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# Some Inequalities Between Moments of Credibility Distributions

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**Abstract**

Fuzzy variables are functions from credibility spaces to the set of real numbers. Based on the expected value of a fuzzy variable, the expected value of function of a simple fuzzy variable is studied. In order further to discuss the mathematical properties of fuzzy variables, some inequalities for fuzzy variables are derived based on the concepts of credibility measures and expected value operators. 
**Keywords:** fuzzy variable, moment, credibility distribution, power mean

## 1 Introduction

The fuzzy set theory was initiated by Zadeh [13] via membership function in 1965, and was well developed and applied in a wide variety of areas, such as control theory, optimizations, intelligent systems, information sciences, and so on. In order to measure the fuzzy event, Zadeh [15] in 1978 proposed the possibility theory which was developed by many other researchers such as Dubois and Prade [1]. Unfortunately, possibility measure is not self-dual. However, a self-dual measure is crucial in both theory and practice. Liu and Liu [4] gave the concept of credibility measure that satisfies the self-dual property in 2002. Li and Liu [2] gave a sufficient and necessary condition for credibility measure in 2006. Liu [6] presented an axiomatic foundation of credibility theory dealing with fuzzy events based on credibility measure in 2004, and refined credibility theory in 2007 [8] as a branch of mathematics for studying the behavior of fuzzy phenomena. Some aspects of study on credibility theory may also be found in [7] and [9].

Fuzzy variables may be defined as functions from credibility spaces to the set of real numbers. The expected value of a fuzzy variable is a very important concept not only in credibility theory but also in fuzzy programming. While expected value of a fuzzy variable can be formularized as a Lebesgue-Stieltjes integral with its credibility distribution, it will be more convenient to compute expected value of fuzzy variable. But generally speaking, expected value of function $f(\xi)$ of fuzzy variable $\xi$ can't be expressed as a Lebesgue-Stieltjes integral with the credibility distribution of $\xi$. Zhu and Ji [17] gave a formula to compute the expected value of function $f(\xi)$ of continuous fuzzy variable $\xi$. In this paper, we also give a formula to compute expected value of function $f(\xi)$ of a simple fuzzy variable $\xi$. In addition, the set of moments of fuzzy variables that uniquely characterizes the distribution under the reasonable conditions is useful in making comparison of two fuzzy variables. Liu [5] presented some inequalities for the moments of fuzzy variables, and Zhu and Liu [16] showed some inequalities for the moments of random fuzzy variables. Based on the mathematical elementary inequalities and the expected value of function of a fuzzy variable, we develop some estimations of

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moments of fuzzy variables when the fuzzy variable is simple or continuous and its support takes values on a finite interval.

The outline of this paper is as follows. In Section 2, we recall some definitions of fuzzy variable, expected value operator, credibility distribution, and so on. In Section 3, some lemmas are expressed. In Section 4, we give a formula to compute expected value of function $f(\xi)$ of a simple fuzzy variable $\xi$. In Section 5, we demonstrate some inequalities on moments of fuzzy variables.

## 2 Some Concepts

In convenience, we recall some useful concepts at first. Let $\Theta$ be a nonempty set, and $\mathcal{P}$ the power set of $\Theta$ (i.e., all subsets of $\Theta$). The triplet $(\Theta, \mathcal{P}, \text{Cr})$ is said to be a credibility space if Cr, called credibility measure [4], is a nonnegative set function defined on $\mathcal{P}$ satisfying

(1) $\text{Cr}\{\Theta\} = 1$,

(2) $\text{Cr}\{A\} \le \text{Cr}\{B\}$ whenever $A \subset B$,

(3) Cr is self-dual, i.e., $\text{Cr}\{A\} + \text{Cr}\{A^c\} = 1$ for any $A \in \mathcal{P}$,

(4) $\text{Cr}\{\bigcup_i A_i\} = \sup_i \text{Cr}\{A_i\}$ for any $A_i \in \mathcal{P}$ with $\sup_i \text{Cr}\{A_i\} < 0.5$.

**Definition 2.1** *(Liu [3] [8]) A fuzzy variable is defined as a function from a credibility space* $(\Theta, \mathcal{P}, \text{Cr})$ *to the set of real numbers.*

**Definition 2.2** *(Liu [3] [8]) Let $\xi$ be a fuzzy variable defined on the credibility space* $(\Theta, \mathcal{P}, \text{Cr})$*. Then its membership function is derived from the credibility measure by*

$$\mu(x) = (2\text{Cr}\{\xi = x\}) \wedge 1, \quad x \in \Re.$$

**Remark 2.1** If the membership function of variable $\xi$ is $\mu$, then for any set $B \subset \Re$,

$$\text{Cr}\{\xi \in B\} = \frac{1}{2}\left(\sup_{x \in B} \mu(x) + 1 - \sup_{x \in B^c} \mu(x)\right)$$

**Definition 2.3** *(Liu [6]) Let $\xi$ be a fuzzy variable defined on the credibility space* $(\Theta, \mathcal{P}, \text{Cr})$*. Then the set*

$$\{\xi(\theta) | \theta \in \Theta, \text{Cr}\{\theta\} > 0\} = \{\xi(\theta) | \theta \in \Theta^+\}$$

*is called the support of $\xi$, where $\Theta^+$ is the kernel of the credibility space* $(\Theta, \mathcal{P}, \text{Cr})$*.*

**Definition 2.4** *(Liu [6]) A fuzzy variable $\xi$ is said to be*
*(a) nonnegative if* $\text{Cr}\{\xi < 0\} = 0$*;*
*(b) positive if* $\text{Cr}\{\xi \le 0\} = 0$*;*
*(c) continuous if* $\text{Cr}\{\xi = x\}$ *is a continuous function of $x$;*
*(d) simple if there exists a finite sequence $x_1, x_2, \cdots, x_n$ such that*

$$\text{Cr}\{\xi \ne x_1, \xi \ne x_2, \cdots, \xi \ne x_n\} = 0.$$

**Definition 2.5** *(Liu and Liu [4]) Let $\xi$ be a fuzzy variable. Then the expected value of $\xi$ is*

$$E[\xi] = \int_0^{+\infty} \text{Cr}\{\xi \ge r\}\text{d}r - \int_{-\infty}^0 \text{Cr}\{\xi \le r\}\text{d}r.$$

**Remark 2.2** *(Liu [3])* Let $\xi$ be a simple fuzzy variable whose membership function is given by

$$\mu(x) = \begin{cases} \mu_1, & \text{if } x = x_1 \\ \mu_2, & \text{if } x = x_2 \\ \vdots & \\ \mu_n, & \text{if } x = x_n. \end{cases}$$

Then the expected value of $\xi$ is

$$E[\xi] = \sum_{i=1}^{n} \omega_i x_i$$

where the weights $\omega_i, i = 1, 2 \cdots, n$ are given by

$$\begin{aligned} \omega_i = \frac{1}{2} \Big( & \max_{1 \le k \le n} (\mu_k \mid x_k \le x_i) - \max_{1 \le k \le n} (\mu_k \mid x_k < x_i) \\ & + \max_{1 \le k \le n} (\mu_k \mid x_k \ge x_i) - \max_{1 \le k \le n} (\mu_k \mid x_k > x_i) \Big) \end{aligned} \tag{2.1}$$

for $i = 1, 2 \cdots n$. It is easy to verify that all $\omega_i \ge 0$ and the sum of all weights is just 1.

**Definition 2.6** *(Liu [3]) For any positive integer $k$, the expected value $E[\xi^k]$ is called the $k$th moment of the fuzzy variable $\xi$.*

**Definition 2.7** *(Liu [3]) The credibility distribution $\Phi : \Re \to [0, 1]$ of a fuzzy variable $\xi$ is defined by*

$$\Phi(x) = \mathrm{Cr}\{\theta \in \Theta | \xi(\theta) \le x\}.$$

## 3 Lemmas

In this section, we present some results, including a formula to compute expected value of function $f(\xi)$ of continuous variable $\xi$, and some elementary inequalities.

**Lemma 3.1** *(Zhu and Ji [17]) Let $\xi$ be a fuzzy variable and its credibility distribution function $\Phi(x)$. Suppose that $f : \Re \to \Re$ is a strictly monotone and continuous function. If*

$$\lim_{x \longrightarrow -\infty} \Phi(x) = 0, \quad \lim_{x \longrightarrow \infty} \Phi(x) = 1 \tag{3.1}$$

*and the Lebesgue-Stieltjes integral $\int_{-\infty}^{\infty} f(x) \mathrm{d}\Phi(x)$ is finite, then we have*

$$E[f(\xi)] = \int_{-\infty}^{\infty} f(x) \mathrm{d}\Phi(x). \tag{3.2}$$

**Remark 3.1** If the support of the fuzzy variable $\xi$ takes values in the finite interval $[a, b]$, then $\Phi(x) = 0$ or 1 according as $x < a$ or $x > b$. That is $\Phi(x)$ satisfies (3.1).

**Remark 3.2** Let $\xi$ be a fuzzy variable whose support takes the values of nonnegative numbers. If the credibility distribution function $\Phi(x)$ satisfies (3.1) and (3.2), and the Lebesgue-Stieltjes integral

$$\int_{-\infty}^{\infty} x^k \mathrm{d}\Phi(x) \tag{3.3}$$

is finite, then we have

$$E[\xi^k] = \int_{0}^{\infty} x^k \mathrm{d}\Phi(x) \tag{3.4}$$

by Lemma 3.1.

**Lemma 3.2** *(R.Sharama et al. [11]) If $r$ is a positive real number and $s$ is any nonzero real number with $r > s$, then for $a \le x \le b$, with $a > 0$, we have*

$$x^r \le \frac{(b^r - a^r)x^s + a^r b^s - a^s b^r}{b^s - a^s} \tag{3.5}$$

*and for $x$ lying outside $(a, b)$ we have*

$$x^r \ge \frac{(b^r - a^r)x^s + a^r b^s - a^s b^r}{b^s - a^s}. \tag{3.6}$$

*If $r$ is a negative real number with $r > s$, then inequality(3.5) holds for $x$ lying outside $(a, b)$ and inequality (3.6)holds for $a \le x \le b$.*

**Lemma 3.3** *(R.Sharama et al. [11]) For $a \le x \le b$ with $a > 0$, we have*

$$x^r \le \frac{(b^r - a^r)\log x + a^r \log b - b^r \log a}{\log b - \log a} \tag{3.7}$$

*and for $x$ lying outside $(a, b)$, we have*

$$x^r \ge \frac{(b^r - a^r)\log x + a^r \log b - b^r \log a}{\log b - \log a} \tag{3.8}$$

*where $r$ is a real number.*

## 4 Expected value of function of a simple fuzzy variable

In this section, we give a formula to compute the expected value of function $f(\xi)$ of simple variable $\xi$, and then define the power mean of the nonnegative fuzzy variable.

**Theorem 4.1** *Let $\xi$ be a simple fuzzy variable whose membership function is given by*

$$\mu(x) = \begin{cases} \mu_1, & \text{if } x = x_1 \\ \mu_2, & \text{if } x = x_2 \\ \quad\vdots & \\ \mu_n, & \text{if } x = x_n. \end{cases}$$

*Suppose that $f : \Re \to \Re$ is a monotone function, then we have*

$$E[f(\xi)] = \sum_{i=1}^{n} \omega_i f(x_i) \tag{4.1}$$

*where the weights $\omega_i, i = 1, 2 \cdots, n$ are given by*

$$\omega_i = \frac{1}{2}\Bigg(\max_{1\le k\le n}(\mu_k | x_k \le x_i) - \max_{1\le k\le n}(\mu_k | x_k < x_i) + \max_{1\le k\le n}(\mu_k | x_k \ge x_i) - \max_{1\le k\le n}(\mu_k | x_k > x_i)\Bigg)$$

*for $i = 1, 2 \cdots n$.*

**Proof.** Without loss of generality, we suppose that $x_1, \cdots, x_n$ are distinct with $x_1 < x_2 < \cdots < x_n$, and $f$ is an increasing function. Hence $f(x_1) \le f(x_2) \le \cdots \le f(x_n)$.

Suppose that there exist some positive integers $p_0, p_1, p_2, \cdots, p_m$ with $1 = p_0 \le p_1 \le p_2 \le \cdots \le p_m = n$, such that

$$\begin{aligned} & f(x_1) = f(x_2) = \cdots = f(x_{p_1}) \\ & < f(x_{p_1+1}) = f(x_{p_1+2}) = \cdots = f(x_{p_2}) \\ & < \cdots < f(x_{p_{m-1}+1}) = f(x_{p_{m-1}+2}) = \cdots = f(x_{p_m}) = f(x_n). \end{aligned}$$

Then the membership function of fuzzy variable $\eta = f(\xi)$ is

$$\mu(x) = \begin{cases} \mu_1^{'} = \max\{\mu_1, \cdots, \mu_{p_1}\}, & \text{if } x = f(x_1) = f(x_2) = \cdots = f(x_{p_1}) \\ \mu_2^{'} = \max\{\mu_{p_1+1}, \cdots, \mu_{p_2}\}, & \text{if } x = f(x_{p_1+1}) = f(x_{p_1+2}) = \cdots = f(x_{p_2}) \\ \vdots & \\ \mu_m^{'} = \max\{\mu_{p_{m-1}+1}, \cdots, \mu_n\}, & \text{if } x = f(x_{p_{m-1}+1}) = f(x_{p_{m-1}+2}) = \cdots = f(x_n). \end{cases}$$

Thus

$$E[\eta] = \sum_{i=1}^{m} \omega_i^{'} f(x_{p_i}),$$

where $x_{p_m} = x_n$, and the weights $\omega_i^{'} (i = 1, 2 \cdots, m)$ are given by

$$\omega_i^{'} = \frac{1}{2}\left( \max_{k \le i} \mu_k^{'} - \max_{k < i} \mu_k^{'} + \max_{k \ge i} \mu_k^{'} - \max_{k > i} \mu_k^{'} \right).$$

For any $i = 1, 2, \cdots, m$, we have

$$\begin{aligned} \omega_{p_{i-1}+1} + \omega_{p_{i-1}+2} + \cdots + \omega_{p_i} &= \frac{1}{2}\left( \max_{k \le p_{i-1}+1} \mu_k - \max_{k < p_{i-1}+1} \mu_k + \max_{k \ge p_{i-1}+1} \mu_k - \max_{k > p_{i-1}+1} \mu_k \right) \\ &\quad + \frac{1}{2}\left( \max_{k \le p_{i-1}+2} \mu_k - \max_{k < p_{i-1}+2} \mu_k + \max_{k \ge p_{i-1}+2} \mu_k - \max_{k > p_{i-1}+2} \mu_k \right) \\ &\quad + \cdots \\ &\quad + \frac{1}{2}\left( \max_{k \le p_i} \mu_k - \max_{k < p_i} \mu_k + \max_{k \ge p_i} \mu_k - \max_{k > p_i} \mu_k \right) \\ &= \frac{1}{2}\left( - \max_{k < p_{i-1}+1} \mu_k + \max_{k \ge p_{i-1}+1} \mu_k + \max_{k \le p_i} \mu_k - \max_{k > p_i} \mu_k \right) \\ &= \frac{1}{2}\left( \max_{k \le i} \mu_k^{'} - \max_{k < i} \mu_k^{'} + \max_{k \ge i} \mu_k^{'} - \max_{k > i} \mu_k^{'} \right) \\ &= \omega_i^{'}. \end{aligned}$$

Therefore

$$E[\eta] = \sum_{i=1}^{m} \omega_i^{'} f(x_{p_i}) = \sum_{i=1}^{n} \omega_i f(x_i).$$

The theorem is proved.

**Example 4.1** Generally speaking, if the function $f(x)$ is not monotone, Theorem 4.1 may not hold. We consider a simple fuzzy variable $\xi$ whose membership function is given by

$$\mu(x) = \begin{cases} 0.6, & \text{if } x = -3 \\ 0.8, & \text{if } x = -2 \\ 0.4, & \text{if } x = 1 \\ 1, & \text{if } x = 4 \end{cases}$$

and $f(x) = x^2$. It is clear that $f(x) = x^2$ is not a monotone function on $\Re$, and the membership function of $f(\xi)$ is given by

$$\mu(x) = \begin{cases} 0.6, & \text{if } x = 9 \\ 0.8, & \text{if } x = 4 \\ 0.4, & \text{if } x = 1 \\ 1, & \text{if } x = 16. \end{cases}$$

However

$$E[f(\xi)] = \int_0^{+\infty} \text{Cr}\{f(\xi) \geq r\} dr = 10.6,$$

$$\sum_{i=1}^{4} \omega_i f(x_i) = 0.3 \times 9 + 0.1 \times 4 + 0 \times 1 + 16 \times 0.6 = 12.7 \neq E[f(\xi)].$$

**Remark 4.1** If the support of a simple fuzzy variable $\xi$ takes values in interval [a,b], and the function $f(x)$ is monotone just in $[a, b]$, then the conclusion in theorem 4.1 is still true.

**Theorem 4.2** *Let $\xi$ be a positive fuzzy variable. For any real number $k$,*

$$M_k = (E[\xi^k])^{\frac{1}{k}}$$

*is an increasing function of $k$ on $(-\infty, 0)$, and $(0, +\infty)$, respectively.*

**Proof.** It follows from (Liu [6]) that, for any convex function, if $E[\xi]$ and $f(E[\xi])$ is finite, then

$$f(E[\xi]) \leq E[f(\xi)].$$

Now we consider $p_2 > p_1 > 0$. Let $f(x) = |x|^{\frac{p_2}{p_1}}$. It is easy to see that $f(x)$ is a convex function. Thus

$$f(E[\xi^{p_1}]) \leq E[f(\xi^{p_1})],$$
$$(E[\xi^{p_1}])^{\frac{p_2}{p_1}} \leq E[(\xi^{p_1})^{\frac{p_2}{p_1}}],$$
$$(E[\xi^{p_1}])^{\frac{1}{p_1}} \leq (E[\xi^{p_2}])^{\frac{1}{p_2}}.$$

Therefore $\left(E[\xi^k]\right)^{\frac{1}{k}}$ is an increasing function for $k > 0$.

If $k < 0$. If $\xi = 0$, then

$$(E[\xi^{-k}])^{\frac{1}{-k}} = \frac{1}{(E[\frac{1}{\xi}]^k)^{\frac{1}{k}}}.$$

So for $k < 0$ the $M_k$ is also increasing. The theorem is proved.

**Definition 4.1** *Let $\xi$ be a positive fuzzy variable. Then power mean of order $k$ for $\xi$ is defined as*

$$M_k = \left(E[\xi^k]\right)^{\frac{1}{k}}, \qquad k \neq 0. \tag{4.2}$$

It follows from Theorem 4.2 that the limitations

$$M_0^+ = \lim_{k\downarrow 0}(E[\xi^k])^{\frac{1}{k}}, \qquad M_0^- = \lim_{k\uparrow 0}(E[\xi^k])^{\frac{1}{k}} \tag{4.3}$$

exist. If $M_0^+ = M_0^-$, then denote $M_0 = \lim_{k\to 0}\left(E[\xi^k]\right)^{\frac{1}{k}}$.

**Example 4.2** If $\xi$ is a positive simple fuzzy variable defined as Remark 2.2, then

$$M_k = \left(\sum_{i=1}^{n}\omega_i x_i^k\right)^{\frac{1}{k}},$$

and

$$M_0 = \lim_{k\to 0}\left(\sum_{i=1}^{n}\omega_i x_i^k\right)^{\frac{1}{k}} = \lim_{k\to 0}\exp\left(\frac{\ln\sum_{i=1}^{n}\omega_i x_i^k}{k}\right)$$
$$= \exp\left(\lim_{k\to 0}\frac{\sum_{i=1}^{n}\omega_i x_i^k \ln x_i}{\sum_{i=1}^{n}\omega_i x_i^k}\right) = x_1^{w_1}x_2^{w_2}\cdots x_n^{w_n}$$

where the weights $\omega_i$ $(i = 1, 2, \cdots, n)$ are given by (2.1).

## 5 Inequalities between moments of fuzzy variables

In this section, we prove some inequalities between moments of fuzzy variable by using Lemma 3.1, 3.2, and Theorem 4.1, 4.2.

**Theoreom 5.1** *(i) Let $\xi$ be a simple fuzzy variable whose support takes values $x_i (i = 1, \cdots, n)$ in the interval $[a, b]$ with $a > 0$. Suppose that $r$ is a positive real number and $s$ is any nonzero real number with $r > s$. Then we have*

$$E[\xi^r] \leq \frac{(b^r - a^r)E[\xi^s] + a^r b^s - a^s b^r}{b^s - a^s} \tag{5.1}$$

*and*

$$E[\xi^r] \geq \frac{(x_j^r - x_{j-1}^r)E[\xi^s] + x_{j-1}^r x_j^s - x_{j-1}^s x_j^r}{x_j^s - x_{j-1}^s} \tag{5.2}$$

*where $j = 2, 3\cdots n$. Furthermore, if $E[\xi^s]$ coincides with one of $x_{j-1}^s$ or $x_j^s$, we get*

$$E[\xi^r] \geq E[\xi^s]^{\frac{r}{s}}. \tag{5.3}$$

*(ii) If $\xi$ is a continuous fuzzy variable whose support takes values in the interval $[a, b]$ with $a > 0$, then the upper bound for $E[\xi^r]$ is given by the inequality* (5.1).

**Proof.** (i) From the Theorem 4.1, we know that $E[\xi^r] = \sum_{i=1}^{n} \omega_i x_i^r$. It is seen that $E[\xi^r]$ can be expressed in terms of $E[\xi^s]$ in the following form :

$$\begin{aligned} E[\xi^r] &= \sum_{i=1}^{n} \omega_i x_i^r + \left[ \frac{x_\beta^r - x_\alpha^r}{x_\beta^s - x_\alpha^s} E[\xi^s] - \frac{x_\beta^s x_\alpha^r - x_\alpha^s x_\beta^r}{x_\beta^s - x_\alpha^s} + \frac{x_\beta^s x_\alpha^r - x_\alpha^s x_\beta^r}{x_\beta^s - x_\alpha^s} - \frac{x_\beta^r - x_\alpha^r}{x_\beta^s - x_\alpha^s} E[\xi^s] \right] \\ &= \left( \frac{x_\beta^r - x_\alpha^r}{x_\beta^s - x_\alpha^s} \right) E[\xi^s] + \frac{x_\beta^s x_\alpha^r - x_\alpha^s x_\beta^r}{x_\beta^s - x_\alpha^s} + \sum_{i=1}^{n} \omega_i \left[ x_i^r - \frac{x_\beta^r - x_\alpha^r}{x_\beta^s - x_\alpha^s} x_i^s + \frac{x_\beta^r x_\alpha^s - x_\alpha^r x_\beta^s}{x_\beta^s - x_\alpha^s} \right] \end{aligned} \tag{5.4}$$

where the weights $\omega_i, i = 1, 2, \cdots, n$ are given by (2.1), and $x_\alpha$ and $x_\beta$ are in $[a, b]$ and distinct. Without loss of generality we can arrange values of the variable such that $a \le x_1 < x_2 < \cdots x_n \le= b$. If we take $x_\alpha = a$ and $x_\beta = b$, then by using the inequality (3.5), we know that the last term in equation (5.4) is negative,

$$\sum_{i=1}^{n} \omega_i \left[ x_i^r - \frac{b^r - a^r}{b^s - a^s} x_i^s + \frac{b^r a^s - a^r b^s}{b^s - a^s} \right] \le 0.$$

So the inequality(5.1) holds. If we take $x_\alpha = x_{j-1}$ and $x_\beta = x_j$, $j = 2, 3, \cdots n$, then each $x_i$ lies outside $(x_{j-1}, x_j)$ and it follows from (3.6) that the last term in equation (5.4) is positive,

$$\sum_{i=1}^{n} \omega_i \left[ x_i^r - \frac{x_j^r - x_{j-1}^r}{x_j^s - x_{j-1}^s} x_i^s + \frac{x_j^r x_{j-1}^s - x_{j-1}^r x_j^s}{x_j^s - x_{j-1}^s} \right] \ge 0.$$

Therefore inequality (5.2) holds. It is also clear that quality in the inequalities (5.1) and (5.2) take equal mark if and only if $n = 2$.

If the value of $E[\xi^s]$ coincides with one of $x_{j-1}^s$ or $x_j^s$, e.g. $E[\xi^s] = x_{j-1}^s$, then from inequality (5.2), we have

$$E[\xi^r] \ge \frac{(x_j^r - x_{j-1}^r)E[\xi^s] + x_{j-1}^r x_j^s - x_{j-1}^s x_j^r}{x_j^s - x_{j-1}^s} = x_{j-1}^r = (x_{j-1}^s)^{\frac{r}{s}} = E[\xi^s]^{\frac{r}{s}}. \tag{5.5}$$

(ii) Let $\xi$ be a continuous fuzzy variable whose support takes values in the interval $[a, b]$ with $a > 0$, then the credibility distribution function $\Phi(x)$ of $\xi$ satisfies (3.1). It follows from the inequality (3.5) that

$$x^r \le \frac{(b^r - a^r)x^s + a^r b^s - a^s b^r}{b^s - a^s}$$

Taking Lebesgue-Stieltjes over the credibility distribution function $\Phi(x)$ in the integral both sides of above inequality and using the properties of integrals, we get

$$\int_a^b x^r \mathrm{d}\Phi(x) \le \frac{(b^r - a^r) \int_a^b x^s \mathrm{d}\Phi(x) + a^r b^s - a^s b^r}{b^s - a^s}.$$

According to Remark 3.1 and 3.2, we have

$$E[\xi^r] \le \frac{(b^r - a^r)E[\xi^s] + a^r b^s - a^s b^r}{b^s - a^s}.$$

Therefore the theorem is proved.

**Remark 5.1** Let $r$ and $s$ be negative real numbers with $r > s$. Then "$\le$" in (5.1) and "$\ge$" in (5.2) are replaced by "$\ge$" and "$\le$" respectively.

**Theoreom 5.2** *For a simple fuzzy variable $\xi$ whose support takes values $x_i (i = 1, \cdots, n)$ in the interval $[a, b]$ with $a > 0$, we have*

$$E[\xi^r] \le \frac{(b^r - a^r)\log M_0 + a^r \log b - b^r \log a}{\log b - \log a} \tag{5.6}$$

*and*

$$E[\xi^r] \ge \frac{(x_j^r - x_{j-1}^r)\log M_0 + x_{j-1}^r \log x_j - x_j^r \log x_{j-1}}{\log x_j - \log x_{j-1}} \tag{5.7}$$

*where $j = 2, 3, \cdots n$, $r$ is a real number and*

$$M_0 = x_1^{w_1} x_2^{w_2} \cdots x_n^{w_n}. \tag{5.8}$$

*Furthermore, if $M_0$ coincides with one of $x_{j-1}$ or $x_j$, we get*

$$E[\xi^r] \ge M_0^r. \tag{5.9}$$

**Proof.** Following the Example 4.2, we know

$$M_0 = \lim_{k\to 0}\left(E[\xi^k]\right)^{\frac{1}{k}} = \lim_{k\to 0}\left(\sum_{i=1}^n \omega_i x_i^k\right)^{\frac{1}{k}} = x_1^{w_1} x_2^{w_2} \cdots x_n^{w_n}.$$

It is seen that $E[\xi^r]$ can be expressed in terms of $E[\xi^s]$ in the following form :

$$\begin{aligned} E[\xi^r] = & \frac{x_\beta^r - x_\alpha^r}{\log x_\beta - \log x_\alpha}\log M_0 + \frac{x_\alpha^r \log x_\beta - x_\beta^r \log x_\alpha}{\log x_\beta - \log x_\alpha} \\ & + \sum_{i=1}^n \omega_i \left[x_i^r - \frac{x_\beta^r - x_\alpha^r}{\log x_\beta - \log x_\alpha}\log x_i + \frac{x_\beta^r \log x_\alpha - x_\alpha^r \log x_\beta}{\log x_\beta - \log x_\alpha}\right] \end{aligned} \tag{5.10}$$

where $x_\alpha$ and $x_\beta$ are in $[a, b]$ and distinct. Without loss of generality we can arrange values of the variable such that $a \le x_1 < x_2 < \cdots x_n \le b$. If we take $x_\alpha = a$ and $x_\beta = b$, then by using the inequality (3.7), we know that the last term in equation (5.10) is negative,

$$\begin{aligned} & \sum_{i=1}^n \omega_i \left[x_i^r - \frac{x_\beta^r - x_\alpha^r}{\log x_\beta - \log x_\alpha}\log x_i + \frac{x_\beta^r \log x_\alpha - x_\alpha^r \log x_\beta}{\log x_\beta - \log x_\alpha}\right] \\ = & \sum_{i=1}^n \omega_i \left[x_i^r - \frac{b^r - a^r}{\log b - \log a}\log x_i + \frac{b^r \log a - a^r \log b}{\log b - \log a}\right] \le 0. \end{aligned}$$

So the inequality (5.6) holds. If we take $x_\alpha = x_{j-1}$ and $x_\beta = x_j$, $j = 2, 3, \cdots n$, then each $x_i$ lies outside $(x_{j-1}, x_j), j = 1, 2, \cdots n$, and it follows from inequality (3.8) that the last term in equation (5.10) is positive,

$$\begin{aligned} & \sum_{i=1}^n \omega_i \left[x_i^r - \frac{x_\beta^r - x_\alpha^r}{\log x_\beta - \log x_\alpha}\log x_i + \frac{x_\beta^r \log x_\alpha - x_\alpha^r \log x_\beta}{\log x_\beta - \log x_\alpha}\right] \\ = & \sum_{i=1}^n \omega_i \left[x_i^r - \frac{x_j^r - x_{j-1}^r}{\log x_j - \log x_{j-1}}\log x_i + \frac{x_j^r \log x_{j-1} - x_{j-1}^r \log x_j}{\log x_j - \log x_{j-1}}\right] \ge 0. \end{aligned}$$

Therefore (5.7) holds. If the value of $M_0$ coincides with one of $x_{j-1}$ or $x_j$, e.g. $M_0 = x_{j-1}$, then from inequality (5.7) we have

$$E[\xi^r] \ge \frac{(x_j^r - x_{j-1}^r)\log M_0 + x_{j-1}^r \log x_j - x_j^r \log x_{j-1}}{\log x_j - \log x_{j-1}} = M_0^r. \tag{5.11}$$

The theorem is proved.

**Theoreom 5.3** *For a continuous fuzzy variable $\xi$ whose support takes values in the interval $[a,b]$ with $a>0$, we have*

$$E[\xi^r] \le \frac{(b^r-a^r)\log M_0 + a^r\log b - b^r\log a}{\log b - \log a}. \tag{5.12}$$

**Proof.** We consider

$$\log(M_0) = \log\left[\lim_{r\to 0}\left(\int_a^b x^r \mathrm{d}\Phi(x)\right)^{\frac{1}{r}}\right] \tag{5.13}$$

$$= \lim_{r\to 0}\left(\frac{\log[\int_a^b x^r\mathrm{d}\Phi(x)]}{r}\right)$$

$$= \lim_{r\to 0}\frac{\int_a^b x^r\log x\mathrm{d}\Phi(x)}{\int_a^b x^r\mathrm{d}\Phi(x)}.$$

According to Lebesgue Dominated Convergence Theorem, we know that

$$\log M_0 = \lim_{r\to 0}\frac{\int_a^b x^r\log x\mathrm{d}\Phi(x)}{\int_a^b x^r\mathrm{d}\Phi(x)} = \int_a^b \log x\mathrm{d}\Phi(x).$$

Taking Lebesgue-Stieltjes integral over credibility distribution function $\Phi(x)$ in the both sides of inequality (3.7) and using the properties of integrals, we get

$$E[\xi^r] = \int_a^b x^r\mathrm{d}\Phi(x) \le \frac{(b^r-a^r)\int_a^b\log x\mathrm{d}\Phi(x) + a^r\log b - b^r\log a}{\log b-\log a}$$

$$= \frac{(b^r-a^r)\log M_0 + a^r\log b - b^r\log a}{\log b - \log a}.$$

The theorem is proved.

By using Theorem 4.1, we can get the following results of fuzzy variables while their proofs are similar to that of the corresponding results for random variables in [12], and hence we omit them here.

**Theoreom 5.4** *For a simple fuzzy variable $\xi$ whose support takes values in the interval $[a,b]$ with $a>0$, if $f(x)$ is a monotone and convex function, then*

$$f(a+b-E[\xi]) \le f(a)+f(b)-E[f(\xi)]. \tag{5.14}$$

*Especially, if $a\le x_1<\cdots<x_n\le b$, then*

$$f(x_1+x_n-E[\xi]) \le f(x_1)+f(x_n)-E[f(\xi)]. \tag{5.15}$$

**Theoreom 5.5** *For a simple fuzzy variable $\xi$ whose support takes values $x_i (i=1,\cdots,n)$ in the interval $[a,b]$ with $a>0$, suppose that $Q_r(a,b,\xi)=(a^r+b^r-M_r^r)^{\frac{1}{r}}$, where $M_r$ is the power mean of order $r$ for $\xi$. Then $Q_0(a,b,\xi)=\lim_{r\to 0}(a^r+b^r-M_r^r)^{\frac{1}{r}}=\frac{ab}{M_0}$, and for $r$.*

## 6 Conclusion

In this paper, we first studied the expected value of function of a simple fuzzy variable and gave a formula to compute it. Then, based on introducing the definition of power mean, we developed some estimations of moments of nonnegative fuzzy variable. We obtained some upper bounds and below bounds for the moments of nonnegative fuzzy variable.

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