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Fuzzy Mathematical Programming: Theory, Applications and Extension

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Abstract

Mathematical programming has been successfully used for years in a variety of problems related to hard systems in which the structure, relations and behaviour are well-defined and quantifiable. Unfortunately, attempts to apply similar means to soft systems have not been generally successful. One of the reasons for this mismatching is the key role played by human judgement and preferences which are subjective, imprecise and not easily quantifiable. Although probabilistic theories claim to model decision making under imprecision, there is qualitatively different kind of indeterminacy which are not covered by these tools, that is: inexactness, ill-definedness, vagueness. The aim of this paper is twofold. First, it takes a general look at core ideas aimed at softening mathematical programming models by making it possible to incorporate non-stochastic imprecision into these models. Second, it extends these ideas to situations where both fuzziness and randomness are under one roof in a mathematical programming setting. The paper ends with some concluding remarks along with lines for further developments in the field of fuzzy mathematical programming.

Keywords: Mathematical programming, fuzzy relationship, fuzzy parameter, fuzzy random variable.

1 Introduction

Many systems to be controlled or simply analyzed include some level of imprecision about the values to assign to some parameters or about the actual design of some of the components of the system.

In this connection, the noted philosopher Nietzche was quoted as saying: "No one is gifted with immaculate perception". This has also been well noticed by the physics Nobel Laureate Feynman who once wrote: "When dealing with a mathematical model, special attention should be paid to imprecision in data".

Zadeh's incompatibility principle: "When the complexity of a system increases, our aptitude to formulate precise and meaningful statements decreases up to a threshold beyond which precision and significance becomes mutually exclusive characteristics" is also telling in this regard. It is of little relevance to inquire about the wrongfulness of a deterministic model. The real question is whether replacing stubbornly imprecise data by fixed ones in a model, does not impige on predictions concerning the phenomenon under investigation. A large amount of evidence [1], [2], [3] exists telling us not to bow to the Hammer principle (When you only have a hammer, you want everything at your hand to be a nail).

Indeed, replacing arbitrarily imprecise data by fixed values in a model would according to the well-known "Garbage in, garbage out" rule, leave no other chance to the model but to churn out meaningless outcomes.

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False certainty is bad science and it can be dangerous if it stunts articulation of critical choices.

Although probability theory claims to model decision making under imprecision [4], there are qualitatively different facets of undeterminacy which are not covered by probabilistic tools.

Situations where doubt arises about the exactness of concepts, correctness of statements and judgements have little to do with occurrence of events, the back-bone of probability Theory. This insight has lead researchers to embark upon the investigation of how to incorporate non-stochastic imprecision into mathematical models. Fuzzy sets theory [4,5], belief theory and, evidence theory [6,7,8] are along this line. The aim of this paper is twofold. First, it takes a general look at core ideas aimed at softening mathematical programming models by making it possible to incorporate fuzzy relationships and fuzzy parameters into these models.

Second, it extends these ideas to situations where both fuzziness and randomness are under one roof in a mathematical programming setting. The remaining of this paper is organized as follows. For the self-containedness of the paper, we present basic notions of fuzzy sets theory along with properties of fuzzy random variables in the following section. Section 3 is devoted to mathematical programming with fuzzy relationships (flexible programming). Mathematical programming with fuzzy parameters are taken up in section 4. Extensions to situations where fuzziness and randomness are combined in the scope of a mathematical program are discussed in section 5. Our approach to this problem is in tune with the general scheme for solving fuzzy stochastic optimization problems [9]. It consists of first handling fuzziness through α-level decomposition and then dealing with randomness via a hybrid method based on Monte-Carlo simulation and semi-infinite mathematical programming techniques.

The paper ends with some concluding remarks along with lines for further developments in the field of fuzzy mathematical programming.

2 Basic notions on Fuzzy sets theory and Fuzzy random variables

2.1 Fuzzy set

The main idea behind a fuzzy set is that of gradual membership to a set without sharp boundary. This idea is in tune with human representation of reality that is more nuanced than clear-cut. Some philosophical-related issues ranging from ontological level to application level via epistemological level may be found elsewhere [10].

In a fuzzy set, the membership degree of an element is expressed by any real number from 0 to 1 rather than the limiting extremes.

More formally, a fuzzy set of a set $A \neq \phi$ is characterized by a membership function $\mu : A \rightarrow [0, 1]$. In what follows a fuzzy set will be identified with its membership function. Moreover, for our purposes, we restrict ourselves to fuzzy sets of the real line $\mathbb{R}$.

2.2 Main notions and operations for fuzzy sets of $\mathbb{R}$

- The support of a fuzzy set μ is the crisp set $\text{supp}(\mu) = \{x \in \mathbb{R} | \mu(x) > 0\}$.
- The kernel of a fuzzy set μ is the crisp set $\text{Ker}(\mu) = \{x \in \mathbb{R} | \mu(x) = 1\}$.
- A fuzzy set μ is said to be normal if $\text{Ker}(\mu) \neq \phi$.
- The α-cut or α-level set of a fuzzy set μ is the crisp set $\mu^\alpha = \{x \in \mathbb{R} | \mu(x) \geq \alpha\}$.
- A fuzzy set μ is said to be convex if $\mu(x)$ is a quasi-concave function.

- A fuzzy number is a normal and convex fuzzy set of $\mathbb{R}$. A fuzzy number is well suited for representing vague data [11].

  For instance the vague datum: "close to five" can be represented by the fuzzy number μ as in Fig 1.

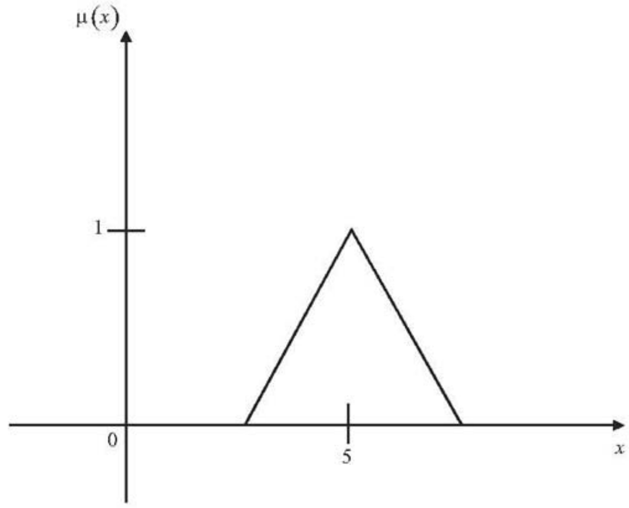

Figure 1: Membership function of the vague data: "close to 5"

- Let μ be a normal fuzzy set of $\mathbb{R}$. A family of subsets of $\mathbb{R}\,\{\vartheta^\alpha | \alpha \in (0,1)\}$ is called a set representation of μ, if and only if:

  (i) $0 < \alpha \leq \beta < 1 \Rightarrow \vartheta^\beta \subseteq \vartheta^\alpha$

  (ii) $\forall t \in \mathbb{R}\ ,\ \mu(t) = \sup\{\alpha I_{\vartheta^\alpha} | \alpha \in (0,1)\}$

  where I_P stands for the characteristic function of P, i.e.

$$I_P(x) = \begin{cases} 1 & \text{if } x \in P \\ 0 & \text{otherwise.} \end{cases}$$

- The following results, the proofs of which may be found elsewhere [12], bridge a gap between a fuzzy number and its set representation.

Theorem 1. Let μ be a fuzzy number. Then $\{\mu^\alpha | \alpha \in [0,1]\}$ is a set representation of μ.

Theorem 2. Let μ be a fuzzy number and $\{\vartheta^\alpha | \alpha \in (0,1)\}$ be its set representation. Then, we have for all

$$\alpha \in (0,1),\ \lim_{r\to\infty} \inf \vartheta_{\left(\alpha + \frac{1}{(2r)(1-\alpha)}\right)} = \inf \mu^\alpha$$

and

$$\lim_{r\to\infty} \sup \vartheta_{\left(\alpha + \frac{1}{(2r)(1-\alpha)}\right)} = \sup \mu^\alpha.$$

- Consider two fuzzy sets of $\mathbb{R}$, μ_1 and μ_2.
  - The complement of μ_1 is defined as $\bar{\mu}_1$ where $\bar{\mu_1}(x) = 1 - \mu_1(x)$.

- The union of μ_1 and μ_2 is defined as $\mu_1 \vee \mu_2$ where $\mu_1 \vee \mu_2(x) = \max(\mu_1(x), \mu_2(x))$.
- The intersection of μ_1 and μ_2 is defined as $\mu_1 \wedge \mu_2$ where $\mu_1 \wedge \mu_2(x) = \min(\mu_1(x), \mu_2(x))$.

2.3 Possibility, Necessity and Credibility Measures

Let Θ be a nonempty set representing the sample space. A possibility measure is a function

$$\text{Pos} : 2^{\Theta} \to [0,1]$$

satisfying the following axioms:

(i) $\text{Pos}\{\Theta\} = 1$,

(ii) $\text{Pos}\{\phi\} = 0$,

(iii) $\text{Pos}\{\bigcup_i A_i\} = \sup_i \text{Pos}\{A_i\}$,

(iv) Let $\{\Theta_k\}_k$ be a family of sets and $\text{Pos}_k : 2^{\Theta k} \to [0,1]$ verify (i) – (iii) and $\Theta = \Theta_1 \times \Theta_2 \ldots \times \Theta_n$. Then for $A \subset \Theta, \text{Pos}\{A\} = \sup_{(\Theta_1,\ldots,\Theta_n)\in A} \min_{1\le k\le n} \text{Pos}_k\{\Theta_k\}$.

In that case we write:

$$\text{Pos} = \text{Pos}_1 \wedge \text{Pos}_2 \wedge \ldots \wedge \text{Pos}_n.$$

Necessity and Credibility measure are obtained from Possibility measure as follows:

$$\text{Nec}\{A\} = 1 - \text{Pos}(A^C)$$

and

$$\text{Cr}\{A\} = \frac{\text{Pos}\{A\} + \text{Nec}\{A\}}{2}$$

where A^c is the complement of A. Details on Possibility, Necessity and Credibility measures may be found elsewhere [13].

2.4 Fuzzy random variable

A fuzzy random variable (frv) on a probability space $(\Omega, \Im, P)$ is a fuzzy-valued function

$$X : \quad \Omega \to \Im(\mathbb{R})$$
$$\omega \to X(\omega)$$

such that for every Borel set B of $\mathbb{R}$ and every $\alpha \in [0,1] : (X^{\alpha})^{-1}(B) \in \Im(\mathbb{R})$ where $\Im(\mathbb{R})$ is the set of fuzzy numbers and X^{α} stands for the set-valued function:

$$X^{\alpha} : \Omega \to 2^{\mathbb{R}}$$

where

$$X^{\alpha}(\omega) = X^{\alpha}_{\omega} = \{x \in \mathbb{R} | X_{\omega}(x) \ge \alpha\}.$$

Saying the above informally, a frv is an appropriate model for rules converting experimental outcomes into fuzzy numbers. Taken literally, this definition accounts adequately for both randomness and fuzziness.

The following result, the proof of which may be found in [14] will be used in the sequel.

Theorem 3 X is a fuzzy random variable if and only if $\forall \alpha \in [0,1], X^{\alpha}_w$ is a random interval i.e.,

$$\forall w \in \Omega,\ X^{\alpha}_w = \left\{ Y \mid Y \text{ is a random variable and } X^{\alpha^-} \le Y \le X^{\alpha^+} \right\}$$

where $X^{\alpha-}$ and $X^{\alpha+}$ are random variables. An interested reader is referred to [14] for properties of frvs.

3 Flexible mathematical programming

A flexible mathematical program is a problem of the form:

$$(1)\begin{cases} \widetilde{\min}\ f(x) \\ \quad g_i(x) \lesssim b_i; \quad i = 1, \dots, m \\ \quad x \in X = \{x \in \mathbb{R}^n | x \geq 0\} \end{cases}$$

where " $\sim$ " means that some leeway may be accepted in the objective and the constraints satisfaction. Such a flexible version of a standard mathematical program may be required when specifying strict satisfaction of constraints leads to inconsistencies which result in the vacuousness of the feasible set. A flexible mathematical program may also be of great help in situations of the more-or-less type.

An interpretation which can go with (P_1) is as follows: Find $x \in X$ such that $f(x)$ be as well as possible below a reasonable level b_0 and such that the constraints $g_i(x) \leq b_i,\ i = 1, \dots, m$ are met as well as possible. Or merely, find $x \in X$ such that:

$$g_i(x) \lesssim b_i; \quad i = 0, 1, \dots, m$$

where $g_0(x) = f(x)$.

A convenient way to represent these soft constraints is through appropriate fuzzy sets of $\mathbb{R}$ the membership functions of which are μ_i; $i = 0, 1, \dots, m$ defined as follows:

$$\begin{array}{lcl} \mu_i(x) & = & 0 \qquad \text{if } g_i(x) > b_i + d_i \\ \mu_i(x) & \in & (0,1) \text{ if } b_i < g_i(x) \leq b_i + d_i \\ \mu_i(x) & = & 1 \qquad \text{if } g_i(x) \leq b_i \end{array}$$

where $d_i (i = 1, \dots, m)$ are subjectively chosen constants for admissible violation. In other terms, $\mu_i(x)$ is equal to 1 if there is no violation in the constraint $g_i(x) \leq b_i$. $\mu_i(x) \in (0,1)$ if the violation in the constraint $g_i(x) \leq b_i$ can be tolerated because its magnitude is less than a reasonable threshold d_i. And $\mu_i(x) = 0$, if the violation in the constraint $g_i(x) \leq b_i$ cannot be accepted. The following simple kind of piecewise function may be used for $\mu_i(x)$.

$$\mu_i(x) = \begin{cases} 1; & \text{if } g_i(x) \leq b_i \\ 1 - \frac{g_i(x) - b_i}{d_i}; & \text{if } b_i < g_i(x) \leq b_i + d_i \\ 0; & \text{if } g_i(x) > b_i + d_i. \end{cases}$$

Decision in a fuzzy environment is given as an option that simultaneously fulfills the goal and the constraints of the problem (see Bellman-Zadeh's confluence principle [15]). Therefore the optimal decision to our problem should be $x^* \in X$ that has the highest membership degree in the fuzzy set intersection of fuzzy sets representing the objective function and the constraints, i.e. x^* should maximize $\mu_D(x) = \min_i \mu_i(x)$.

The resulting problem is then:

$$(P_1)' = \begin{cases} \max \min \mu_i(x) \\ \quad x \in \bigcap_{i=0}^m \text{Supp } \mu_i. \end{cases}$$

This problem is equivalent to the following mathematical program:

$$(P_1)' = \begin{cases} \max \lambda \\ \quad \lambda \leq 1 - \frac{g_i(x) - b_i}{d_i}; \quad i = 0, 1, \dots, m \\ \quad x \geq 0. \end{cases}$$

The last program can be solved by existing mathematical programming software like LINDO, LINGO, CPLEX, XPRESS. It is worthmentioning that using min to translate the semantic meaning of the connective "and" may be a too pessimistic approach. In decision problems where such a ultra pessimistic attitude cannot be accepted, a compensatory operator can be used. For instance μ_D can be defined as follows:

$$\mu_D(x) = \gamma \min_i \mu_i(x) + (1-\gamma) \min\left(1, \sum_i \mu_i(x)\right)$$

[16]. Where $\delta \in (0,1)$ is a coefficient of compensation. Moreover other membership functions more tuned to the situation at hand may be used instead of the piecewise functions defined here. See for instance the paper by Leberling [17] where hyperbolic membership functions are considered.

3.1 Mathematical programming with fuzzy parameters

3.1.1 Problem formulation

A common paradigm in application of mathematical programming models is that all involved parameters are fixed known data. In many practical situations such an assumption turns out to be unreliable. Consider for example, the case where second members of a mathematical program are demands which are given in the form of fuzzy numbers.

In this section we describe models for addressing the presence of fuzzy data in mathematical programming problems. Assume that a mathematical program is given in the following form

$$(P_2) = \begin{cases} \max f(x, \tilde{a}) \\ \quad g_i\left(x, \tilde{b}_i\right) \leq \tilde{c}_i;\ i = 1, \ldots, m \\ \quad x \in R_n \end{cases}$$

where $\tilde{a}$ is a fuzzy vector and $\tilde{b_i}(i = 1, m)$, $\tilde{c_i}(i = 1, \ldots, m)$ are fuzzy numbers. Owing to the presence of fuzzy quantities, (P_2) is an ill-defined problem and the notion of "optimum optimorum" does not apply.

Putting (P_2) in deterministic terms, in a way not to badly caricature the original problem, may proceed along the following lines.

3.1.2 Solving (P_2) by deffuzification

Here a deterministic version of (P_2) is obtained by replacing involved fuzzy quantities by appropriate real values in their respective supports. The most frequently used values [18] are either α-level sets or kernels of fuzzy quantities under consideration. The resulting problem is a standard mathematical program and existing mathematical programming software listed in §3.1 apply. In the above described approach, the possibility distributions of fuzzy quantities under consideration are not taken into account. Therefore such an approach is not in tune with the minimum uncertainty principle that tells us not to ignore available knowledge in solving a mathematical problem under uncertainty. The above deffuzification approach can be of use only when supports of involved fuzzy quantities are not too large.

3.1.3 Uncertainty-constrained approach for (P_2)

Here the deterministic counterpart of (P_2) is obtained through uncertainty measures. An approach which is reminiscent to the stochastic chance-constrained programming approach is as follows. Consider as a deterministic version of (P_2) the following mathematical program:

$$(P_2) = \begin{cases} \max_x \max \lambda \\ \quad F_M \{f(x, \tilde{a}) \geq \lambda\} \geq \alpha \\ \quad F_M \left\{ g_i(x, \tilde{b}_i) \leq \tilde{c}_i;\ i = 1, \ldots, m \right\} \geq \beta \\ \quad x \in R^n \end{cases}$$

where α, β are fixed thresholds and F_M is some uncertainty measure, like possibility, necessity or credibility. A hybrid intelligent algorithm [19] for the case where F_M is the credibility measure is as follows:

Step 1. Generate training data for uncertain function approximation by fuzzy random simulations.

Step 2. Train a neural network to approximate the uncertain functions according to the generated training data.

Step 3. Initialize pop-size chromosomes in which the trained neural network may be used to check the feasibility.

Step 4. Update the chromosomes by crossover and mutation operations in which the feasibility of offspring may be checked by the trained neural network.

Step 5. Calculate the objective values of all chromosomes by the trained neural network.

Step 6. Compute the fitness of each chromosome according to the objective values.

Step 7. Select the chromosomes by spinning the roulette wheel.

Step 8. Repeat the fourth to seventh steps for a given number of cycles.

Step 9. Report the best chromosomes as the satisfactory solution.

4 Extension to situations where fuzziness and randomness co-occur in a mathematical programming setting

4.1 Problem formulation

In this section, we restrict ourselves to the linear programming case. Extension to nonlinear case is straightforward inasmuch functions defining the feasible set are monotonous. Consider the optimization problem:

$$(P_3) \begin{cases} \min \tilde{c}x \\ \quad \sum_{j=1}^{n} \tilde{a}_{ij}\ x_j \leq \tilde{b}_i \\ \quad x_j \geq 0;\ j = 1, n \end{cases}$$

where $\tilde{c}_j$, $\tilde{a}_{ij}$, $\tilde{b}_i$ are fuzzy random variables on $(\Omega, \Im, P)$ (see §2.4).

Problems of this type arise naturally in concrete situations [21], when a decision maker needs to couple subjective perceptions with hard statistical data.

Using the more possible value (V_p) as a summarizing functional of the objective function, (P_3) takes the form:

$$(P_3')\begin{cases}\min V_p(\tilde{c})x\\ \quad\sum_{j=1}^{n}\tilde{a}_{ij}\ x_j\le\tilde{b}_i\\ \quad x_j\ge 0;\ j=1,\dots,n\end{cases}$$

where $V_p(\tilde{c})=(V_p(\tilde{c}_1)\dots,V_p(\tilde{c}_n))$ and $V_p(\tilde{c}_j)=\arg\max\tilde{c}_{jw}(t)$.

4.2 Solving (P_3')

To solve (P_3') we interpret inequality between two fuzzy random variables of $\mathbb{R}$ as follows:

$$\tilde{m}\le\tilde{n}\ \text{iff}\ \tilde{m}^\alpha\le\tilde{n}^\alpha\ \forall\alpha\in(0,1]. \qquad (1)$$

In this case $\tilde{m}^\alpha$ and $\tilde{n}^\alpha$ are intervals denoted by $[\underline{m}^\alpha,\ \bar{m}^\alpha]$ and $[\underline{n}^\alpha,\ \bar{n}^\alpha]$, respectively. By Moore's results on interval arithmetic, (1) can read:

$$\tilde{m}\le\tilde{n}\Leftrightarrow\bar{m}^\alpha\le\underline{n}^\alpha,\ \forall\alpha\in(0,1]. \qquad (2)$$

By virtue of (1), (P_3') takes the form:

$$(P_3'')\begin{cases}\min V_p(\tilde{c})x\\ \quad\sum_{j=1}^{n}\tilde{a}_{ij}^\alpha\ x_j\le\tilde{b}_i^\alpha,\ \forall\alpha\in(0,1]\\ \quad x_j\ge 0;\ j=1,\dots,n.\end{cases}$$

Moreover by Theorem 3, we have that $\tilde{a}_{ij}^\alpha=[\underline{a}_{ij}^\alpha,\ \bar{a}_{ij}^\alpha]$ and $\tilde{b}_i^\alpha=[\underline{b}_i^\alpha,\ \bar{b}_i^\alpha]$ where $\underline{a}_{ij}^\alpha$, $\bar{a}_{ij}^\alpha$, $\underline{b}_i^\alpha$, $\bar{b}_i^\alpha$ are random variables.

By (2), (P_3'') can be written as follows:

$$(P_3''')\begin{cases}\min V_p(\tilde{c})x\\ \quad\sum_{j=1}^{n}\bar{a}_{ij}^\alpha\ x_j\le\underline{b}_i^\alpha;i=1,\dots,m,\ \forall\alpha\in(0,1]\\ \quad x_j\ge 0.\end{cases}$$

(P_3''') is a semi-infinite mathematical program with stochastic coefficients. We now describe a procedure based on Monte-Carlo simulation for solving this mathematical program.

4.3 A procedure for solving (P_3''')

4.3.1 Reformulation of (P_3''')

For the sake of clarity (P_3''') can be written:

$$(P_4)\begin{cases}\min F(x,\ \zeta)\\ \quad G_\alpha(x,\eta)\le 0,\ \alpha\in(0,1]\\ \quad x_j\ge 0\end{cases}$$

where

$$\begin{aligned}F(x,\zeta)&=V_p(\tilde{c})x,\\ G_{j\alpha}(x,\eta)&=\sum\bar{a}_{ij}^\alpha x_j-\underline{b}_i^\alpha.\end{aligned}$$

In what follows $\Lambda=(\zeta,\ \eta)$ is a random vector on $(\Omega,\Im,P)$ and we use the following notations:

$$F(x,\Lambda)=F(x,\zeta),$$
$$G_{i\alpha}(x,\Lambda)=G_{i\alpha}(x,\eta).$$

4.3.2 Sample average approximate problem associated to (P_4)

Suppose that we can generate a sample of N replications of the random vector $\Lambda = (\zeta, \eta)$ and let $\Lambda^1, \Lambda^2, \ldots, \Lambda^N$ be a particular realization of the pertaining random sample.

A sample average approximate (SAA) problem associated to (P_4) is:

$$(P_4)_N \begin{cases} \min \hat{f}_N(x) \\ \quad x \in X_N \end{cases}$$

where

$$\begin{aligned} \hat{f}_N(x) &= \frac{1}{N}\sum_{j=1}^{N} F(x, \Lambda^j) \\ \hat{G}_{i\alpha}(x)_N &= \frac{1}{N}\sum_{j=1}^{N} G_{i\alpha}(x, \Lambda^j) \end{aligned}$$

and $T = [0, 1]$.

It can be shown [22] that under mild assumptions the optimal value of $(P_4)_N$ tends to the optimal value of (P_4) with probability 1.

4.3.3 Cutting-plane algorithm for solving $(P_4)_N$

Before describing the procedure (CPALGO) for solving $(P_4)_N$, we need the following notations.

- $(T^j)_j$ is a family of finite subsets of T such that:$s < r \Rightarrow T^s \subset T^r$.
- $X_N^j = \{x \in \mathbb{R}^n | G_{i\alpha N}(x) \leq 0;\ i = 1, m;\ \alpha \in T,\ x \geq 0\}$.
- $(P_4^j)_N$ stands for the following mathematical program:

$$\begin{cases} \min \hat{f}_N(x) \\ \quad x \in X_N^j. \end{cases}$$

- M is a large natural number.
- $L^j = \max_{t \in T^j} \hat{G}_{tN}(x^j)$.

The CPALGO procedure is as follows:

Step 1: Fix M, put $j = 1$.

Step 2: Choose $T^j \subset T$.

Step 3: Solve $(P_4^j)_N$ and denote its solution by x^j.

Step 4: Find L^j.

Step 5: If $L^j \leq 0$, put $x^j = x^*$. Otherwise go to step 6.

Step 6: Put $j = j + 1$.

Step 7: If $j \geq M$, go to step 9. Otherwise go to step 8.

Step 8: Take $r > j$ and put $j = r$. Go to Step 2.

Step 9: Find a limit point of $\{x_j\}_{j=1}^{M}$, let x^ℓ be this limit point. Put $x^* = x^\ell$.

Step 10: Print x^* is optimal solution of $(P_4)_N$.

Step 11: Stop.

A justification of the stopping criterion of this procedure is given by the following result [23].

Theorem 3. Assume $X_N \neq \phi$, X_N^1 is compact and $\hat{f}_N$ is continuous then the following statements hold true:

(i) If $L^j < 0$, then x^j is a solution of $(P_4)_N$.

(ii) Any limit point of $(x^j)_j$ generated by CPALGO is an optimal solution of $(P_4)_N$.

4.3.4 General procedure for (P_4)

We now describe a general procedure for solving (P_4). This procedure is a hybrid of sample average approximation technique and cutting-plane method. The detail of this procedure is given in Fig. 2.

4.3.5 Numerical example

Consider the linear program:

$$(P_5)\begin{cases}\min \sum_{j=1}^{6} \tilde{c}_j x_j \\ \text{subject to:} \\ \quad \sum_{j=1}^{6} \tilde{a}_{ij} x_j \geq \tilde{b}_i \\ \quad |x_i| \leq 10;\ i = 1,\ldots,6\end{cases}$$

where $\tilde{a}_{1j}$ and $\tilde{b}_1$ are fuzzy random variables where:

$$\underline{a}_{11}^{\alpha} = \alpha; \quad \underline{a}_{12}^{\alpha} = \zeta_2\alpha^2; \quad \underline{a}_{13}^{\alpha} = \zeta_3^{2}\alpha^3$$
$$\underline{a}_{14}^{\alpha} = \zeta_4^{3}\alpha^4, \quad \underline{a}_{15}^{\alpha} = (1-\zeta_5), \quad \underline{a}_{16}^{\alpha} = (1-\zeta_6)\alpha$$
$$\bar{b}_1^{\alpha} = 2e^{\alpha}; \quad V_p(\tilde{c}_1) = 1; \quad V_p(\tilde{c}_2) = \frac{1}{\zeta_2}$$
$$V_p(\tilde{c}_3) = \frac{1}{2\zeta_3}; \quad V_p(\tilde{c}_4) = \frac{1}{3\zeta_4}; \quad V_p(\tilde{c}_5) = \frac{1}{4\zeta_5}; \quad V_p(\tilde{c}_6) = \frac{1}{5\zeta_6}$$

and $\zeta_2, \zeta_3, \ldots, \zeta_6$ are random variables whose distributions are as follows:

$$\zeta_2 \rightsquigarrow \bigcup(3,4);\ \zeta_3 \rightsquigarrow N(5,1),\ \zeta_4 \rightsquigarrow \text{Exp}(6),$$
$$\zeta_5 \rightsquigarrow N(4,1);\ \zeta_6 \rightsquigarrow \text{Exp}(5).$$

Using the general procedure described in §4.3.4 with $\varepsilon = 0.5$ we obtain the following results

N	Optimal solution x^*_N	Optimal value of the objective function ν^*_N
1 000	(-9.89,-1.42,6.47,0.24,-5.30,9.75)	-41.38
5 000	(-9.74,-7.54,-2.93,0.71,8.87,-9.17)	-36.67
9 000	(-9.95,-8.28,-1.31,0.38,1.66,-9,18)	-22
13 000	(-9.99,-8.28,-1.56,0.58,0.07,-9.84)	-22.47

As $0.47 < 0.5, (-9.99, -8.28, -1.56, 0.58, 0.07, -9.84)$ is considered as the optimal solution of (P_5).

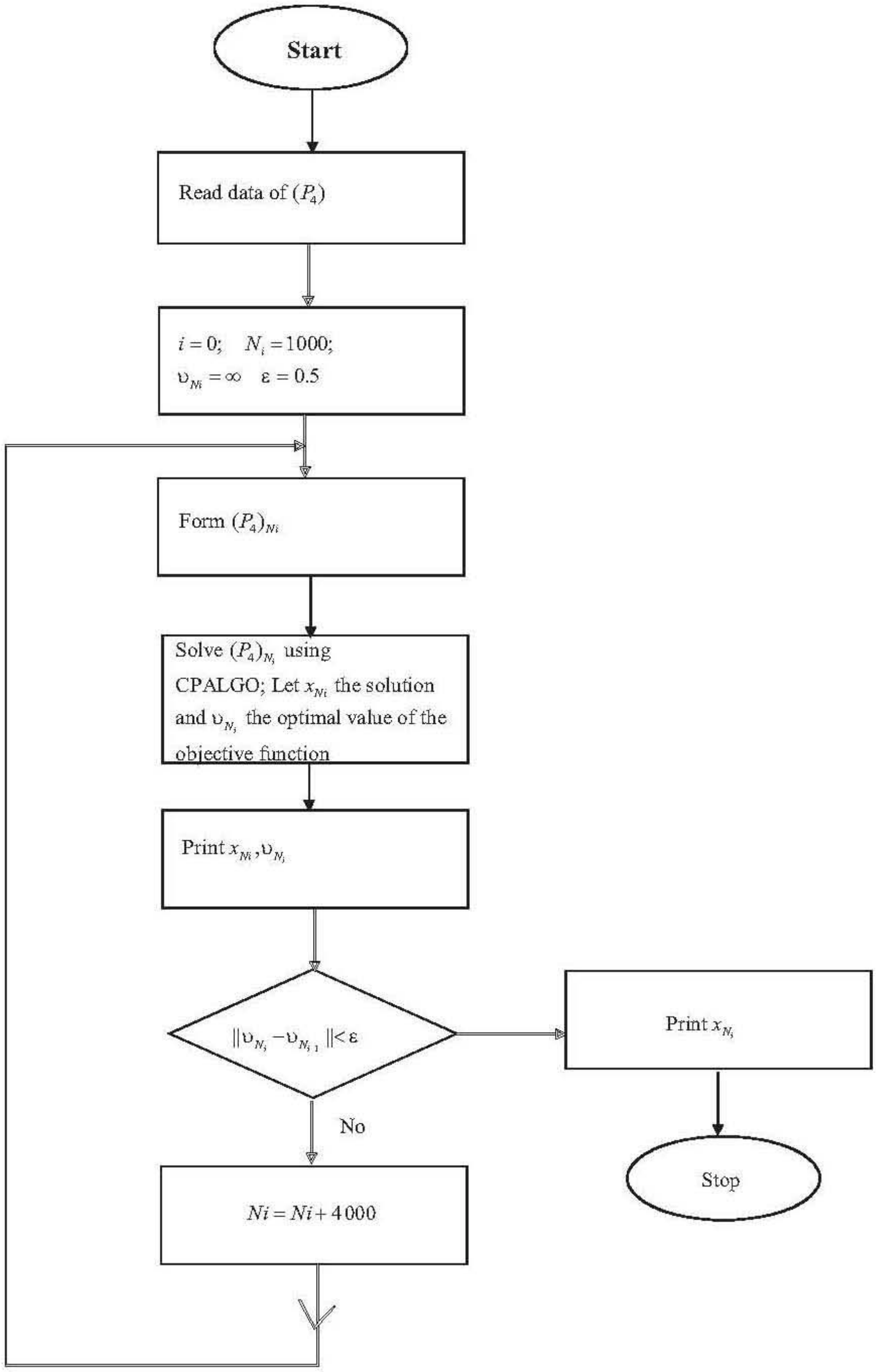

Figure 2: General procedure for solving (P_4)

5 Concluding remarks

The methods and models which have been often used in Operations have been primarily hard or crisp, i.e. the solutions were considered to be either feasible or unfeasible, either above a certain aspiration level or below. This dichotomous structure of methods very often force the modeller to approximate real problem situations of the more-or-less type by yes-or-no type models, the solutions of which might turn out not to be the solutions to the real problems.

In this paper we have discussed how Fuzzy sets theory may be of great help while handling situations where an optimization problem includes vaguely defined relationships or imprecise parameters due to subjective human evaluation or to inconsistent or incomplete evidence. The framework has then be extended to hybrid situations where fuzziness and randomness co-occur in an optimization setting. Fuzzy mathematical programming has found numerous applications in, e.g. media selection in advertising [24], air pollution regulation [25], water resource management [26], portfolio selection [27]. Among lines for further developments we may mention the following.

- A deep understanding of the following questions in a way to deal efficiently and effectively with Fuzzy Stochastic Optimization problems.
  - How should we compare fuzzy random or random fuzzy variables?
  - How should we define preference ordering between fuzzy random variables?
  - How should we interpret and deal with inequality relations involving fuzzy random variables or random fuzzy variables?
- Full implementation of the procedure described in §4.3.

Let us hope that successful developments in the above mentioned directions will proceed in the near future, thus bridging the gap between the language used for fuzzy stochastic optimization techniques and the language used by potential users of these techniques.

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