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On the Continuity and Convexity Analysis of the Expected Value Function of a Fuzzy Mapping

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Abstract

In the optimization problem under a fuzzy environment, the objective function might be given with fuzzy coefficients. In this paper, fuzzy variables are used to characterize these coefficients, and such a function is referred as a fuzzy mapping. In order to make a decision in the fuzzy sense, we study some properties of fuzzy mapping. Since the convex analysis plays an important role in the studies of optimization problems, this paper discusses the continuity and convexity of the expected value function of a fuzzy mapping. We prove that the continuity and convexity can be inherited after calculating the expected value on the hypotheses of monotonicity and the upper semi-continuous of the membership functions of independent fuzzy variables. An application in a retailer's optimization problem illuminates how to study programming in a fuzzy environment quantitatively and qualitatively. © 2007 World Academic Press, UK. All rights reserved.
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1 Introduction

In many system-analysis areas, a model has to be set up using data that is only approximately known. Fuzzy sets theory makes it possible. Since Zadeh [20] first proposed the concept of fuzzy set, it has been well developed and applied in a wide variety of real problems. In order to measure a fuzzy event, Zadeh [21] initialized the concept of possibility measure, which was developed by several researchers such as Nahmias [13], Kaufman and Gupta [4], Zimmermann [22], Dubois and Prade [2] and Liu [6]. Since the possibility measure is not self-dual, Liu and Liu [9] proposed the concept of credibility measure, which is a self-dual measure, and defined the expected value of a fuzzy variable based on the credibility measure. A framework of credibility theory was given by Liu [8].

Fuzzy programming offers powerful means of handling optimization problems with fuzzy factors. Many researchers such as Zimmermann [22], Yazenin [18] [19], Sakawa [15] and Tanaka *et al* [17] applied fuzzy sets theory to optimization problems successfully. A detailed survey on fuzzy optimization was made in 1989 by Luhandjula [12]. Especially, Liu and Liu [9] presented a concept of expected value operator of a fuzzy variable and provided a spectrum of fuzzy expected value model (EVM). In EVM, the decision is to optimize the expected value of the objective function under constraints. The convexity of the fuzzy expected value model was discussed in their paper. Liu [11] introduced the fuzzy programming with recourse problem and discussed the convexity of the model.

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In order to model a fuzzy decentralized decision-making problem, Gao and Liu [3] introduced the fuzzy expected value multilevel programming and chance-constrained multilevel programming.

It is well known that the convex analysis [14] is important for quantitative and qualitative studies and helps to find optimal solutions in operation research. Since the continuity and convexity of a function are the basics for convex analysis, it would be significant for us to consider these properties of the expected value function of a fuzzy mapping. Then some results of the convex analysis can be used to study the programming and help to find solutions under a fuzzy environment.

This paper is organized as follows. In Section 2, preliminaries of some basic notions and several results which can be used in this paper are given. In Section 3, we consider the continuity and convexity of the expected value function of a fuzzy mapping, and several examples are given corresponding with the results. In Section 4, an application is given on the retailer's optimization problem.

2 Preliminaries

Let Θ be a nonempty set, $\mathcal{P}(\Theta)$ the power set of Θ. A possibility measure [13] is a set function

$$\text{Pos} : \mathcal{P}(\Theta) \to [0, 1],$$

which satisfies the following conditions:

Axiom 1. $\text{Pos}\{\Theta\} = 1$, $\text{Pos}\{\phi\} = 0$;

Axiom 2. $\text{Pos}\{\cup_i A_i\} = \sup_i \text{Pos}\{A_i\}$ for any collection $\{A_i\}$ in $\mathcal{P}(\Theta)$;

Axiom 3. Let Θ_i be nonempty sets on which $\text{Pos}_i\{\cdot\}$ satisfy the first two axioms, $i = 1, 2, \ldots, n$, respectively, and $\Theta = \Theta_1 \times \Theta_2 \times \ldots \times \Theta_n$. Then

$$\text{Pos}\{A\} = \sup_{(\theta_1,\theta_2,\ldots,\theta_n)\in A} \text{Pos}_1\{\theta_1\} \wedge \text{Pos}_2\{\theta_2\} \wedge \ldots \wedge \text{Pos}_n\{\theta_n\}$$

for each $A \in \mathcal{P}(\Theta)$. In that case we write $\text{Pos} = \text{Pos}_1 \wedge \text{Pos}_2 \wedge \ldots \wedge \text{Pos}_n$.

Then the triplet $(\Theta, \mathcal{P}(\Theta), \text{Pos})$ is called a possibility space [16]. Let the kernel of a possibility space $(\Theta, \mathcal{P}(\Theta), \text{Pos})$, $\{\theta \in \Theta \mid \text{Pos}\{\theta\} > 0\}$, be denoted by Θ^+.

Definition 1 (Liu and Liu [9]) *Let $(\Theta, \mathcal{P}(\Theta), \text{Pos})$ be a possibility space and A a set in $\mathcal{P}(\Theta)$. Then the credibility measure of A is defined by*

$$\text{Cr}\{A\} = \frac{1}{2} \left(\text{Pos}\{A\} + 1 - \text{Pos}\{A^c\}\right), \tag{1}$$

where A^c is the complement of A.

Definition 2 (Nahmias [13], Liu and Liu [6]) *A fuzzy variable ξ is defined as a function from the possibility space $(\Theta, \mathcal{P}(\Theta), \text{Pos})$ to the set of real numbers and its membership function is derived from the possibility measure by*

$$\mu_\xi(x) = \text{Pos}\{\theta \in \Theta \mid \xi(\theta) = x\}, \quad x \in \mathcal{R}. \tag{2}$$

Let $\xi(\Theta^+)$ denote the collection $\{x \mid \text{Pos}\{\theta \in \Theta \mid \xi(\theta) = x\} > 0\}$. A fuzzy variable ξ is said to be bounded if for any $\alpha \in (0, 1]$, $\{x \in \mathcal{R} \mid \mu_\xi(x) \geq \alpha\}$ is a nonempty bounded subset of $\mathcal{R}$.

Definition 3 (Nahmias [13]) *The fuzzy variables* $\xi_1, \xi_2, \ldots, \xi_n$ *are said to be independent if and only if*

$$\mathrm{Pos}\{\xi_i \in B_i, i = 1, 2, \ldots, n\} = \min_{1 \leq i \leq n} \mathrm{Pos}\{\xi_i \in B_i\} \tag{3}$$

for any sets $B_1, B_2, \ldots, B_n$ *of* $\mathcal{R}$.

Definition 4 (Liu and Liu [9]) *Suppose* ξ *is a fuzzy variable. Then the expected value of* ξ *is defined by*

$$E[\xi] = \int_0^{+\infty} \mathrm{Cr}\{\xi \geq r\}\mathrm{d}r - \int_{-\infty}^{0} \mathrm{Cr}\{\xi \leq r\}\mathrm{d}r \tag{4}$$

provided that at least one of the two integrals is finite.

Definition 5 (Liu and Liu [10]) *Let* ξ *be a fuzzy variable on the possibility space* $(\Theta, \mathcal{P}(\Theta), \mathrm{Pos})$. *Then the* α*-optimistic value, denoted by* $\xi^U(\alpha)$, *of* ξ *is defined as*

$$\xi^U(\alpha) = \sup\{r \mid \mathrm{Pos}\{\theta \in \Theta \mid \xi(\theta) \geq r\} \geq \alpha\}, \quad \alpha \in (0, 1],$$

and the α*-pessimistic value, denoted by* $\xi^L(\alpha)$, *of* ξ *is defined as*

$$\xi^L(\alpha) = \inf\{r \mid \mathrm{Pos}\{\theta \in \Theta \mid \xi(\theta) \leq r\} \geq \alpha\}, \quad \alpha \in (0, 1].$$

Lemma 1 (Liu and Liu [10]) *Let* ξ *be a bounded fuzzy variable defined on the possibility space* $(\Theta, \mathcal{P}(\Theta), \mathrm{Pos})$. *Then we have*

$$E[\xi] = \frac{1}{2}\int_0^1 \left[\xi^U(\alpha) + \xi^L(\alpha)\right] \mathrm{d}\alpha.$$

Definition 6 (Liu [7]) *Let* $f : \mathcal{R}^n \to \mathcal{R}$ *be a measurable function, and* ξ_i *fuzzy variables defined on the possibility spaces* $(\Theta_i, \mathcal{P}(\Theta_i), \mathrm{Pos}_i)$, $i = 1, 2, \ldots, n$, *respectively. Then the function of fuzzy variables, denoted by* $\xi = f(\xi_1, \xi_2, \ldots, \xi_n)$, *is a fuzzy variable defined on the product possibility space* $(\Theta, \mathcal{P}(\Theta), \mathrm{Pos})$, *where* $\Theta = \prod_{i=1}^n \Theta_i$, $\mathrm{Pos} = \bigwedge_{i=1}^n \mathrm{Pos}_i$ *and*

$$\xi(\theta_1, \theta_2, \ldots, \theta_n) = f\left(\xi_1(\theta_1), \xi_2(\theta_2), \ldots, \xi_n(\theta_n)\right)$$

for any $(\theta_1, \theta_2, \ldots, \theta_n) \in \Theta$.

Definition 7 (Aubin [1]) *Suppose* X *is a topological space and* $f : X \to \mathcal{R}$ *a real-valued function. We say that* f *is lower semi-continuous at* $\boldsymbol{x}_0 \in X$ *if for all* $\lambda < f(\boldsymbol{x}_0)$, *there exists a neighborhood* $B(\boldsymbol{x}_0, \delta)$, *where* $\delta > 0$, *such that*

$$\lambda \leq f(\boldsymbol{x}), \quad \forall \boldsymbol{x} \in B(\boldsymbol{x}_0, \delta). \tag{5}$$

We shall say that f *is lower semi-continuous if it is lower semi-continuous at every point of* X. *A function* f *is upper semi-continuous if* $-f$ *is lower semi-continuous, and* f *is continuous if and only if* f *is lower semi-continuous and upper semi-continuous.*

Lemma 2 (Aubin [1]) *A function* f *from* X *to* $\mathcal{R}$ *is lower semi-continuous (or upper semi-continuous) at* $x_0 \in X$ *if and only if*

$$\liminf_{x \to x_0} f(x) \geq f(x_0) \quad \left(or \quad \limsup_{x \to x_0} f(x) \leq f(x_0)\right).$$

Theorem 1 *Let ξ be a fuzzy variable with an upper semi-continuous membership function, $\xi^U(\alpha)$ and $\xi^L(\alpha)$ the α-optimistic value and the α-pessimistic value of ξ, respectively. Then we have*

$$\mathrm{Pos}\left\{\xi = \xi^U(\alpha)\right\} \geq \alpha, \quad \forall \alpha \in (0,1] \tag{6}$$

and

$$\mathrm{Pos}\left\{\xi = \xi^L(\alpha)\right\} \geq \alpha, \quad \forall \alpha \in (0,1]. \tag{7}$$

Proof. It follows from Definition 5 that

$$\mathrm{Pos}\{\xi \geq \xi^U(\alpha) + \epsilon\} < \alpha \quad \text{and} \quad \mathrm{Pos}\{\xi \geq \xi^U(\alpha) - \epsilon\} \geq \alpha, \quad \forall \epsilon > 0, \forall \alpha \in (0,1].$$

Since the membership function of ξ is upper semi-continuous, we can deduce that

$$\begin{aligned}
\mathrm{Pos}\{\xi = \xi^U(\alpha)\} &\geq \limsup_{r \to \xi^U(\alpha)} \mathrm{Pos}\{\xi = r\} \\
&= \lim_{\epsilon \to 0^+} \sup_{\xi^U(\alpha)-\epsilon \leq r \leq \xi^U(\alpha)+\epsilon} \mathrm{Pos}\{\xi = r\} \\
&= \lim_{\epsilon \to 0^+} \left(\left(\sup_{\xi^U(\alpha)-\epsilon \leq r \leq \xi^U(\alpha)+\epsilon} \mathrm{Pos}\{\xi = r\} \right) \vee \left(\sup_{\xi^U(\alpha)+\epsilon < r} \mathrm{Pos}\{\xi = r\} \right) \right) \\
&= \lim_{\epsilon \to 0^+} \sup_{\xi^U(\alpha)-\epsilon \leq r} \mathrm{Pos}\{\xi = r\} \\
&= \lim_{\epsilon \to 0^+} \mathrm{Pos}\{\xi \geq \xi^U(\alpha) - \epsilon\} \\
&\geq \alpha.
\end{aligned}$$

The inequality $\mathrm{Pos}\left\{\xi = \xi^L(\alpha)\right\} \geq \alpha$ can be proved similarly.

Remark 1 *If the membership function is not upper semi-continuous, the result may not be right. Consider the fuzzy variable ξ with membership function*

$$\mu_\xi(x) = \begin{cases} x, & x \in [0,1] \\ 2 - x, & x \in (1, 1.5) \cup (1.5, 2] \\ 0, & x = 1.5. \end{cases} \tag{8}$$

By Definition 5, we have that $\xi^U(0.5) = \sup\{r \mid \mathrm{Pos}\{\theta \in \Theta \mid \xi(\theta) \geq r\} \geq 0.5\} = 1.5$, whilst $\mathrm{Pos}\{\xi = \xi^U(0.5)\} = \mathrm{Pos}\{\xi = 1.5\} = 0$.

Definition 8 *A function $f : \mathcal{R}^n \to \mathcal{R}$ is said to be nondecreasing (nonincreasing, respectively) with respect to x_i if and only if for each fixed $\boldsymbol{x}_{-i} \in \mathcal{R}^{n-1}$, where $\boldsymbol{x}_{-i}$ denotes the vector without x_i, $f(\boldsymbol{x}_{-i}, x_i)$ is nondecreasing (nonincreasing, respectively) with respect to x_i. If $f(\boldsymbol{x})$ is nondecreasing or nonincreasing with respect to x_i, then $f(\boldsymbol{x})$ is monotonic with respect to x_i.*

Example 1 *Consider the function $f(x,y) = xy$ with $x \in [-1,1]$ and $y \in [1,2]$. For each $y_0 \in [1,2]$, $f(x,y_0) = xy_0$ is nondecreasing with respect to x, we say $f(x,y) = xy$ is monotonic with respect to x. Since $f(x_0,y) = x_0 y$ is nonincreasing with respect to y when $x_0 \in [-1,0]$ and nondecreasing with respect to y when $x_0 \in [0,1]$, $f(x,y)$ is not monotonic with respect to y.*

The definition of fuzzy mapping can be given as follows.

Definition 9 *A fuzzy mapping is a mapping from a topological space X to a collection of fuzzy variables.*

3 The Expected Value Function of a Fuzzy Mapping

In practical analysis, we often come across such functions, some of whose parameters may be of fuzziness. Usually the expected values or the values with the highest possibility of the parameters are considered. However, it will be more accurate for us to use fuzzy variables to characterize such parameters. For example, if the coefficient b is fuzzy in the function $f(x) = x + b$, we may use a fuzzy variable ξ to depict it, and the mapping is denoted by $f(x, \xi) = x + \xi$ as a fuzzy mapping. For simple depiction, a function $f(x, u) = x + u$ is introduced by substituting the variable u for the parameter b in the function $f(x) = x + b$.

More generally, let $\mathcal{F}(\Theta_i, \mathcal{P}(\Theta_i), \text{Pos}_i)$ denote the collection of all bounded fuzzy variables defined on the possibility spaces $(\Theta_i, \mathcal{P}(\Theta_i), \text{Pos}_i)$, $i = 1, 2, \ldots, n$, respectively, $\xi_i \in \mathcal{F}(\Theta_i, \mathcal{P}(\Theta_i), \text{Pos}_i)$, $\boldsymbol{\xi} = (\xi_1, \xi_2, \ldots, \xi_n)$ a fuzzy vector, X a topological space and $f : X \times \prod_{i=1}^{n} \xi_i(\Theta_i) \to \mathcal{R}$ a bounded measurable function. Thus $f(\boldsymbol{x}, \boldsymbol{\xi}) = f(\boldsymbol{x}, \xi_1, \xi_2, \ldots, \xi_n)$ is a fuzzy mapping from X to $\mathcal{F}(\Theta, \mathcal{P}(\Theta), \text{Pos})$, where $\Theta = \prod_{i=1}^{n} \Theta_i$ and $\text{Pos} = \bigwedge_{i=1}^{n} \text{Pos}_i$. Let $E[f(\boldsymbol{x}, \boldsymbol{\xi})]$, $f^U(\boldsymbol{x}, \boldsymbol{\xi}; \alpha)$ and $f^L(\boldsymbol{x}, \boldsymbol{\xi}; \alpha)$ denote the expected value function, the α-optimistic value function and the α-pessimistic value function of $f(\boldsymbol{x}, \boldsymbol{\xi})$, $\alpha \in (0, 1]$, respectively, i.e., for each $\boldsymbol{x}_0 \in X$, $E[f(\boldsymbol{x}_0, \boldsymbol{\xi})]$, $f^U(\boldsymbol{x}_0, \boldsymbol{\xi}; \alpha)$ and $f^L(\boldsymbol{x}_0, \boldsymbol{\xi}; \alpha)$ are the expected value, the α-optimistic value and the α-pessimistic value of the fuzzy variable $f(\boldsymbol{x}_0, \boldsymbol{\xi})$, respectively.

In this section, the continuity and convexity of the function $E[f(\boldsymbol{x}, \boldsymbol{\xi})]$ are studied, since they play crucial roles in quantitative and qualitative studies of optimization problem under a fuzzy environment. First, some properties of the function of fuzzy variables are discussed.

Theorem 2 *Let ξ_i be independent fuzzy variables defined on the possibility spaces $(\Theta_i, \mathcal{P}(\Theta_i), \text{Pos}_i)$ with upper semi-continuous membership functions, $i = 1, 2, \ldots, n$, respectively and $f : X \subset \mathcal{R}^n \to \mathcal{R}$ a measurable function. If $f(\boldsymbol{x})$ is monotonic with respect to x_i on $\xi_i\left(\Theta_i^+\right)$, $i = 1, 2, \ldots, n$, respectively, then we have*

(i) $f^U(\boldsymbol{\xi})(\alpha) = f\left(\xi_1^V(\alpha), \xi_2^V(\alpha), \ldots, \xi_n^V(\alpha)\right)$, *where $\xi_i^V(\alpha) = \xi_i^U(\alpha)$, if $f(x_1, x_2, \ldots, x_n)$ is nondecreasing with respect to x_i, $\xi_i^V(\alpha) = \xi_i^L(\alpha)$, otherwise, $\alpha \in (0, 1]$;*

(ii) $f^L(\boldsymbol{\xi})(\alpha) = f\left(\xi_1^{\overline{V}}(\alpha), \xi_2^{\overline{V}}(\alpha), \ldots, \xi_n^{\overline{V}}(\alpha)\right)$, *where $\xi_i^{\overline{V}}(\alpha) = \xi_i^L(\alpha)$, if $f(x_1, x_2, \ldots, x_n)$ is nondecreasing with respect to x_i, $\xi_i^{\overline{V}}(\alpha) = \xi_i^U(\alpha)$, otherwise, $\alpha \in (0, 1]$,*

where $f^U(\boldsymbol{\xi})(\alpha)$ and $f^L(\boldsymbol{\xi})(\alpha)$ denote the α-optimistic value and the α-pessimistic value of the fuzzy variable $f(\boldsymbol{\xi})$, respectively.

Proof. Without loss of generality, suppose f is nondecreasing with the first k variables and nonincreasing with the last $n - k$ variables. Then we want to prove that for all $\alpha \in (0, 1]$,

$$f^U(\boldsymbol{\xi})(\alpha) = f\left(\xi_1^U(\alpha), \xi_2^U(\alpha), \ldots, \xi_k^U(\alpha), \xi_{k+1}^L(\alpha), \ldots, \xi_n^L(\alpha)\right) \tag{9}$$

and

$$f^L(\boldsymbol{\xi})(\alpha) = f\left(\xi_1^L(\alpha), \xi_2^L(\alpha), \ldots, \xi_k^L(\alpha), \xi_{k+1}^U(\alpha), \ldots, \xi_n^U(\alpha)\right). \tag{10}$$

For any $\alpha \in (0, 1]$, since ξ_i are independent with each other, it follows from Definition 3 and

Theorem 1 that

$$\begin{aligned}
&\text{Pos}\left\{f(\boldsymbol{\xi}) \geq f\left(\xi_1^U(\alpha), \xi_2^U(\alpha), \ldots, \xi_k^U(\alpha), \xi_{k+1}^L(\alpha), \ldots, \xi_n^L(\alpha)\right)\right\} \\
&\geq \text{Pos}\left\{\xi_1 \geq \xi_1^U(\alpha); \xi_2 \geq \xi_2^U(\alpha); \ldots; \xi_k \geq \xi_k^U(\alpha); \xi_{k+1} \leq \xi_{k+1}^L(\alpha); \ldots; \xi_n \leq \xi_n^U(\alpha)\right\} \\
&= \left(\wedge_{i=1}^k \text{Pos}\left\{\xi_i \geq \xi_i^U(\alpha)\right\}\right) \wedge \left(\wedge_{i=k+1}^n \text{Pos}\left\{\xi_i \leq \xi_i^L(\alpha)\right\}\right) \\
&\geq \left(\wedge_{i=1}^k \text{Pos}\left\{\xi_i = \xi_i^U(\alpha)\right\}\right) \wedge \left(\wedge_{i=k+1}^n \text{Pos}\left\{\xi_i = \xi_i^L(\alpha)\right\}\right) \\
&\geq \alpha.
\end{aligned}$$

Following Definition 5, we get

$$f^U(\boldsymbol{\xi})(\alpha) \geq f\left(\xi_1^U(\alpha), \xi_2^U(\alpha), \ldots, \xi_k^U(\alpha), \xi_{k+1}^L(\alpha), \ldots \xi_n^L(\alpha)\right).$$

Suppose $f^U(\boldsymbol{\xi})(\alpha) > f\left(\xi_1^U(\alpha), \xi_2^U(\alpha), \ldots, \xi_k^U(\alpha), \xi_{k+1}^L(\alpha), \ldots \xi_n^L(\alpha)\right)$, which implies

$$\text{Pos}\left\{f(\boldsymbol{\xi}) > f\left(\xi_1^U(\alpha), \xi_2^U(\alpha), \ldots, \xi_k^U(\alpha), \xi_{k+1}^L(\alpha), \ldots, \xi_n^L(\alpha)\right)\right\} \geq \alpha$$

by Definition 5. So there exists $\boldsymbol{x}^* = (x_1^*, x_2^*, \ldots, x_n^*)$, such that

$$f(\boldsymbol{x}^*) > f\left(\xi_1^U(\alpha), \xi_2^U(\alpha), \ldots, \xi_k^U(\alpha), \xi_{k+1}^L(\alpha), \ldots, \xi_n^L(\alpha)\right) \quad \text{and} \quad \text{Pos}\{\theta \in \Theta \mid \boldsymbol{\xi}(\theta) = \boldsymbol{x}^*\} \geq \alpha.$$

According to the monotonicity of $f(\boldsymbol{x})$, we can deduce that there must exist some $i \leq k$ and $x_i^* > \xi_i^U(\alpha)$ or some $i > k$ and $x_i^* < \xi_i^L(\alpha)$. It follows from Definition 5 that if $x_i^* > \xi_i^U(\alpha)$ then we have

$$\text{Pos}\{\theta \in \Theta \mid \boldsymbol{\xi}(\theta) = \boldsymbol{x}^*\} \leq \text{Pos}\{\theta_i \in \Theta_i \mid \xi_i(\theta_i) = x_i^*\} < \alpha.$$

If $x_i^* < \xi_i^L(\alpha)$, we also have

$$\text{Pos}\{\theta \in \Theta \mid \boldsymbol{\xi}(\theta) = \boldsymbol{x}^*\} \leq \text{Pos}\{\theta_i \in \Theta_i \mid \xi_i(\theta_i) = x_i^*\} < \alpha.$$

Those are in contradiction with $\text{Pos}\{\theta \in \Theta \mid \boldsymbol{\xi}(\theta) = \boldsymbol{x}^*\} \geq \alpha$, such that (9) holds. The assertion that $f^L(\boldsymbol{\xi})(\alpha) = f\left(\xi_1^L(\alpha), \xi_2^L(\alpha), \ldots, \xi_k^L(\alpha), \xi_{k+1}^U(\alpha), \ldots, \xi_n^U(\alpha)\right)$ can be proved similarly.

Since the α-optimistic value and the α-pessimistic value of the function of fuzzy variables can be given by Theorem 2, the expected value of such fuzzy variable function can be calculated directly following from Lemma 1. The following examples illustrate the result.

Example 2 *Consider the function $f(x_1, x_2, x_3) = e^{x_1} + x_2^3 - x_3$ with $x_1, x_2, x_3 \in \mathcal{R}$. It is obviously that f is increasing with respect to x_1, x_2 and decreasing with respect to x_3. Let ξ_i be defined on the possibility spaces $(\Theta_i, \mathcal{P}(\Theta_i), \text{Pos}_i)$, where $\Theta_i = [0, 2]$,*

$$\text{Pos}_i\{\theta_i\} = \begin{cases} \theta_i, & 0 \leq \theta_i \leq 1 \\ 2 - \theta_i, & 1 < \theta_i \leq 2, \end{cases}$$

$i = 1, 2, 3$, respectively. Let $\xi_1(\theta_1) = \theta_1$, $\xi_2(\theta_2) = 1 + \theta_2$ and $\xi_3(\theta_3) = 2 + \theta_3$, $\theta_i \in [0, 2]$. Then ξ_1, ξ_2 and ξ_3 are independent bounded fuzzy variables with upper semi-continuous membership functions and equal to the triangular fuzzy variables $(0, 1, 2), (1, 2, 3)$ and $(2, 3, 4)$, respectively. It follows from Theorem 2 that

$$f^U(\xi_1, \xi_2, \xi_3)(\alpha) = e^{\xi_1^U(\alpha)} + \left(\xi_2^U(\alpha)\right)^3 - \xi_3^L(\alpha) = e^{2-\alpha} + (3 - \alpha)^3 - (2 + \alpha)$$

and

$$f^L(\xi_1,\xi_2,\xi_3)(\alpha) = e^{\xi_1^L(\alpha)} + \left(\xi_2^L(\alpha)\right)^3 - \xi_3^U(\alpha) = e^\alpha + (1+\alpha)^3 - (4-\alpha),$$

for all $\alpha \in (0,1]$. *Following Lemma 1, we get*

$$\begin{aligned} E[f(\xi_1,\xi_2,\xi_3)] &= \frac{1}{2}\int_0^1 \left(f^U(\xi_1,\xi_2,\xi_3)(\alpha) + f^L(\xi_1,\xi_2,\xi_3)(\alpha)\right) \mathrm{d}\alpha \\ &= \frac{1}{2}\int_0^1 \left[e^{2-\alpha} + (3-\alpha)^3 - (2+\alpha) + e^\alpha + (1+\alpha)^3 - (4-\alpha)\right] \mathrm{d}\alpha \\ &= \frac{13+e^2}{2}. \end{aligned}$$

Example 3 *Let* ξ_1, ξ_2 *and* ξ_3 *be given as those in Example 2,* $\alpha \in (0,1]$ *and the function*

$$f(x_1,x_2,x_3) = x_2 e^{x_1} + x_1 x_3$$

with $x_1, x_2, x_3 \in \mathcal{R}$. *Then*

$$\begin{aligned} f^U(\xi_1,\xi_2,\xi_3)(\alpha) &= \xi_2^U(\alpha) e^{\xi_1^U(\alpha)} + \xi_1^U(\alpha)\xi_3^U(\alpha) = (3-\alpha)e^{2-\alpha} + (2-\alpha)(4-\alpha), \\ f^L(\xi_1,\xi_2,\xi_3)(\alpha) &= \xi_2^L(\alpha) e^{\xi_1^L(\alpha)} + \xi_1^L(\alpha)\xi_3^L(\alpha) = (1+\alpha)e^{\alpha} + \alpha(2+\alpha) \end{aligned}$$

and

$$\begin{aligned} E[f(\xi_1,\xi_2,\xi_3)] &= \frac{1}{2}\int_0^1 \left(f^U(\xi_1,\xi_2,\xi_3)(\alpha) + f^L(\xi_1,\xi_2,\xi_3)(\alpha)\right) \mathrm{d}\alpha \\ &= \frac{1}{2}\int_0^1 \left[(3-\alpha)e^{2-\alpha} + (2-\alpha)(4-\alpha) + (1+\alpha)e^\alpha + \alpha(2+\alpha)\right] \mathrm{d}\alpha \\ &= \frac{20}{3} + 2e^2. \end{aligned}$$

Now for the fuzzy mapping $f(\boldsymbol{x},\boldsymbol{\xi})$, we have the following results on the hypothesis:

(P): ξ_i are independent bounded fuzzy variables with upper semi-continuous membership functions and the function $f(\boldsymbol{x}, u_1, u_2, \ldots, u_n)$ is monotonic with respect to u_i on $\xi_i(\Theta_i^+)$, $i = 1, 2, \ldots, n$, respectively.

Theorem 3 *Consider the fuzzy mapping* $f(\boldsymbol{x},\boldsymbol{\xi})$*, satisfying the hypothesis* ***(P)****. Furthermore, if* $f\left(\boldsymbol{x}, \xi_1^V(\alpha), \xi_2^V(\alpha), \ldots, \xi_n^V(\alpha)\right)$ *and* $f\left(\boldsymbol{x}, \xi_1^{\overline{V}}(\alpha), \xi_2^{\overline{V}}(\alpha), \ldots, \xi_n^{\overline{V}}(\alpha)\right)$ *are lower semi-continuous for all* $\alpha \in (0,1]$*, where* $\xi_i^V(\alpha)$ *and* $\xi_i^{\overline{V}}(\alpha)$ *are given as those in Theorem 2, then* $E[f(\boldsymbol{x},\boldsymbol{\xi})]$ *is lower semi-continuous with respect to* $\boldsymbol{x}$*.*

Proof. For any $\alpha \in (0,1]$ and fixed $\boldsymbol{x}_0 \in X$, it follows from Theorem 2 and Lemma 2 that

$$\begin{aligned} \liminf_{\boldsymbol{x}\to\boldsymbol{x}_0} f^U(\boldsymbol{x},\boldsymbol{\xi};\alpha) &= \liminf_{\boldsymbol{x}\to\boldsymbol{x}_0} f\left(\boldsymbol{x}, \xi_1^V(\alpha), \xi_2^V(\alpha), \ldots, \xi_n^V(\alpha)\right) \\ &\geq f\left(\boldsymbol{x}_0, \xi_1^V(\alpha), \xi_2^V(\alpha), \ldots, \xi_n^V(\alpha)\right) \\ &= f^U(\boldsymbol{x}_0,\boldsymbol{\xi};\alpha), \end{aligned} \tag{11}$$

so $f^U(\boldsymbol{x},\boldsymbol{\xi};\alpha)$ is lower semi-continuous at $\boldsymbol{x}_0$. Furthermore $f^U(\boldsymbol{x},\boldsymbol{\xi};\alpha)$ is lower semi-continuous with respect to $\boldsymbol{x}$, $\forall\alpha \in (0,1]$.

The assertion that $f^L(\boldsymbol{x},\boldsymbol{\xi};\alpha)$ is lower semi-continuous with respect to $\boldsymbol{x}$ for all $\alpha \in (0,1]$ can be proved similarly. Since $E[f(\boldsymbol{x},\boldsymbol{\xi})] = \frac{1}{2}\int_0^1 \left[f^U(\boldsymbol{x},\boldsymbol{\xi};\alpha) + f^L(\boldsymbol{x},\boldsymbol{\xi};\alpha)\right] \mathrm{d}\alpha$, it follows that $E[f(\boldsymbol{x},\boldsymbol{\xi})]$ is lower semi-continuous when both $f^U(\boldsymbol{x},\boldsymbol{\xi};\alpha)$ and $f^L(\boldsymbol{x},\boldsymbol{\xi};\alpha)$ are lower semi-continuous for all $\alpha \in (0,1]$.

Remark 2 *Similarly, $E[f(\boldsymbol{x},\boldsymbol{\xi})]$ is upper semi-continuous (continuous, respectively) on the conditions that the functions $f\left(\boldsymbol{x},\xi_1^V(\alpha),\xi_2^V(\alpha),\ldots,\xi_n^V(\alpha)\right)$ and $f\left(\boldsymbol{x},\xi_1^{\overline{V}}(\alpha),\xi_2^{\overline{V}}(\alpha),\ldots,\xi_n^{\overline{V}}(\alpha)\right)$ are upper semi-continuous (continuous, respectively) for all $\alpha\in(0,1]$.*

Furthermore, since $\mathrm{Pos}\left\{\xi=\xi^U(\alpha)\right\}\geq\alpha$ and $\mathrm{Pos}\left\{\xi=\xi^L(\alpha)\right\}\geq\alpha$, then $\xi_i^U(\alpha),\xi_i^L(\alpha)\in\xi_i(\Theta^+)$, for all $\alpha\in(0,1]$. Then we have the following remark.

Remark 3 *If the function $f(\boldsymbol{x},\boldsymbol{u})$ is lower semi-continuous (upper semi-continuous and continuous, respectively) with respect to $\boldsymbol{x}$ for each $\boldsymbol{u}\in\boldsymbol{\xi}(\Theta^+)$, then $f^U(\boldsymbol{x},\boldsymbol{\xi};\alpha)$, $f^L(\boldsymbol{x},\boldsymbol{\xi};\alpha)$ and $E[f(\boldsymbol{x},\boldsymbol{\xi})]$ are lower semi-continuous (upper semi-continuous and continuous, respectively) functions with respect to $\boldsymbol{x}$, $\forall\alpha\in(0,1]$.*

Proposition 1 *Let X be a compact subset of a topological space, the fuzzy mapping $f(\boldsymbol{x},\boldsymbol{\xi})$ satisfy the conditions in Theorem 3. Then the following problem has a solution $\bar{\boldsymbol{x}}\in X$, i.e.,*

$$E[f(\bar{\boldsymbol{x}},\boldsymbol{\xi})]=\inf_{\boldsymbol{x}\in X}E[f(\boldsymbol{x},\boldsymbol{\xi})].$$

Example 4 *Let ξ_1,ξ_2 and ξ_3 be given as those in Example 2, $g_1(x)$ and $g_2(x)$ lower semi-continuous with $g_1(x)>0$ and $g_2(x)>0$ for all $x\in[0,+\infty)$. Consider the continuity of the expected value function of the fuzzy mapping*

$$f(x,\xi_1,\xi_2,\xi_3)=\max\{x,\xi_3\}\cdot\xi_1\cdot g_1(x)+\xi_2\cdot(\xi_3+g_2(x)),\quad x\in[0,\infty).$$

Firstly, let

$$f(x,u_1,u_2,u_3)=\max\{x,u_3\}\cdot u_1\cdot g_1(x)+u_2\cdot(u_3+g_2(x)),$$

where $x\in[0,+\infty)$, $u_1\in\xi_1(\Theta^+)=(0,2)$, $u_2\in\xi_2(\Theta^+)=(1,3)$ and $u_3\in\xi_3(\Theta^+)=(2,4)$. It is obviously that $f(x,u_1,u_2,u_3)$ is increasing with respect to u_i on $\xi_i(\Theta^+)$, $i=1,2,3$, respectively. Since for each $u_1'\in\xi_1(\Theta^+)$, $u_2'\in\xi_2(\Theta^+)$ and $u_3'\in\xi_3(\Theta^+)$, $\max\{x,u_3'\}$ is continuous with respect to x. Thus the function $f(x,u_1',u_2',u_3')$ is lower semi-continuous with respect to x. It follows from Remark 3 that $f^U(x,\xi_1,\xi_2,\xi_3;\alpha)$, $f^L(x,\xi_1,\xi_2,\xi_3;\alpha)$ and $E[f(x,\xi_1,\xi_2,\xi_3)]$ are lower semi-continuous with respect to x, $\forall\alpha\in(0,1]$. Furthermore, it follows from Theorem 2 and Lemma 1 that for all $\alpha\in(0,1]$,

$$\begin{aligned}f^U(x,\xi_1,\xi_2,\xi_3;\alpha)&=\max\left\{x,\xi_3^U(\alpha)\right\}\cdot\xi_1^U(\alpha)\cdot g_1(x)+\xi_2^U(\alpha)\cdot\left(\xi_3^U(\alpha)+g_2(x)\right)\\&=\max\{x,4-\alpha\}\cdot(2-\alpha)g_1(x)+(3-\alpha)(4-\alpha+g_2(x))\\&=\begin{cases}(4-\alpha)(2-\alpha)g_1(x)+(3-\alpha)(g_2(x)+4-\alpha), & x<4-\alpha\\ x(2-\alpha)g_1(x)+(3-\alpha)(g_2(x)+4-\alpha), & x\geq 4-\alpha,\end{cases}\\f^L(x,\xi_1,\xi_2,\xi_3;\alpha)&=\max\left\{x,\xi_3^L(\alpha)\right\}\cdot\xi_1^L(\alpha)\cdot g_1(x)+\xi_2^L(\alpha)\cdot\left(\xi_3^L(\alpha)+g_2(x)\right)\\&=\max\{x,2+\alpha\}\cdot\alpha\cdot g_1(x)+(1+\alpha)(2+\alpha+g_2(x))\\&=\begin{cases}(2+\alpha)\alpha\cdot g_1(x)+(1+\alpha)(2+\alpha+g_2(x)), & x<2+\alpha\\ x\cdot\alpha\cdot g_1(x)+(1+\alpha)(2+\alpha+g_2(x)), & x\geq 2+\alpha\end{cases}\end{aligned}$$

and

$$\begin{aligned}E[f(x,\xi_1,\xi_2,\xi_3)]&=\frac{1}{2}\int_0^1\left(f^U(x,\xi_1,\xi_2,\xi_3;\alpha)+f^L(x,\xi_1,\xi_2,\xi_3;\alpha)\right)\mathrm{d}\alpha\\&=\begin{cases}\dfrac{10}{3}g_1(x)+2g_2(x)+\dfrac{13}{2}, & x\in[0,2)\\ \left(\dfrac{1}{12}x^3-\dfrac{1}{2}x^2+x+\dfrac{8}{3}\right)g_1(x)+2g_2(x)+\dfrac{13}{2}, & x\in[2,4)\\ xg_1(x)+2g_2(x)+\dfrac{13}{2}, & x\in[4,\infty).\end{cases}\end{aligned}$$

It is easy to check that $f^U(x,\xi_1,\xi_2,\xi_3;\alpha)$, $f^L(x,\xi_1,\xi_2,\xi_3;\alpha)$ *and* $E[f(x,\xi_1,\xi_2,\xi_3)]$ *are lower semi-continuous with respect to* x, $\forall\alpha\in(0,1]$, *since* $g_1(x)$ *and* $g_2(x)$ *are lower-semi-continuous with respect to* x.

Theorem 4 *Consider the fuzzy mapping* $f(\boldsymbol{x},\boldsymbol{\xi})$, *satisfying the hypothesis* (***P***). *Furthermore, if* $f\left(\boldsymbol{x},\xi_1^V(\alpha),\xi_2^V(\alpha),\ldots,\xi_n^V(\alpha)\right)$ *and* $f\left(\boldsymbol{x},\xi_1^{\overline{V}}(\alpha),\xi_2^{\overline{V}}(\alpha),\ldots,\xi_n^{\overline{V}}(\alpha)\right)$ *are convex for all* $\alpha\in(0,1]$, *where* $\xi_i^V(\alpha)$ *and* $\xi_i^{\overline{V}}(\alpha)$ *are given as those in Theorem 2, then* $E[f(\boldsymbol{x},\boldsymbol{\xi})]$ *is convex with respect to* $\boldsymbol{x}$.

Proof. For each fixed $\boldsymbol{x}_0\in X$, it follows from Theorem 2 that

$$f^U(\boldsymbol{x}_0,\boldsymbol{\xi};\alpha)=f\left(\boldsymbol{x}_0,\xi_1^V(\alpha),\xi_2^V(\alpha),\ldots,\xi_n^V(\alpha)\right)$$

and

$$f^L(\boldsymbol{x}_0,\boldsymbol{\xi};\alpha)=f\left(\boldsymbol{x}_0,\xi_1^{\overline{V}}(\alpha),\xi_2^{\overline{V}}(\alpha),\ldots,\xi_n^{\overline{V}}(\alpha)\right)$$

for each $\alpha\in(0,1]$. So $f^U(\boldsymbol{x},\boldsymbol{\xi};\alpha)$ and $f^L(\boldsymbol{x},\boldsymbol{\xi};\alpha)$ are convex functions with respect to $\boldsymbol{x}$, $\forall\alpha\in(0,1]$. Then for each $\lambda\in[0,1]$ and $\boldsymbol{x}_1,\boldsymbol{x}_2\in X$, we have

$$\begin{aligned}
&E[f(\lambda\boldsymbol{x}_1+(1-\lambda)\boldsymbol{x}_2,\boldsymbol{\xi})]\\
&=\frac{1}{2}\int_0^1\left[f^U(\lambda\boldsymbol{x}_1+(1-\lambda)\boldsymbol{x}_2,\boldsymbol{\xi};\alpha)+f^L(\lambda\boldsymbol{x}_1+(1-\lambda)\boldsymbol{x}_2,\boldsymbol{\xi};\alpha)\right]\mathrm{d}\alpha\\
&\le\frac{1}{2}\int_0^1\left[\lambda\left(f^U(\boldsymbol{x}_1,\boldsymbol{\xi};\alpha)+f^L(\boldsymbol{x}_1,\boldsymbol{\xi};\alpha)\right)+(1-\lambda)\left(f^U(\boldsymbol{x}_2,\boldsymbol{\xi};\alpha)+f^L(\boldsymbol{x}_2,\boldsymbol{\xi};\alpha)\right)\right]\mathrm{d}\alpha\\
&=\frac{\lambda}{2}\int_0^1\left[f^U(\boldsymbol{x}_1,\boldsymbol{\xi};\alpha)+f^L(\boldsymbol{x}_1,\boldsymbol{\xi};\alpha)\right]\mathrm{d}\alpha+\frac{1-\lambda}{2}\int_0^1\left[f^U(\boldsymbol{x}_2,\boldsymbol{\xi};\alpha)+f^L(\boldsymbol{x}_2,\boldsymbol{\xi};\alpha)\right]\mathrm{d}\alpha\\
&=\lambda E[f(\boldsymbol{x}_1,\boldsymbol{\xi})]+(1-\lambda)E[f(\boldsymbol{x}_2,\boldsymbol{\xi})].
\end{aligned}$$

So the function $E[f(\boldsymbol{x},\boldsymbol{\xi})]$ is convex with respect to $\boldsymbol{x}$.

Remark 4 *Similarly,* $E[f(\boldsymbol{x},\boldsymbol{\xi})]$ *is concave on the conditions that* $f\left(\boldsymbol{x},\xi_1^V(\alpha),\xi_2^V(\alpha),\ldots,\xi_n^V(\alpha)\right)$ *and* $f\left(\boldsymbol{x},\xi_1^{\overline{V}}(\alpha),\xi_2^{\overline{V}}(\alpha),\ldots,\xi_n^{\overline{V}}(\alpha)\right)$ *are concave for all* $\alpha\in(0,1]$.

Remark 5 *If* $f(\boldsymbol{x},\boldsymbol{u})$ *is convex (concave, respectively) for each* $\boldsymbol{u}\in\boldsymbol{\xi}(\Theta^+)$, *then* $f^U(\boldsymbol{x},\boldsymbol{\xi};\alpha)$, $f^L(\boldsymbol{x},\boldsymbol{\xi};\alpha)$ *and* $E[f(\boldsymbol{x},\boldsymbol{\xi})]$ *are convex (concave, respectively),* $\forall\alpha\in(0,1]$.

Example 5 *Let* ξ_1,ξ_2 *and* ξ_3 *be given as those in Example 2 and the fuzzy mapping*

$$f(x,\xi_1,\xi_2,\xi_3)=\frac{e^{\xi_1}\xi_2^3}{1+x\xi_3},$$

where $x\in X\subset\mathcal{R}^+$. *It is obviously that for all* $x_0\in X$, $f(x_0,\xi_1,\xi_2,\xi_3)\in\mathcal{F}(\Theta,\mathcal{P}(\Theta),\mathrm{Pos})$ *and the function* $f(x,u_1,u_2,u_3)$ *is increasing with respect to* u_1 *and* u_2 *on* $\xi_1(\Theta_1)$ *and* $\xi_2(\Theta_2)$, *respectively, and decreasing with respect to* u_3 *on* $\xi_3(\Theta_3)$. *Since for each* $\theta=(\theta_1,\theta_2,\theta_3)\in\Theta$, $f(x,\xi_1(\theta_1),\xi_2(\theta_2),\xi_3(\theta_3))$ *is convex with respect to* x, *then* $E[f(x,\xi_1,\xi_2,\xi_3)]$ *is convex with respect to* x.

Example 6 *The utility function for a risk averse agent [5] can be given as*

$$U(x) = \frac{1 - e^{-rx}}{r}$$

with $x \in \mathcal{R}$ *and* $r > 0$*, where* r *is a measure of the agent's degree of risk aversion. Since the degree of one's risk aversion is fuzziness and hard to be calculated, it would be more reasonable for us to depict the degree by a positive fuzzy variable* ξ_r*, i.e.,* $\text{Pos}\{\xi_r \le 0\} = 0$*. Let*

$$U(x, \xi_r) = \frac{1 - e^{-\xi_r x}}{\xi_r} \quad \text{and} \quad U(x, u) = \frac{1 - e^{-ux}}{u}.$$

It is easy to check that $U(x_0, u)$ *is decreasing with respect to* u*, for each* $x_0 \in \mathcal{R}$*, and* $U(x, u_0)$ *a concave function, for each* $u_0 > 0$*. By Remark 5 we can deduce that* $E[U(x, \xi_r)]$ *is a concave function.*

4 An Application in the Retailer's Optimization Problem

Consider an optimization problem a retailer usually faces. The retailer does such a kind of business that he makes an order of one item product from one supplier and sells them to customers. Assume that the retailer is risk-neutral; the ordering quantity does not affect the ordering price per unit; no discount is allowed for the retailer to sell the goods; no reorder is allowed when lack of goods; there is a salvage value for unsold goods. Furthermore, if the retailer chooses to give up the business, he has an outside opportunity utility level W^*. In the programming, we have the following notations.

Notation

x	ordering quantity made by the retailer (units)
s	quantity of the product the retailer can sell (units)
c	ordering price per unit (\$)
p	retail price per unit (\$)
v	salvage value per unit (\$)
$W(x)$	the profit for the retailer (\$)

We have the relation of the parameters that $0 < v < c < p$, $0 \le s$ and $0 \le x \le M$, where M can be explained as the quantity of the product provided by the supplier. Suppose M is great enough. Since the ability of the retailer to sell the product is fuzziness and hard to be characterized, it would be more reasonable for us to use fuzzy variable ξ_s to depict the quantity of the goods he can sell. Assume that ξ_s is a bounded fuzzy variable with upper semi-continuous membership function defined on the possibility space $(\Theta, \mathcal{P}(\Theta), \text{Pos})$. Since the quantity of the goods the retailer can sell must be greater than or equal to 0, we have $\text{Pos}\{\xi_s < 0\} = 0$. Thus, when the retailer makes an order x, his profit $W(x)$ is a fuzzy variable on the possibility space $(\Theta, \mathcal{P}(\Theta), \text{Pos})$. The profit for the retailer can be calculated as the fuzzy mapping

$$\begin{aligned} W(x, \xi_s) &= p \min\{x, \xi_s\} + v(x - \min\{x, \xi_s\}) - cx \\ &= (p - v) \min\{x, \xi_s\} - (c - v)x. \end{aligned} \tag{12}$$

Let the function

$$W(x, u) = (p - v) \min\{x, u\} - (c - v)x. \tag{13}$$

It is obviously that $W(x, u)$ is increasing with respect to u, and for each $u_0 \ge 0$, $W(x, u_0)$ is continuous and concave with respect to x. It follows from Remark 3 and Remark 5 that $W^U(x, \xi_s; \alpha)$, $W^L(x, \xi_s; \alpha)$ and $E[W(x, \xi_s)]$ are continuous and concave with respect to x.

In the programming, suppose the retailer has a constraint that the possibility of the event that the profit is less than W^*, is less than α_0, where W^* is his outside opportunity utility level, and α_0 his critical level. Then the optimization problem can be depicted as the following programming model,

$$\begin{cases} \max\limits_{x} E[W(x,\xi_s)] \\ \quad s.t. \\ \qquad \text{Pos}\{W(x,\xi_s) < W^*\} < \alpha_0 \\ \qquad 0 \le x \le M. \end{cases} \tag{14}$$

Obviously, $\text{Pos}\{W(x,\xi_s) < W^*\} < \alpha_0$ is equal to $W^L(x,\xi_s;\alpha_0) \ge W^*$ by Definition 5. Since all the sections of upper semi-continuous function are closed, we deduce that the set of x which satisfies the constraints, denoted by $D(\alpha_0, W^*)$, is closed. If $D(\alpha_0, W^*)$ is empty, then the retailer would not do the business and with a reserve income W^*. Furthermore, if the set is nonempty, since $E[W(x,\xi_s)]$ is continuous and concave with respect to x, we can deduce, by Proposition 1, that there exists an x^*, which maximizes $E[W(x,\xi_s)]$ under the constraints.

If the fuzzy variable ξ_s is given as a triangular fuzzy variable and equals $(s-\Delta_1, s, s+\Delta_2)$, where $\Delta_1 > 0$, $\Delta_2 > 0$ and $s \ge \Delta_1$, then we have $\xi_s^U(\alpha) = s + (1-\alpha)\Delta_2$ and $\xi_s^L(\alpha) = s - (1-\alpha)\Delta_1$, $\forall \alpha \in (0,1]$. It follows from Theorem 2 and Lemma 1 that

$$\begin{aligned} W^U(x,\xi_s;\alpha) &= (p-v)\min\{x, \xi_s^U(\alpha)\} - (c-v)x \\ &= \begin{cases} (p-c)x, & x < s + (1-\alpha)\Delta_2 \\ (p-v)(s+(1-\alpha)\Delta_2) - (c-v)x, & x \ge s + (1-\alpha)\Delta_2, \end{cases} \end{aligned} \tag{15}$$

$$\begin{aligned} W^L(x,\xi_s;\alpha) &= (p-v)\min\{x, \xi_s^L(\alpha)\} - (c-v)x \\ &= \begin{cases} (p-c)x, & x < s - (1-\alpha)\Delta_1 \\ (p-v)(s-(1-\alpha)\Delta_1) - (c-v)x, & x \ge s - (1-\alpha)\Delta_1 \end{cases} \end{aligned} \tag{16}$$

and

$$E[W(x,\xi_s)] = \begin{cases} (p-c)x, & x \le s - \Delta_1 \\ \dfrac{1}{4\Delta_1}\big(-(p-v)(s-\Delta_1)^2 \\ \qquad +2((p-v)s + (p-2c+v)\Delta_1)x - (p-v)x^2\big), & s - \Delta_1 < x \le s \\ \dfrac{1}{4\Delta_2}\big(-(p-v)(s^2 - 2s\Delta_2 + \Delta_1\Delta_2) \\ \qquad +2((p-v)s + (p-2c+v)\Delta_2)x - (p-v)x^2\big), & s < x \le s + \Delta_2 \\ (p-v)\left(s + \dfrac{1}{4}(\Delta_2 - \Delta_1)\right) - (c-v)x, & x > s + \Delta_2. \end{cases} \tag{17}$$

Obviously that $W^U(x,\xi_s;\alpha), W^L(x,\xi_s;\alpha)$ and $E[W(x,\xi_s)]$ are continuous and concave functions. Then the program (14) can be solved through classical analysis.

Since $\max_{x\ge 0} W^L(x,\xi_s;\alpha_0) = (p-c)(s-(1-\alpha_0)\Delta_1)$, if it is less than W^*, the set of x satisfying the constraints, $D(\alpha_0, W^*)$, is empty. Thus the retailer would not choose to do the business and with a reserve income W^*. Otherwise, we have

$$D(\alpha_0, W^*) = \left[\frac{W^*}{p-c}, \frac{(p-v)(s-(1-\alpha_0)\Delta_1) - W^*}{c-v}\right]. \tag{18}$$

The programming model can be simplified as

$$\max_{x \in D(\alpha_0, W^*)} E[W(x,\xi_s)]. \tag{19}$$

Specifically, let $p = 10$, $c = 7$, $v = 6$, $W^* = 500$, $\alpha_0 = 0.4$, $s = 200$, $\Delta_1 = \Delta_1 = 50$, i.e., $\xi_s = (150, 200, 250)$. Since $\max_{x\geq 0} W^L(x, \xi_s; \alpha_0) = 510 > 500$, the equations (17) and (18) can rewrite as

$$E[W(x, \xi_s)] = \begin{cases} 3x, & x \leq 150 \\ -450 + 9x - \dfrac{1}{50}x^2, & 150 < x \leq 250 \\ 800 - x, & x > 250 \end{cases} \tag{20}$$

and $D(\alpha_0, W^*) = [500/3, 180]$, respectively. Solving the program (19), we get the optimal ordering quantity $x^* = 180$ and the maximum expected profit under constraints $E[W(x^*, \xi_s)] = 522$.

5 Conclusion

This paper studies the continuity and convexity of the expected value function of the fuzzy mapping $f(\boldsymbol{x}, \boldsymbol{\xi})$ to provide a way for convex analysis of the programming under a fuzzy environment. Firstly, we give the definition of fuzzy mapping as a mapping from a topological space to a collection of fuzzy variables. Secondly, some properties of fuzzy variable function are surveyed. With the assumptions of monotonicity and the independence of fuzzy variables with upper semi-continuous membership functions, the critical value of a fuzzy variable function, which is also a fuzzy variable, can be solved by the function of the critical values of fuzzy variables. Then, based on the properties of fuzzy variable function, we study the continuity and convexity of the α-optimistic value function, the α-pessimistic value function and the expected value function of a fuzzy mapping, respectively. We have the results that the continuity and convexity can be inherited after calculating the α-optimistic value, the α-pessimistic value and the expected value, respectively, if the function $f(\boldsymbol{x}, \boldsymbol{u})$ is monotonic with respect to $\boldsymbol{u}$ and the fuzzy variables ξ_i are bounded with upper semi-continuous membership function and independent with each other.

In the retailer's optimization problem within competition, the quantity of the product he can sell is more reasonable to be considered as a endogenous variable than exogenous variable, since the quantity is dependent on the ability of the retailer. Thus we can consider the quantity as a positive fuzzy variable. Then, following from the properties of fuzzy mapping, the optimization problem can be solved classically.

This paper also gives a method to calculate the expected value of the function of fuzzy variables. This might be useful for quantitative and qualitative studies in the optimization problem under a fuzzy environment. However, the results are given on the fuzzy mapping with restrictions. Considering the limitation, we should do further research for more general fuzzy mappings.

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