Journal of Uncertain Systems
Vol.1, No.3, pp.185-200, 2007
Online at: www.jus.org.uk

On the Continuity and Absolute Continuity of Credibility Functions*

Shuming Wang†, Ying Liu, and Xiao-Dong Dai
College of Mathematics and Computer Science, Hebei University, Baoding 071002, Hebei, China

Received March 15 2007; Accepted 20 June 2007

Abstract

The objective of this paper is to study some new analytical properties of credibility functions. We first discuss the continuity of credibility functions, and establish the sufficient and necessary conditions for the right continuity and left continuity of credibility functions. Furthermore, we study the absolute continuity of credibility functions and obtain some sufficient conditions for the absolute continuity of credibility functions. The results obtained in this paper are useful in fuzzy optimization theory.
Keywords: Fuzzy variable, credibility function, continuity, absolute continuity

1 Introduction

Possibility measure was proposed by Zadeh [16] in 1978 to measure a fuzzy event. Since then, possibility theory has been studied by many researchers [1, 3, 15], and become a useful tool to deal with possibilistic uncertainty. Many successful applications of possibility theory can be found in the field of fuzzy optimization [2, 7, 12, 14]. In addition, a self-dual set function, called credibility measure was presented by Liu and Liu [9] based on possibility measure, which can be regarded as the counterpart of probability measure in fuzzy decision systems. Based on credibility measure, an axiomatic approach, called credibility theory [6, 8], was studied extensively. From a measure-theoretic viewpoint, credibility theory provides a theoretical foundation for fuzzy programming [5, 10, 11], just like the role of probability theory in stochastic programming [4].

It is known that, in probability theory, the probability distribution function of a random variable X, $F(x) = \Pr\{X \le x\}$, is a right continuous function. However, in credibility theory, since credibility measure is nonadditive, the credibility function of a fuzzy variable ξ, $\Phi_L(x) = \mathrm{Cr}\{\xi \le x\}$, is neither left continuous nor right continuous [6]. In many fuzzy programming problems [5, 10, 11], their objective functions and constraints consist of credibility functions and critical value functions which actually are the inverse functions of credibility functions, the continuity of objective and constraints may be useful in solving these programming problems. Therefore, to study the continuity of credibility functions will be a meaningful work in both theory and practice.

On the other hand, recall that in probability theory, a random variable X is continuous if and only if the probability distribution function of X is absolutely continuous, which can be written as

$$F(x) = \int_{-\infty}^{x} f(t)\mathrm{d}t,$$

where $f(x)$ is the probability density function of X. However, in credibility theory, we don't have the result as above. Let's see an example given by Liu [6]. Assume ξ is a continuous fuzzy variable

*This work was supported by Program for One Hundred Excellent and Innovative Talents in Colleges and Universities of Hebei Province, and the National Natural Science Foundation of China Grant No.70571021.
†Corresponding author: smwang@mail.hbu.edu.cn.

(i.e., possibility distribution is continuous) with the following possibility distribution is defined by

$$\mu(x) = \begin{cases} g(x), & \text{if } x \in [0,1] \\ g(2-x), & \text{if } x \in (1,2] \\ 0, & \text{otherwise} \end{cases}$$

where $g(x)$ is the Cantor function defined on $[0,1]$. It is easy to check that the credibility function $\Phi_L(x) = \text{Cr}\{\xi \le x\}$ is a singular function. Therefore, the credibility density function does not exist. That is $\Phi(x)$ is not absolutely continuous. This example shows that the credibility function does not necessarily absolutely continuous even though the fuzzy variable is continuous.

Motivated by these considerations, the purpose of this paper is to investigate the continuity of credibility functions and to establish the conditions for the absolute continuity of credibility functions.

The paper is organized as follows. Section 2 recalls some basic concepts in credibility theory. In Section 3, we discuss the continuity of credibility functions. Section 4 studies the absolute continuity of credibility functions. Finally, a brief summary is drawn in Section 5.

2 Basic Concepts

Given a universe Γ, let Pos be a set function defined on $\mathcal{P}(\Gamma)$. The set function Pos is said to be a possibility measure if it satisfies the following conditions

($P1$) Pos($\emptyset$)=0, and $\text{Pos}(\Gamma) = 1$;

($P2$) $\text{Pos}\left(\bigcup_{i\in I} A_i\right) = \sup_{i\in I} \text{Pos}(A_i)$ for any subclass $\{A_i \mid i \in I\}$ of $\mathcal{P}(\Gamma)$, where I is an arbitrary index set.

By the definition of possibility measure, we know it is a lower continuous set function. Based on possibility measure, a self-dual set function Cr, called credibility measure, was formally defined as follows:

Definition 2.1 ([9]) *Let A be a subset of Γ. The credibility measure of A, denoted* Cr(A)*, is defined as*

$$\text{Cr}(A) = \frac{1}{2}\left(1 + \text{Pos}(A) - \text{Pos}(A^c)\right) \tag{1}$$

where A^c is the complement of A.

A credibility measure has the following properties:

(Cr1) Cr($\emptyset$)=0, and $\text{Cr}(\Gamma) = 1$.

(Cr2) Monotonicity: $\text{Cr}(A) \le \text{Cr}(B)$ for all $A, B \subset \Gamma$ with $A \subset B$.

(Cr3) Self-duality: $\text{Cr}(A) + \text{Cr}(A^c) = 1$ for all $A \subset \Gamma$.

(Cr4) Subadditivity: $\text{Cr}(A \cup B) \le \text{Cr}(A) + \text{Cr}(B)$ for all $A, B \subset \Gamma$.

A function $\xi : \Gamma \to \Re$ is said to be a fuzzy variable defined on Γ [6]. For any $x \in \Re$, the possibility $\text{Pos}\{\gamma \in \Gamma \mid \xi(\gamma) = x\}$ is said to be the possibility distribution of the fuzzy variable ξ, denoted by μ.

Let ξ be a fuzzy variable with possibility distribution μ. Then we say ξ is right continuous [resp., left continuous, upper semicontinuous or lower semicontinuous] iff μ is right continuous [resp., left continuous, upper semicontinuous or lower semicontinuous].

Furthermore, the possibility of the event $\{\xi \geq x\}$ is given by

$$\text{Pos}\{\xi \geq x\} = \sup_{t \geq x} \mu(t), \tag{2}$$

and from Definition 2.1, the credibility of $\{\xi \geq x\}$ is given by

$$\text{Cr}\{\xi \geq x\} = \frac{1}{2}(1 + \text{Pos}\{\xi \geq x\} - \text{Pos}\{\xi < x\}). \tag{3}$$

Using credibility measure, the credibility function is defined as follows.

Definition 2.2 ([6]) *Let ξ be a fuzzy variable defined on Γ. The credibility function of ξ is defined by*

$$\Phi_L(x) = \text{Cr}\{\gamma \in \Gamma \mid \xi(\gamma) \leq x\} \tag{4}$$

or

$$\Phi_U(x) = \text{Cr}\{\gamma \in \Gamma \mid \xi(\gamma) \geq x\} \tag{5}$$

for every $x \in \Re$.

3 The Continuity of Credibility Functions

Before studying the properties of credibility functions, in this section, we first discuss the properties of possibility functions.

3.1 The Continuity of Possibility Functions

Proposition 3.1 *Let ξ be a fuzzy variable with possibility distribution $\mu(x)$. Then for any $x \in \Re$, we have*

(1) *If $\mu(x)$ is right continuous, then* $\text{Pos}\{\xi > x\} = \text{Pos}\{\xi \geq x\}$.

(2) *If $\mu(x)$ is left continuous, then* $\text{Pos}\{\xi < x\} = \text{Pos}\{\xi \leq x\}$.

Proof. We use a proof by contradiction. Suppose

$$\text{Pos}\{\xi \geq x\} = \mu(x) \vee \text{Pos}\{\xi > x\} > \text{Pos}\{\xi > x\}.$$

Then we have

$$\mu(x) > \text{Pos}\{\xi > x\},$$

which implies there exists a positive number ε_0 such that $\mu(x) > \text{Pos}\{\xi > x\} + \varepsilon_0$. Therefore, by $\text{Pos}\{\xi > x\} = \sup_{t>x} \mu(t)$, for any $\delta > 0$, when t is a point such that $0 < t - x < \delta$, we have $\mu(x) > \mu(t) + \varepsilon_0$, which is a contradiction with the right continuity of $\mu(x)$. The proof is complete.

Proposition 3.2 *Let ξ be a lower semicontinuous fuzzy variable. Then for any $x \in \Re$,*

(1) $\text{Pos}\{\xi < x\} = \text{Pos}\{\xi \leq x\}$*, and*

(2) $\text{Pos}\{\xi > x\} = \text{Pos}\{\xi \geq x\}$.

Proof. We use a proof by contradiction. Suppose

$$\text{Pos}\{\xi \le x\} = \mu(x) \vee \text{Pos}\{\xi < x\} > \text{Pos}\{\xi < x\}.$$

Then we have

$$\mu(x) > \text{Pos}\{\xi < x\},$$

which deduces there is a positive number ε_0 such that $\mu(x) > \text{Pos}\{\xi < x\} + \varepsilon_0$. Furthermore, by $\text{Pos}\{\xi < x\} = \sup_{t<x} \mu(t)$, for any $\delta > 0$ with $0 < x - t < \delta$, one has $\mu(x) > \mu(t) + \varepsilon_0$, which is a contradiction with the lower semicontinuity of $\mu(x)$. The proposition is proved.

Proposition 3.3 *For any fuzzy variable ξ we have*

(1) $\text{Pos}\{\xi < x\}$ *is left continuous, and*

(2) $\text{Pos}\{\xi > x\}$ *is right continuous.*

Proof. We are to prove for any $x \in \Re$ and any sequence $\{\varepsilon_n\}$ with $\varepsilon_n \downarrow 0$ the following limit

$$\lim_{n\to\infty} \text{Pos}\{\xi < x - \varepsilon_n\} = \text{Pos}\{\xi < x\}$$

holds. Since for any $\{\varepsilon_n\}$, $\varepsilon_n \downarrow 0$, we have $\{\xi \in (-\infty, x - \varepsilon_n)\} \uparrow \{\xi \in (-\infty, x)\}$. It follows from the lower semicontinuity of Pos that

$$\lim_{n\to\infty} \text{Pos}\{\xi < x - \varepsilon_n\} = \text{Pos}\left(\bigcup_{n=1}^{\infty} \{\xi \in (-\infty, x - \varepsilon_n)\}\right) = \text{Pos}\{\xi < x\}.$$

The proof is complete.

Corollary 3.1 *For any fuzzy variable ξ, we have*

(1) *If ξ is a left continuous fuzzy variable, then* $\text{Pos}\{\xi \le x\}$ *is left continuous.*

(2) *If ξ is a right continuous fuzzy variable, then* $\text{Pos}\{\xi \ge x\}$ *is right continuous.*

(3) *If ξ is a lower semicontinuous fuzzy variable, then* $\text{Pos}\{\xi \le x\}$ *is left continuous and* $\text{Pos}\{\xi \ge x\}$ *is right continuous.*

Proof. From assertions (2) in Proposition 3.1 and (1) in Proposition 3.3, we can obtain the assertion (1) immediately. The assertion (2) is obvious from assertions (1) in Proposition 3.1 and (2) in Proposition 3.3. We now prove assertion (3), on one hand, by assertion (1) in Proposition 3.2 and assertion (1) in Proposition 3.3, we get $\text{Pos}\{\xi \le x\}$ is left continuous; on the other hand, assertions (2) in Proposition 3.2 and (2) in Proposition 3.3 imply that $\text{Pos}\{\xi \ge x\}$ is right continuous.

Proposition 3.4 *Let ξ be a fuzzy variable. Then*

(1) $\text{Pos}\{\xi \le x\}$ *is right continuous provided ξ is right continuous;*

(2) $\text{Pos}\{\xi \ge x\}$ *is left continuous provided ξ is left continuous.*

Proof. For assertion (1), it suffices to prove for any $x_0 \in \Re$, $\text{Pos}\{\xi \le x\}$ is right continuous at x_0 provided $\mu(x)$ is right continuous. In fact, for any $\varepsilon > 0$, there is $\delta > 0$ such that $|\mu(x)-\mu(x_0)| < \varepsilon/2$ whenever $0 < x - x_0 < \delta$. Thus for all x with $0 < x - x_0 < \delta$, we have

$$|\text{Pos}\{\xi \le x\} - \text{Pos}\{\xi \le x_0\}| = |\text{Pos}\{\xi \le x_0\} \vee \text{Pos}\{x_0 < \xi \le x\} - \text{Pos}\{\xi \le x_0\}|.$$

If $\text{Pos}\{x_0 < \xi \le x\} \le \text{Pos}\{\xi \le x_0\}$, then

$$|\text{Pos}\{\xi \le x\} - \text{Pos}\{\xi \le x_0\}| = 0.$$

If $\text{Pos}\{x_0 < \xi \le x\} > \text{Pos}\{\xi \le x_0\}$, then, by $\text{Pos}\{x_0 < \xi \le x\} = \sup_{x_0<t\le x} \mu(t)$, for the above $\varepsilon > 0$, there exists $t_0 \in (x_0, x]$ such that

$$\text{Pos}\{\xi \le x_0\} < \text{Pos}\{x_0 < \xi \le x\} < \mu(t_0) + \frac{\varepsilon}{2} < \mu(x_0) + \varepsilon \le \text{Pos}\{\xi \le x_0\} + \varepsilon,$$

which implies

$$|\text{Pos}\{\xi \le x\} - \text{Pos}\{\xi \le x_0\}| < \varepsilon.$$

It follows that $\text{Pos}\{\xi \le x\}$ is right continuous.

Proposition 3.5 *Let ξ be an upper semicontinuous fuzzy variable. Then we have*

(1) $\text{Pos}\{\xi \le x\}$ *is right continuous, and*

(2) $\text{Pos}\{\xi \ge x\}$ *is left continuous.*

Proof. To prove that $\text{Pos}\{\xi \le x\}$ is right continuous, it suffices to show that for any $x_0 \in \Re$, $\text{Pos}\{\xi \le x\}$ is right continuous at x_0 provided $\mu(x)$ is upper semicontinuous. In fact, for any $\varepsilon > 0$, there is $\delta > 0$ such that $\mu(x) < \mu(x_0) + \varepsilon/2$ whenever $|x - x_0| < \delta$. Thus, as $0 < x - x_0 < \delta$, we have

$$|\text{Pos}\{\xi \le x\} - \text{Pos}\{\xi \le x_0\}| = |\text{Pos}\{\xi \le x_0\} \vee \text{Pos}\{x_0 < \xi \le x\} - \text{Pos}\{\xi \le x_0\}|\,.$$

If $\text{Pos}\{x_0 < \xi \le x\} \le \text{Pos}\{\xi \le x_0\}$, then

$$|\text{Pos}\{\xi \le x\} - \text{Pos}\{\xi \le x_0\}| = 0.$$

If $\text{Pos}\{x_0 < \xi \le x\} > \text{Pos}\{\xi \le x_0\}$, then, by $\text{Pos}\{x_0 < \xi \le x\} = \sup_{x_0<t\le x} \mu(t)$, for above $\varepsilon > 0$, there exists $t_0 \in (x_0, x]$ such that

$$\text{Pos}\{\xi \le x_0\} < \text{Pos}\{x_0 < \xi \le x\} < \mu(t_0) + \frac{\varepsilon}{2} < \mu(x_0) + \varepsilon \le \text{Pos}\{\xi \le x_0\},$$

which implies

$$|\text{Pos}\{\xi \le x\} - \text{Pos}\{\xi \le x_0\}| < \varepsilon.$$

It follows that $\text{Pos}\{\xi \le x\}$ is right continuous.

Remark 3.1 *For Propositions 3.1, 3.2, 3.3, 3.4 and 3.5, we only prove the first assertion, the second one can be proved in a similar way.*

Proposition 3.6 *Let ξ be a fuzzy variable with the possibility distribution $\mu(x)$. Then* $\text{Pos}\{\xi \le x\}$ *is left continuous at x_0 if and only if $\mu(x_0) \le \sup_{t<x_0} \mu(t)$*

Proof. *Sufficiency:* Since $\mu(x_0) \le \sup_{t<x_0} \mu(t) = \mathrm{Pos}\{\xi < x_0\}$, we have

$$\mathrm{Pos}\{\xi < x_0\} = \mathrm{Pos}\{\xi \le x_0\}.$$

For any sequence $\{\varepsilon_n\}$, $\varepsilon_n \downarrow 0$, one has

$$\{\xi \le x_0 - \varepsilon_n\} \uparrow \{\xi < x_0\},$$

and

$$\mathrm{Pos}\left(\bigcup_{n=1}^{\infty} \{\xi \le x_0 - \varepsilon_n\}\right) = \mathrm{Pos}\{\xi < x_0\}.$$

That is

$$\lim_{x \to x_0-} \mathrm{Pos}\{\xi \le x\} = \mathrm{Pos}\{\xi < x_0\} = \mathrm{Pos}\{\xi \le x_0\}.$$

The sufficiency of the proposition is proved.

Necessity: We use a proof by contradiction. Suppose $\mu(x_0) > \sup_{t<x_0} \mu(t) = \mathrm{Pos}\{\xi < x_0\}$. Then

$$\mathrm{Pos}\{\xi < x_0\} < \mathrm{Pos}\{\xi \le x_0\}.$$

Since for the above sequence $\{\varepsilon_n\}$, we have

$$\mathrm{Pos}\left(\bigcup_{n=1}^{\infty} \{\xi \le x_0 - \varepsilon_n\}\right) = \mathrm{Pos}\{\xi < x_0\}.$$

It follows from the supposition that

$$\lim_{x \to x_0-} \mathrm{Pos}\{\xi \le x\} = \mathrm{Pos}\{\xi < x_0\} < \mathrm{Pos}\{\xi \le x_0\},$$

which is a contradiction with the left continuity of $\mathrm{Pos}\{\xi \le x\}$ at x_0. The necessity is proved.

Similarly, for the possibility function $\mathrm{Pos}\{\xi \ge x\}$, we have the following result.

Proposition 3.7 *Let ξ be a fuzzy variable. Then $\mathrm{Pos}\{\xi \ge x\}$ is right continuous at x_0 if and only if $\mu(x_0) \le \sup_{x>x_0} \mu(x)$.*

3.2 The Continuity of Credibility Functions

The intent of this subsection is to study the continuity of credibility functions. We will show that under some wild conditions, the credibility function of a fuzzy variable is right continuous, left continuous and continuous, respectively.

Theorem 3.1 *Let ξ be a fuzzy variable with the possibility distribution $\mu(x)$. Then the credibility function $\Phi_L(x)$ is right continuous at $x_0 \in \Re$ if and only if the following inequality*

$$\limsup_{t \to x_0+} \mu(x) \le \sup_{x \le x_0} \mu(x)$$

holds.

Proof. *Sufficiency:* On the one hand, if

$$\limsup_{x \to x_0+} \mu(x) = \inf_{\delta>0} \sup_{0<x-x_0<\delta} \mu(x) \le \mathrm{Pos}\{\xi \le x_0\} = \sup_{x \le x_0} \mu(x),$$

then for any $\varepsilon > 0$, there is an $\eta > 0$ such that

$$\sup_{0<x-x_0<\eta} \mu(x) - \varepsilon < \inf_{\delta>0} \sup_{0<x-x_0<\delta} \mu(x) \leq \text{Pos}\{\xi \leq x_0\}.$$

Thus, for all x with $0 < x - x_0 < \eta$, we have

$$\text{Pos}\{x_0 < \xi \leq x\} - \varepsilon \leq \sup_{0<x-x_0<\eta} \mu(x) - \varepsilon < \text{Pos}\{\xi \leq x_0\},$$

which implies

$$\text{Pos}\{\xi \leq x_0\} \leq \text{Pos}\{\xi \leq x\} < \text{Pos}\{\xi \leq x_0\} + \varepsilon,$$

i.e.,

$$|\text{Pos}\{\xi \leq x\} - \text{Pos}\{\xi \leq x_0\}| < \varepsilon.$$

It follows that $\text{Pos}\{\xi \leq x\}$ is right continuous at x_0.

On the other hand, $\text{Pos}\{\xi > x\}$ is right continuous from assertion (2) in Proposition 3.3. It follows that

$$\Phi_L(x) = \frac{1}{2}(1 + \text{Pos}\{\xi \leq x\} - \text{Pos}\{\xi > x\})$$

is also right continuous at x_0. The sufficiency is proved.

Necessity: Let $\Phi_L(x)$ be right continuous at x_0. By assertion (2) in Proposition 3.3 and

$$\text{Pos}\{\xi \leq x\} = 2\Phi_L(x) + \text{Pos}\{\xi > x\} - 1,$$

we know that $\text{Pos}\{\xi \leq x\}$ is also right continuous at x_0. Suppose

$$\sup_{x\leq x_0} \mu(x) < \limsup_{x\to x_0+} \mu(x) = \inf_{\delta>0} \sup_{0<x-x_0<\delta} \mu(x).$$

Then there is a $\varepsilon_0 > 0$ such that

$$\inf_{\delta>0} \sup_{0<x-x_0<\delta} \mu(x) > \text{Pos}\{\xi \leq x_0\} + 2\varepsilon_0,$$

which implies for any $\delta > 0$, we have

$$\sup_{0<x-x_0<\delta} \mu(x) > \text{Pos}\{\xi \leq x_0\} + 2\varepsilon_0.$$

Thus, for the above given ε_0 and any $\delta > 0$, there is $x_\delta^* \in (x_0, x_0 + \delta)$ such that

$$\mu(x_\delta^*) + \varepsilon_0 > \sup_{0<x-x_0<\delta} \mu(x) > \text{Pos}\{\xi \leq x_0\} + 2\varepsilon_0.$$

As a consequence,

$$\text{Pos}\{x_0 < \xi \leq x_\delta^*\} \geq \mu(x_\delta^*) > \text{Pos}\{\xi \leq x_0\} + \varepsilon_0,$$

and

$$\text{Pos}\{\xi \leq x_\delta^*\} \geq \text{Pos}\{x_0 < \xi \leq x_\delta^*\} > \text{Pos}\{\xi \leq x_0\} + \varepsilon_0,$$

which is a contradiction with the right continuity of $\text{Pos}\{\xi \leq x\}$ at x_0. Therefore, we can obtain

$$\limsup_{x\to x_0+} \mu(x) \leq \sup_{x\leq x_0} \mu(x).$$

The necessity is proved.

Corollary 3.2 *If fuzzy variable ξ is right continuous or upper semicontinuous, then the credibility function $\Phi_L(x)$ is right continuous.*

Proof. Assume $\mu(x)$ is the possibility distribution of fuzzy variable ξ. On the one hand, by the definition of the right continuity of $\mu(x)$, we can deduce the following inequality

$$\limsup_{t\to x+}\mu(t)=\mu(x)\le\sup_{t\le x}\mu(t) \tag{6}$$

holds for any $x\in\Re$.

On the other hand, the upper semicontinuity of $\mu(x)$ implies the following inequality

$$\limsup_{t\to x+}\mu(t)\le\limsup_{t\to x}\mu(t)\le\mu(x)\le\sup_{t\le x}\mu(t) \tag{7}$$

holds for any $x\in\Re$.

Thus, by Theorem 3.1, Eq.(6) and Eq.(7) implies the desired result.

Theorem 3.2 *If fuzzy variable ξ is both left continuous and lower semicontinuous, then the credibility function $\Phi_L(x)$ is left continuous.*

Proof. If $\mu(x)$ is left continuous, then it follows from the assertion (2) in Corollary 3.1 that $\text{Pos}\{\xi\le x\}$ is left continuous. By (2) in Proposition 3.4, we know that $\text{Pos}\{\xi\ge x\}$ is left continuous, and by (2) in Proposition 3.2, $\text{Pos}\{\xi>x\}$ is left continuous. Consequently, it follows from

$$\Phi_L(x)=\frac{1}{2}(1+\text{Pos}\{\xi\le x\}-\text{Pos}\{\xi>x\})$$

that $\Phi_L(x)$ is also left continuous. The proof of the theorem is complete.

Theorem 3.3 *Let ξ be a fuzzy variable with the possibility distribution $\mu(x)$. Then credibility function $\Phi_U(x)$ is left continuous at $x_0\in\Re$ if and only if inequality*

$$\limsup_{x\to x_0-}\mu(x)\le\sup_{x\ge x_0}\mu(x)$$

holds.

Proof. *Sufficiency:* On one hand, if

$$\limsup_{x\to x_0-}\mu(x)=\inf_{\delta>0}\sup_{0<x_0-t<\delta}\mu(t)\le\sup_{x\ge x_0}\mu(x)=\text{Pos}\{\xi\ge x_0\},$$

then for any $\varepsilon>0$, there exists $\eta>0$ such that

$$\sup_{0<x_0-t<\eta}\mu(t)-\varepsilon<\inf_{\delta>0}\sup_{0<x_0-t<\delta}\mu(t)<\text{Pos}\{\xi\ge x_0\}.$$

Therefore, for all x with $0<x_0-x<\eta$, we have

$$\text{Pos}\{x\le\xi<x_0\}-\varepsilon\le\sup_{0<x_0-x<\eta}\mu(x)-\varepsilon<\text{Pos}\{\xi\ge x_0\},$$

which implies

$$\text{Pos}\{\xi\ge x_0\}\le\text{Pos}\{\xi\ge x\}<\text{Pos}\{\xi\ge x_0\}+\varepsilon,$$

i.e.,

$$|\text{Pos}\{\xi \geq x\} - \text{Pos}\{\xi \geq x_0\}| < \varepsilon.$$

It follows that $\text{Pos}\{\xi \geq x\}$ is left continuous at x_0.

On the other hand, noting that $\text{Pos}\{\xi < x\}$ is left continuous, we have

$$\Phi_U(x) = \frac{1}{2}(1 + \text{Pos}\{\xi \geq x\} - \text{Pos}\{\xi < x\})$$

is also left continuous. The sufficiency is proved.

Necessity: Let $\Phi_U(x)$ be left continuous. It follows from the assertion (1) in Proposition 3.3 and

$$\text{Pos}\{\xi \geq x\} = 2\Phi_U(x) + \text{Pos}\{\xi < x\} - 1.$$

that $\text{Pos}\{\xi \geq x\}$ is left continuous. Suppose

$$\text{Pos}\{\xi \geq x_0\} = \sup_{x \geq x_0} \mu(x) < \limsup_{x \to x_0-} \mu(x) = \inf_{\delta > 0} \sup_{0 < x_0 - x < \delta} \mu(x).$$

Then there is a $\varepsilon_0 > 0$ such that

$$\inf_{\delta > 0} \sup_{0 < x_0 - x < \delta} \mu(x) > \text{Pos}\{\xi \geq x_0\} + 2\varepsilon_0,$$

which implies for any $\delta > 0$, we have

$$\sup_{0 < x_0 - x < \delta} \mu(x) > \text{Pos}\{\xi \geq x_0\} + 2\varepsilon_0.$$

Thus, for the above ε_0 , there exists $x_\delta^* \in (x_0 - \delta, x_0)$ such that

$$\mu(x_\delta^*) + \varepsilon_0 > \sup_{0 < x_0 - x < \delta} \mu(x) > \text{Pos}\{\xi \geq x_0\} + 2\varepsilon_0.$$

As a consequence,

$$\text{Pos}\{x_\delta^* \leq \xi < x_0\} \geq \mu(x_\delta^*) > \text{Pos}\{\xi \geq x_0\} + \varepsilon_0,$$

and

$$\text{Pos}\{\xi \geq x_\delta^*\} \geq \text{Pos}\{x_\delta^* \leq \xi < x_0\} > \text{Pos}\{\xi \geq x_0\} + \varepsilon_0,$$

which is a contradiction with the left continuity of $\text{Pos}\{\xi \geq x\}$ at x_0. Thus, we deduce

$$\limsup_{x \to x_0-} \mu(x) \leq \sup_{x \geq x_0} \mu(x).$$

The necessity is proved.

Corollary 3.3 *If fuzzy variable ξ is left continuous or upper semicontinuous, then the credibility function $\Phi_U(x)$ is left continuous.*

Proof. The proof is similar to that of Corollary 3.2.

Theorem 3.4 *If fuzzy variable ξ is both right continuous and lower semicontinuous, then the credibility function $\Phi_U(x)$ is right continuous.*

Proof. Since $\mu(x)$ is right continuous, it follows from assertion (3) in Corollary 3.1 that $\text{Pos}\{\xi \geq x\}$ is right continuous. By (1) in Proposition 3.4, we deduce that $\text{Pos}\{\xi \leq x\}$ is right continuous; by assertion (1) in Proposition 3.2, we know $\text{Pos}\{\xi < x\}$ is right continuous. Therefore, by Eq. (3), $\Phi_U(x)$ is right continuous. The proof of the theorem is complete.

Theorem 3.5 *For any fuzzy variable ξ we have*

$$\Phi_L(x) = \mathrm{Cr}\{\xi < x\} \tag{8}$$

and

$$\Phi_U(x) = \mathrm{Cr}\{\xi > x\} \tag{9}$$

hold except on an at most countable subset of $\Re$.

Moreover, if ξ is a lower semicontinuous fuzzy variable, then Eq. (8) and Eq. (9) hold for every $x \in \Re$.

Proof. We now prove $\Phi_U(x) = \mathrm{Cr}\{\xi > x\}$ e.c.. Since both $\Phi_U(x)$ and $\mathrm{Cr}\{\xi > x\}$ are monotone and bounded functions on $\Re$, the discontinuity set of $\Phi_U(x)$ or $\mathrm{Cr}\{\xi > x\}$ is at most countable. To prove $\Phi_U(x) = \mathrm{Cr}\{\xi > x\}$ e.c., it suffice to prove Eq.(9) hold on the continuity set of $\Phi_U(x)$ and $\mathrm{Cr}\{\xi > x\}$.

Let $x_0 \in \Re$ be a continuity point of $\Phi_U(x)$ and $\mathrm{Cr}\{\xi > x\}$. On one hand, If $\Phi_U(x) = \mathrm{Cr}\{\xi \geq x_0\} \leq 1/2$, then $\mathrm{Cr}\{\xi > x_0\} \leq 1/2$. It follows from the continuity of $\Phi_U(x)$ at x_0 and the lower continuity of possibility measure that

$$\begin{aligned}
\Phi_U(x_0) &= \lim_{n\to\infty} \Phi_U(x_0+\varepsilon_n) = \frac{1}{2}\lim_{n\to\infty}(\mathrm{Pos}\{\xi \geq x_0+\varepsilon_n\}) \\
&= \frac{1}{2}\mathrm{Pos}\left(\bigcup_{n=1}^{\infty}\{\xi \geq x_0+\varepsilon_n\}\right) \\
&= \frac{1}{2}\mathrm{Pos}\{\xi > x_0\} = \mathrm{Cr}\{\xi > x_0\},
\end{aligned}$$

where $\{\varepsilon_n\}$ is a nonincreasing sequence of positive numbers which converges to 0.

On the other hand, if $\Phi_U(x) = \mathrm{Cr}\{\xi \geq x_0\} > 1/2$, then for any sequence $\{\varepsilon_n\} \downarrow 0$, we have $\mathrm{Cr}\{\xi > x_0 - \varepsilon_n\} \geq \mathrm{Cr}\{\xi \geq x_0\}$. By the continuity of $\mathrm{Cr}\{\xi > x\}$ at x_0, we have

$$\mathrm{Cr}\{\xi > x_0\} = \lim_{n\to\infty} \mathrm{Cr}\{\xi > x_0 - \varepsilon_n\} \geq \mathrm{Cr}\{\xi \geq x_0\} > \frac{1}{2}.$$

Moreover, Combining the continuity of $\mathrm{Cr}\{\xi > x\}$ at x_0 and the lower continuity of possibility measure gives

$$\begin{aligned}
\mathrm{Cr}\{\xi > x_0\} &= \lim_{n\to\infty} \mathrm{Cr}\{\xi > x_0-\varepsilon_n\} = 1 - \frac{1}{2}\lim_{n\to\infty}(\mathrm{Pos}\{\xi \leq x_0-\varepsilon_n\}) \\
&= 1 - \frac{1}{2}\mathrm{Pos}\left(\bigcup_{n=1}^{\infty}\{\xi \leq x_0-\varepsilon_n\}\right) \\
&= 1 - \frac{1}{2}\mathrm{Pos}\{\xi < x_0\} \\
&= \mathrm{Cr}\{\xi \geq x_0\} = \Phi_U(x_0).
\end{aligned}$$

As a consequence, we obtain $\Phi_U(x) = \mathrm{Cr}\{\xi > x\}$ on the continuity set of $\Phi_U(x)$ and $\mathrm{Cr}\{\xi > x\}$. Similarly, we can prove Eq.(8) holds on the continuity set of $\Phi_L(x)$ and $\mathrm{Cr}\{\xi < x\}$. The first part of the theorem is proved.

For the lower semicontinuous fuzzy variable ξ,we only prove $\Phi_L(x) = \mathrm{Cr}\{\xi < x\}$, the equality $\Phi_U(x) = \mathrm{Cr}\{\xi > x\}$ can be proved similarly. By the assertions (1) and (2) in Proposition 3.2, for any $x \in \Re$, we have

$$\mathrm{Pos}\{\xi \leq x\} = \mathrm{Pos}\{\xi < x\}, \text{ and } \mathrm{Pos}\{\xi > x\} = \mathrm{Pos}\{\xi \geq x\}.$$

Since

$$\begin{aligned}\Phi_L(x) &= \frac{1}{2}(1+\text{Pos}\{\xi \le x\} - \text{Pos}\{\xi > x\}) \\ &= \frac{1}{2}(1+\text{Pos}\{\xi < x\} - \text{Pos}\{\xi \ge x\}) \\ &= \text{Cr}\{\xi < x\},\end{aligned}$$

the desired result is valid. The proof of the theorem is complete.

Theorem 3.6 *Let ξ be a lower semicontinuous fuzzy variable with the possibility distribution $\mu(x)$. If*

$$\limsup_{x\to x_0+} \mu(x) \le \sup_{x\le x_0} \mu(x),$$

and

$$\limsup_{x\to x_0-} \mu(x) \le \sup_{x\ge x_0} \mu(x),$$

then the credibility functions $\Phi_L(x)$ and $\Phi_U(x)$ are all continuous at x_0.

Proof. We only prove $\Phi_L(x)$ is continuous at x_0, the assertion for $\Phi_U(x)$ can be proved similarly. Since ξ is a lower semicontinuous fuzzy variable, by Theorem 3.5, for every $x \in \Re$, we have

$$\Phi_L(x) = \text{Cr}\{\xi < x\} = 1 - \Phi_U(x). \tag{10}$$

On one hand, since $\limsup_{x\to x_0+} \mu(x) \le \sup_{x\le x_0} \mu(x)$, by Theorem 3.1, we get $\Phi_L(x)$ is right continuous at x_0.

On the other hand, since $\limsup_{x\to x_0-} \mu(x) \le \sup_{x\ge x_0} \mu(x)$, we obtain $\Phi_U(x)$ is left continuous at x_0 from Theorem 3.3. Thus, by Eq. (10), $\Phi_L(x) = 1 - \Phi_U(x)$ is continuous at x_0. The proof of the theorem is complete.

Example 3.1 *Let ξ be a fuzzy variable with the following possibility distribution*

$$\mu(x) = \begin{cases} \frac{3}{4}, & \text{if } x \le 0 \\ -\frac{1}{8}x + \frac{3}{4}, & \text{if } 0 < x < 2 \\ \frac{1}{4}, & \text{if } x = 2 \\ \frac{1}{3}x - \frac{1}{3}, & \text{if } 2 < x \le 4 \\ 1, & \text{if } x > 4. \end{cases}$$

Since $\mu(x)$ is lower semicontinuous,

$$\limsup_{x\to 2+} \mu(x) = \frac{1}{3} < \sup_{x\le 2} \mu(x) = \frac{3}{4}, \textit{ and } \limsup_{x\to 2-} \mu(x) = \frac{1}{2} < \sup_{x\ge 2} \mu(x) = 1,$$

by Theorem 3.6, both $\Phi_L(x)$ and $\Phi_G(x)$ are continuous at $x = 2$.

4 The Absolute Continuity of Credibility Functions

In this section, we will establish the conditions that the credibility functions are absolutely continuous. First of all, since the credibility function Φ_L is a nondecreasing and bounded nonnegative function on $\Re$, according to the absolute continuity of a real function, we can obtain the following general result.

Theorem 4.1 *Let ξ be a fuzzy variable with the credibility function $\Phi_L(x)$. If $\Phi_L(x)$ is continuous on $\Re$, and $\Phi'_L(x)$ exists e.c. on $\Re$, and $\lim_{x\to-\infty}\Phi_L(x)=0$, then $\Phi_L(x)$ is absolutely continuous on $\Re$, and*

$$\Phi_L(x)=\int_{-\infty}^{x}\Phi'_L(t)\mathrm{d}t$$

for any $x\in\Re$, where $\Phi'_L(x)$ is integrable on $\Re$.

Proof. Since $\Phi_L(x)$ is nondecreasing on $\Re$, for every $a>0$, $\Phi'_L(x)$ is Lebesgue integrable on $[-a,a]$. By the supposition, we know that for any $x\in[-a,a]$,

$$\Phi_L(x)=\int_{-a}^{x}\Phi'_L(t)\mathrm{d}t+\Phi_L(-a).$$

Letting $a\to\infty$, we have

$$\Phi_L(x)=\lim_{a\to\infty}\int_{-a}^{x}\Phi'_L(t)\mathrm{d}t$$

for any real x since $\Phi_L(-a)\to 0$. However, noting that

$$\Phi'_L(x)\ge 0 \ \text{ and } \ \Phi'_L(x)I_{(-n,n)}\uparrow\Phi'(x)$$

for all $x\in\Re$, By the Lebesuge monotone convergence theorem, we obtain

$$\begin{aligned}\int_{-\infty}^{\infty}\Phi'_L(t)\mathrm{d}t &= \int_{-\infty}^{\infty}\lim_{n\to\infty}\Phi'_L(t)I_{(-n,n)}\mathrm{d}t\\ &= \lim_{n\to\infty}\int_{-\infty}^{\infty}\Phi'_L(t)I_{(-n,n)}\mathrm{d}t\\ &= \lim_{n\to\infty}\int_{-n}^{n}\Phi'_L(t)\mathrm{d}t\\ &= \lim_{n\to\infty}V_{-n}^{n}\Phi_L\le 1\end{aligned}$$

where I_A is the characteristic function of set A, and $V_{-n}^{n}\Phi_L$ is the total variation of Φ_L on $[-n,n]$.

Therefore, $\Phi'_L(x)$ is Lebesuge integrable on $\Re$. Since

$$\Phi_L(x)=\lim_{n\to\infty}\int_{-n}^{x}\Phi'_L(t)\mathrm{d}t=\lim_{n\to\infty}\int_{-\infty}^{x}\Phi'_L(t)I_{(-n,\infty)}\mathrm{d}t,$$

the Lebesgue monotone convergence theorem implies

$$\Phi_L(x)=\int_{-\infty}^{x}\Phi'_L(t)\mathrm{d}t,$$

and $\Phi_L(x)$ is absolutely continuous on $\Re$. The proof of the theorem is complete.

In the following, we will establish the conditions on the possibility distribution that the credibility function Φ_L is absolutely continuous. In the following propositions, we always assume that ξ is the considered fuzzy variable with possibility distribution $\mu(x)$. In addition, we denote the right and left derivative of possibility function $\mathrm{Pos}\{\xi\le x\}$ at x_0 by ${P_L}'_+(x_0)$ and ${P_L}'_-(x_0)$, respectively; and, the right and left derivatives of possibility function $\mathrm{Pos}\{\xi\ge x\}$ at x_0 are denoted by ${P_U}'_+(x_0)$ and ${P_U}'_-(x_0)$, respectively.

Proposition 4.1 *Suppose $\mu(x)$ is continuous at $x_0 \in \Re$ and there is $\delta_0 > 0$ such that $\mu(x)$ is monotone on $(x_0, x_0+\delta_0)$. If the right derivative of $\mu(x)$ at x_0 (i.e., $\mu'_+(x_0)$) exists, then the right derivative of* $\text{Pos}\{\xi \le x\}$ *at x_0 also exists, and*

$$P_L{}'_+(x_0) = \begin{cases} \mu'_+(x_0), & \textit{if } \sup_{x\le x_0}\mu(x) = \mu(x_0) \textit{ and } \mu'_+(x_0) > 0 \\ 0, & \textit{otherwise.} \end{cases}$$

Proof. On the one hand, if $\sup_{x\le x_0}\mu(x) > \mu(x_0)$, then the continuity of $\mu(x)$ at x_0 implies

$$\text{Pos}\{\xi \le x_0\} > \text{Pos}\{x_0 \le \xi \le x_0 + h\}$$

for any small number $h > 0$. Therefore,

$$\begin{aligned} P_L{}'_+(x_0) &= \lim_{h\to 0^+}(\text{Pos}\{\xi \le x_0+h\} - \text{Pos}\{\xi \le x_0\})/h \\ &= \lim_{h\to 0^+}(\text{Pos}\{\xi \le x_0\} \vee \text{Pos}\{x_0 \le \xi \le x_0+h\} - \text{Pos}\{\xi \le x_0\})/h \\ &= 0. \end{aligned}$$

On the other hand, if $\sup_{x\le x_0}\mu(x) = \mu(x_0)$, we consider in the following the cases $\mu'_+(x_0) > 0$, $\mu'_+(x_0) < 0$ and $\mu'_+(x_0) = 0$, respectively.

For the case $\mu'_+(x_0) > 0$, it follows from the definition of $\mu'_+(x_0)$ that $\mu(x_0) < \mu(x_0+\eta)$ for any small number $\eta > 0$. Then the assumptions imply that $\mu(x)$ is nondecreasing on $[x_0, x_0+\delta_0)$. Hence

$$\begin{aligned} P_L{}'_+(x_0) &= \lim_{h\to 0^+}(\text{Pos}\{\xi \le x_0+h\} - \text{Pos}\{\xi \le x_0\})/h \\ &= \lim_{h\to 0^+}(\text{Pos}\{\xi \le x_0\} \vee \text{Pos}\{x_0 \le \xi \le x_0+h\} - \text{Pos}\{\xi \le x_0\})/h \\ &= \lim_{h\to 0^+}(\mu(x_0+h) - \mu(x_0))/h = \mu'_+(x_0). \end{aligned}$$

For the case $\mu'_+(x_0) < 0$, the definition of $\mu'_+(x_0)$ implies there is $\eta > 0$ such that $\mu(x) \le \mu(x_0)$ for any $x \in [x_0, x_0+\eta)$, i.e., $\mu(x_0) = \text{Pos}\{x_0 \le \xi < x_0+\eta\}$. Thus

$$\begin{aligned} P_L{}'_+(x_0) &= \lim_{h\to 0^+}(\text{Pos}\{\xi \le x_0+h\} - \text{Pos}\{\xi \le x_0\})/h \\ &= \lim_{h\to 0^+}(\mu(x_0) - \mu(x_0))/h = 0. \end{aligned}$$

Finally, if $\mu'_+(x_0) = 0$, by the analysis of the two cases mentioned above, we can always obtain $P_L{}'_+(x_0) = 0$.

Combining the above gives

$$P_L{}'_+(x_0) = \begin{cases} \mu'_+(x_0), & \text{if } \sup_{x\le x_0}\mu(x) = \mu(x_0) \text{ and } \mu'_+(x_0) > 0 \\ 0, & \text{otherwise.} \end{cases}$$

The theorem is proved.

Proposition 4.2 *Assume that $\mu(x)$ is continuous at x_0 and there is $\delta_0 > 0$ such that $\mu(x)$ is monotone on $(x_0-\delta_0, x_0)$. If the left derivative of $\mu(x)$ at x_0 exists, then the left derivative of* $\text{Pos}\{\xi \le x\}$ *at x_0 also exists, and*

$$P_L{}'_-(x_0) = \begin{cases} \mu'_-(x_0), & \textit{if } \sup_{t\le x}\mu(t) < \mu(x_0) \textit{ for any } x < x_0 \\ 0, & \textit{otherwise.} \end{cases}$$

Proof. If $\sup_{x\le x_0}\mu(x)>\mu(x_0)$, by the continuity of $\mu(x)$ at x_0, there is $\eta>0$ such that

$$\mathrm{Pos}\{\xi\le x_0-\eta\}=\mathrm{Pos}\{\xi\le x_0\}>\mu(x_0).$$

Thus,

$$P_L{}'_-(x_0)=\lim_{h\to 0^+}(\mathrm{Pos}\{\xi\le x_0\}-\mathrm{Pos}\{\xi\le x_0-h\})/h=0. \tag{11}$$

On the other hand, if $\sup_{x\le x_0}\mu(x)=\mu(x_0)$, the argument breaks down into the following two cases.

Case 1: There is $\eta>0$ such that $\mathrm{Pos}\{\xi\le x_0-\eta\}=\mu(x_0)$. In this case, we can easily get

$$P_L{}'_-(x_0)=0. \tag{12}$$

Case 2: $\sup_{t\le x}\mu(t)<\mu(x_0)$ for any $x<x_0$. In this case, by the assumption, there is δ_0 such that $\mu(x)$ is nondecreasing on $(x_0-\delta_0,x_0]$. Since $\mathrm{Pos}\{\xi\le x_0-\delta_0\}<\mu(x_0)$ and $\mu(x)$ is continuous at x_0, there exists $\delta^*\in(0,\delta_0)$ such that $\mathrm{Pos}\{\xi\le x_0-\delta_0\}<\mu(x_0-\delta^*)$.

Thus, we obtain

$$\mathrm{Pos}\{\xi\le x_0-\delta\}=\mathrm{Pos}\{\xi\le x_0-\delta_0\}\vee\mathrm{Pos}\{x_0-\delta_0<\xi\le x_0-\delta\}=\mu(x_0-\delta)$$

for all $\delta\in(0,\delta^*)$. Therefore

$$\begin{aligned}P_L{}'_-(x_0)&=\lim_{h\to 0^+}(\mathrm{Pos}\{\xi\le x_0\}-\mathrm{Pos}\{\xi\le x_0-h\})/h\\&=\lim_{h\to 0^+}(\mu(x_0)-\mu(x_0-h))/h=\mu'_-(x_0).\end{aligned}$$

The desired result follows from Eqs. (11), (12) and (13), which completes the proof of the theorem.

Similarly, for the differentiation of $\mathrm{Pos}\{\xi\ge x\}$, we have the following propositions.

Proposition 4.3 *Suppose $\mu(x)$ is continuous at x_0 and there is $\delta_0>0$ such that $\mu(x)$ is monotone on $(x_0,x_0+\delta_0)$. If $\mu'_+(x_0)$ exists, then $P_U{}'_+(x_0)$ also exists, and*

$$P_U{}'_+(x_0)=\begin{cases}\mu'_+(x_0), & \text{if } \sup_{t\ge x}\mu(t)<\mu(x_0) \text{ for any } x>x_0\\ 0, & \text{otherwise.}\end{cases}$$

Proposition 4.4 *Suppose $\mu(x)$ is continuous at x_0 and there is $\delta_0>0$ such that $\mu(x)$ is monotone on $(x_0-\delta_0,x_0)$. If $\mu'_-(x_0)$ exists, then $P_U{}'_-(x_0)$ also exists, and*

$$P_U{}'_-(x_0)=\begin{cases}\mu'_-(x_0), & \text{if } \sup_{x\ge x_0}\mu(x)=\mu(x_0) \text{ and } \mu'_-(x_0)<0\\ 0, & \text{otherwise.}\end{cases}$$

A real-valued function $f(x)$ is said to be piecewise monotone on $\Re$ if and only if for any $x\in\Re$, there is δ_x such that $f(x)$ is monotone on $(x-\delta_x,x)$ and $(x,x+\delta_x)$, respectively. Assuming the fuzzy variables with piecewise monotone possibility distributions, we have the following theorem.

Theorem 4.2 *Assume that ξ is a continuous fuzzy variable with the possibility distribution μ and the credibility function Φ_L. If μ is piecewise monotone on $\Re$, both the right and left derivatives of μ exist (possibly infinite) e.c., and $\lim_{x\to-\infty}\mu(x)=0$, then Φ_L is absolutely continuous, and*

$$\Phi_L(x)=\int_{-\infty}^{x}\Phi'_L(t)\mathrm{d}t$$

for any $x\in\Re$.

Proof. On the one hand, by the assumptions and Propositions 4.1 and 4.2, we can deduce both ${P_L}'_+(x)$ and ${P_L}'_-(x)$ exist e.c.. Since the set of all points, that ${P_L}'_+(x)$ and ${P_L}'_-(x)$ exist but not equal is at most countable. That is the derivative of $\text{Pos}\{\xi \le x\}$, $P'_L(x)$, exists e.c..

On the other hand, by Propositions 4.3, 4.4 and the assumptions, we know that $P'_U(x)$ exists e.c..

By Propositions 4.1 and 4.2, if $P'_L(x)$ exists, it equals to $\mu'(x)$ or zero; and by Propositions 4.3 and 4.4, we know that if $P'_U(x)$ exists, it equals to $\mu'(x)$ or zero as well. Moreover, we know if $\mu'(x) \neq 0$, it follows from Proposition 4.1 and 4.3 that the following equalities

$$P'_L(x) = \mu'(x) = P'_U(x) \tag{13}$$

do not hold, which implies $P'_L(x) - P'_U(x)$ always exists. Since $\mu(x)$ is continuous on $\Re$, we have $\text{Pos}\{\xi \ge x\} = \text{Pos}\{\xi > x\}$ for any $x \in \Re$. It follows from

$$\begin{aligned}\Phi_L(x) &= \frac{1}{2}(1 + \text{Pos}\{\xi \le x\} - \text{Pos}\{\xi > x\}) \\ &= \frac{1}{2}(1 + \text{Pos}\{\xi \le x\} - \text{Pos}\{\xi \ge x\})\end{aligned}$$

that $\Phi'_L(x) = (P'_L(x) - P'_U(x))/2$ exists e.c..

By $\lim_{x\to-\infty} \mu(x) = 0$, we can easily check that $\lim_{x\to-\infty} \Phi_L(x) = 0$. Thus, by Theorem 4.1, we have

$$\Phi_L(x) = \int_{-\infty}^{x} \Phi'_L(t)\mathrm{d}t$$

for any $x \in \Re$. The proof of the theorem is complete.

By Theorem 4.2, it is easy to check the following corollary is valid .

Corollary 4.1 *Let ξ be a continuous fuzzy variable with the possibility distribution μ and the credibility function Φ_L. If μ is piecewise monotone and differentiable e.c.on $\Re$, and* $\lim_{x\to-\infty} \mu(x) = 0$, *then for any $x \in \Re$,*

$$\Phi_L(x) = \int_{-\infty}^{x} \Phi'_L(t)\mathrm{d}t.$$

Remark 4.1 *Under the conditions of Theorems 4.1 and 4.2, we have $\Phi_U = 1 - \Phi_L$. Therefore, if Φ_L is absolutely continuous and*

$$\Phi_L(x) = \int_{-\infty}^{x} \Phi'_L(t)\mathrm{d}t,$$

then Φ_U is also absolutely continuous and

$$\Phi_U(x) = \int_{x}^{\infty} \Phi'_L(t)\mathrm{d}t.$$

5 Conclusions

In this paper, we discussed some new analytical properties of the credibility functions and obtained the following major new results:

(*i*) The sufficient and necessary condition for the right (resp., left) continuity of the credibility function Φ_L (resp., Φ_U) was established.

(*ii*) The sufficient condition for the left (resp., right) continuity and continuity of the credibility function Φ_L (resp., Φ_U) was given.

(*iii*) The sufficient conditions for the absolute continuity of the credibility functions were obtained.

References

[1] D. Dubois, H. Prade, *Possibility Theory,* Plenum Press, New York, 1988.

[2] M. Inuiguchi, H. Ichihashi, Y. Kume, Modality constrained programming problems: A unified approach to fuzzy mathematical programming problems in the setting of possibility theory, *Information Sciences,* Vol.67, 93–126, 1993.

[3] G. J. Klir, On fuzzy-set interpretation of possibility theory, *Fuzzy Sets Syst.,* Vol.108, 263–273, 1999.

[4] P. Kall, S. W. Wallace, *Stochastic Programming,* Wiley, Chichester, 1994.

[5] B. Liu, *Theory and Practice of Uncertain Programming,* Physica-Verlag, Heidelberg, 2002.

[6] B. Liu, *Uncertainty Theory: An Introduction to Its Axiomatic Foundations,* Springer-Verlag, Berlin, 2004.

[7] B. Liu, Toward fuzzy optimization without mathematical ambiguity, *Fuzzy Optimization and Decision Making,* Vol.1, 43–63, 2002.

[8] B. Liu, A survey of credibility theory, *Fuzzy Optimization and Decision*, Vol.5, 387–408, 2006.

[9] B. Liu, Y.-K. Liu, Expected value of fuzzy variable and fuzzy expected value models, *IEEE Trans. Fuzzy Syst.* Vol.10, 445–450, 2002.

[10] Y.-K. Liu, Fuzzy programming with recourse, *Int. J. Uncertainty Fuzziness Knowl.-Based Syst.*, Vol.13, No.4, 381–413, 2005,

[11] Y. K. Liu, S. Wang, *Theory of Fuzzy Random Optimization*, China Agricultural University Press, Beijing, 2006.

[12] Y.-J. Lai, C.-L. Hwang, *Fuzzy Mathematical Programming: Methods and Applications,* Springer-Verlag, New York, 1992.

[13] H. L. Royden, *Real Analysis,* Macmillan, New York, 1968.

[14] M. Sakawa, *Fuzzy Sets and Interactive Multiobjective Optimization,* Plenum, New York, 1993.

[15] R. R. Yager, On the specificity of a possibility distribution, *Fuzzy Sets Syst.*, Vol.50, 279–292, 1992.

[16] L. A. Zadeh, Fuzzy sets as a basis for a theory of possibility, *Fuzzy Sets Syst.*, Vol.1, 3–28, 1978.