

Theorem 8 *Assume that the family* $(G_p)_{p\geq 1}$ *is a log-convolution semi-group with Laplace exponent*

$$\psi(\lambda) = -\frac{\ln \mathbb{E}\left[e^{\lambda \ln R_p}\right]}{\ln p} = \beta\lambda + \int_0^\infty \left(1 - e^{-\lambda x}\right) \nu(dx).$$

If $\psi(1/2) = 1$*, then there exists a Markov martingale* $\mathcal{X}_t$ *starting at zero with transition probabilities* $P_{s,t}(x, dy)$ *given by (18) and (17) the marginal distributions of which are Gaussian with mean zero and variance* t*.*

Basic properties of the process $\mathcal{X}_t$ can then be obtained (see [8] for details). In particular, we have the following result.

Theorem 9 *The (predictable) quadratic variation of* $\mathcal{X}_t$ *is*

$$\langle \mathcal{X}, \mathcal{X} \rangle_t = \delta t + (1-\delta) \int_0^t \frac{\mathcal{X}_s^2}{s} ds,$$

where $\delta = \psi(1)/2$*.*

Further, the process $\mathcal{X}_t$ *is continuous if and only if* $R_{s,t} \equiv s/t$*, in which case* $\mathcal{X}_t$ *is a standard Brownian motion.*

We end this section with some explicit constructions. These fall into two categories according to whether or not $G_p(\{1\}) = 0$ uniformly in $p > 1$. Indeed, if R_p has distribution G_p, then

$$\mathbb{E}\left[e^{\lambda \ln R_p}\right] = \mathbb{P}[R_p = 1] + \mathbb{E}[e^{\lambda \ln R_p}, R_p < 1]$$

and

$$\gamma(p) = \mathbb{P}[R_p = 1] = \lim_{\lambda\uparrow\infty} L_p(\lambda) = \lim_{\lambda\uparrow\infty} \exp\left(-\psi(\lambda)\ln p\right).$$

That is, uniformly in $p > 1$,

$$\gamma(p) = 0 \Leftrightarrow \lim_{\lambda\uparrow\infty} \psi(\lambda) = \infty.$$

3.5.3 The Case $\gamma(p) > 0$

Proposition 10 *Let G_p be a log-convolution semi-group for which $\gamma(p) = G_p(\{1\}) > 0$, γ is differentiable at 1 and $\lim_{\lambda\downarrow 0} \psi(\lambda) = 0$. Then the infinitesimal generator of $\mathcal{X}_t$ on the set of C_0^2-functions is given by $A_0 f(x) = \frac{1}{2} f''(x)$ and for $s > 0$,*

$$A_s f(x) = \frac{x}{2s} f'(x) + \frac{-\gamma'(1)}{2s} \int [f(x+z) - f(x)] \int_{[0,1)} \phi((\sqrt{r}-1)x, s(1-r), z)\bar{G}(dr)dz,$$

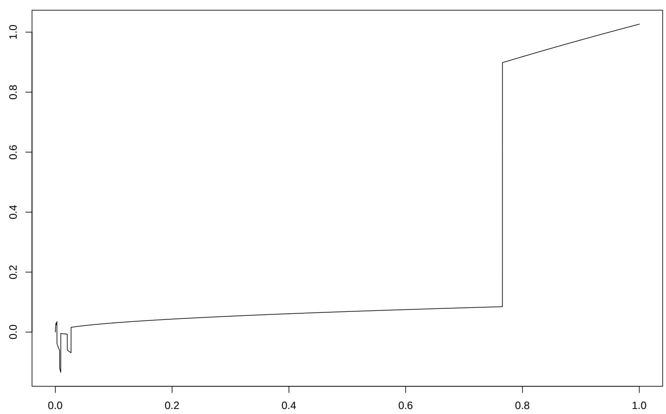
where $\bar{G}(dr) = \lim_{p\downarrow 1}(G_p(dr \cap [0,1))/G_p([0,1)))$ is a probability measure on $[0,1)$, and the limit is understood in the weak sense.

This result explains in detail the behaviour of the process. The process $\mathcal{X}$ starts off as a Brownian motion. While in positive territory, $\mathcal{X}_t$ continuously drifts upwards and has jumps that tend to be negative. In negative region, the reverse occurs; $\mathcal{X}_t$ drifts downwards and has (on average) positive jumps.

Example 11 *Log-Poisson distribution, namely $-\ln R_p$ is distributed as a Poisson random variable with parameter $c \ln p$, with $c = 1/(1 - e^{-1/2})$. This corresponds to the case $\gamma(p) = p^{-c}$, $\beta = 0$, $\nu(dx) = c\varepsilon_1(dx)$, $\psi(\lambda) = c(1 - e^{-\lambda})$ and*

$$A_s f(x) = \frac{x}{2s} f'(x) + \frac{c}{2s} \int [f(x+z) - f(x)]\phi(-x/c, s(1-e^{-1}), z)dz.$$

From the form of the generator we deduce that the process jumps at the rate of $\frac{c}{2s}$ with a size distributed as a Gaussian random variable with mean $-\frac{x}{c}$ and variance $s(1 - e^{-1})$. A simulation of a path of such a process is given below.



3.5.4 The Case $\gamma(p) = 0$

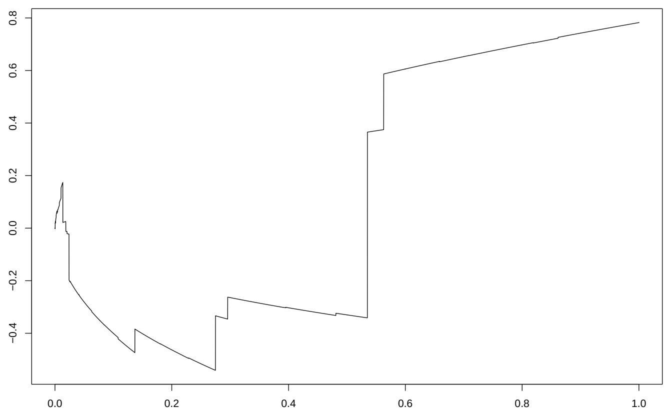
Theorem 12 *Assume that $\beta = 0$. For any polynomial f and any $s > 0$,*

$$A_s f(x) = \frac{x}{2s} f'(x) + \frac{1}{2s} \int [f(x+y) - f(x)] \int_0^{+\infty} \phi((e^{-\omega/2} - 1)x, s(1 - e^{-\omega}), y) \nu(d\omega) dy. \qquad (21)$$

Example 13 *Inverse Log-Gamma distribution, namely $-\ln R_p$ is distributed as Gamma(a, b) with $a = 1/\ln\left(1 + \frac{1}{2b}\right)$. That is R_p has density*

$$g_p(r) = \frac{b^{a \ln p}}{\Gamma(a \ln p)} (-\ln r)^{a \ln p - 1} r^{b-1}, \quad 0 < r < 1.$$

This corresponds to $\beta = 0$, $\nu(dx) = ax^{-1}e^{-bx}dx$ and $\psi(\lambda) = a \ln\left(1 + \frac{\lambda}{b}\right)$. A simulation of a path of such a process is given below.



References

[1] Black, F. and M. Scholes, The pricing of options and corporate liabilities, *Journal of Political Economy*, vol.81, no.3, pp.637-654, 1973.

[2] Brace, A., B. Goldys, F. Klebaner and R. Wormersley, Market model of stochastic implied volatility with application to BGM model, Working paper S01–1, Department of Statistics, University of NSW, 2001.

[3] Brace, A., B. Goldys, J. van der Hoek and R. Wormersley, Markovian models in implied volatility framework, *Statistics preprints*, University of NSW, 2002.

[4] Carr, P., A survey of preference free option valuation with stochastic volatility, *April Risk Conference*, Paris, 2000.

[5] Danilo, M. and V. Spokoinyi, Estimation of time dependent volatility via local change point analysis, *Ann. Statist.*, vol.32, no.2, pp.577–602, 2004.

[6] Goldentyer, L., F. C. Klebaner and R. Liptser, Tracking volatility, *Problems of Information Transmition*, vol.41, pp.32–50, 2005.

[7] Hamza, K. and F. C. Klebaner, On nonexistence of non-constant volatility in the Black-Scholes formula, *Discrete and Continuous Dynamical Systems*, vol.6, pp.829–834, 2006.

[8] Hamza, K. and F. C. Klebaner, A family of non-Gaussian martingales with Gaussian marginals, *J. Applied Math. and Stochastic Analysis,* vol. 2007, Article ID 92723, 19 pages, 2007. doi:10.1155/2007/92723.

[9] Fouque, J-P., G. Papanicolau and R. Sircar, *Derivatives in Financial Markets with Stochastic Volatility*, Cambridge Universisty Press, 2000.

[10] Hull, J. and A. White, The pricing of options on assets with stochastic volatilities, *Journal of Finance*, vol.42, pp.281–300, 1987.

[11] Kallsen, J. and A. N. Shiryaev, The cumulant process and Esscher's change of measure, *Finance Stochast*, vol.6, pp.397–428, 2002.

[12] Klebaner, F. C., *Introduction to Stochastic Calculus with Applications*, Imperial College Press, 2nd Ed., 2005.

[13] Madan, D. P. and M. Yor, Making markov martingales meet marginals: with explicit constructions, *Bernoulli*, vol.8, pp.509–536, 2002.

[14] Schonbucher, P. J., A market model for stochastic implied volatility, *Phil. Trans. of the Royal Soc.*, Series A, vol.357, pp.2071–2092, 1999.

[15] Shiryaev, A. N., *Essential of Stochastic Finance*, World Scientific, 1999.

[16] Stein, E. M. and J. C. Stein, Stock price distributions with stochastic volatility – an analytic approach, *Review of Financial Studies*, vol.4, pp.727–752, 1991.