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Credibilistic Game with Fuzzy Information*

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Abstract

Harsanyi's work on games with incomplete information is by use of probabilistic approach and constitutes one of the most important developments in game theory. This paper presents a spectrum of credibilistic game, in which the incomplete information is interpreted as fuzzy variables and processed via credibilistic approach. As the leadoff, a strategic game with fuzzy payoffs is discussed by presenting three types of credibilistic equilibria as well as their existence theorems.
Keywords: Game; fuzzy variable; credibility measure; credibilistic equilibrium

1 Introduction

As a study of mathematical models of conflict and cooperation between intelligent rational decision-makers, game theory may be said to begin in the 1920s with some preliminary works (e.g., Borel [6][7], von Neumann [36]), and to be established as a systematic theory in 1944 due to the great seminal work [37] by von Neumann and Morgenstern. Thereafter, the power of game theory was gradually proved by a prolific development of important applications. The readers may consult the books [22][41][42] and the book series [44] for detailed exposition of game theory.

In many real-world economic, political and other social situations, the participants often lack full information about some important aspects of the game they are playing. As Harsanyi [21] pointed out, "the player may lack full information about the other players' (or even their own) payoffs, about the physical or social resources..." Traditionally, the incomplete information is modeled as random variables and processed by probabilistic methods. For instance, following Bayesian approach, Harsanyi [18]-[20], Aumann [2], and Myerson [33] developed Bayesian game, which constitutes one of the most important developments in game theory. By use of satisficing methods, Blau [5], Cassidy *et al* [12], Charnes *et al* [13] discussed the solution of games with random payoffs. Using methods from the statistical mechanics of disordered systems, some researchers analyzed the properties of bimatrix games with random payoffs (e.g., Berg [4], Ein-Dor and Kanter [15] and Roberts [43]).

However, many situations of interest contain few historical records available for probabilistic reasoning because of the complicated decision environments. In such situations, fuzzy set theory offers an appropriate and powerful alternative to deal with imperfect or incomplete information in games. With it, we can make use of human experiences, subjective judgements and intuitions, and specify the incomplete information as fuzzy variables, thus introducing an interesting topic: games with fuzzy information (e.g., fuzzy payoffs, fuzzy beliefs). Recently, games with fuzzy payoffs [1][8] have received an ever increasing attention. Many researchers defined and analyzed equilibrium solutions via some defuzzifying methods (e.g., possibility measure and necessity measure [30][31],

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membership degree [38]-[40]) and fuzzy relation [45]). Games with fuzzy parameters were also discussed by Gao and Liu [17].

Recently, credibility theory was founded and refined by Liu [24][25] as a branch of mathematics for dealing with the behavior of fuzzy phenomena. Credibility theory has received increasing attention and has been applied widely in the area of fuzzy optimization and decision-making [26]. In this paper, we introduce the credibilistic game with fuzzy information. That is, the incomplete information in games is modeled by fuzzy variables and processed by credibilistic approach. For example, when the players lack full information about the other players' (or even their own) payoff functions, payoffs may be specified as fuzzy variables according to an expert system. If the fuzzy payoffs are known to all players, then according to appropriate optimal criteria, the game can be solved by credibilistic approaches. Other cases of game with incomplete information can also be discussed with similar methods. As the leadoff, we only consider a strategic game with fuzzy payoffs. By appropriate decision criteria in credibility theory, we define three types of equilibria called *credibilistic equilibria*. We also give the existence theorems of the proposed credibilistic equilibria.

This paper is arranged as follows. In Section 2, we recall some basic results of credibility theory with some credibilistic approaches. Then in Section 3, we discuss a strategic game with fuzzy payoffs by presenting three credibilistic equilibria as well as their existence theorems. Lastly, we give a concluding remark.

2 Preliminaries

Liu and Liu [27] proposed the concept of credibility measure. Then credibility theory was founded and refined by Liu [24][25] as a mathematical framework for dealing with fuzzy phenomena. Let ξ be a fuzzy variable with membership function μ, and B a set of real numbers. The credibility measure Cr is defined by

$$\mathrm{Cr}\{\xi \in B\} = \frac{1}{2}\left(\sup_{x\in B}\mu(x) + 1 - \sup_{x\in B^c}\mu(x)\right). \tag{1}$$

The above formula is also called credibility inversion theorem.

Definition 2.1 (Liu and Gao [29]) *The fuzzy variables $\xi_1, \xi_2, \cdots, \xi_m$ are said to be independent if*

$$\mathrm{Cr}\left\{\bigcap_{i=1}^{m}\{\xi_i \in B_i\}\right\} = \min_{1\le i\le m}\mathrm{Cr}\{\xi_i \in B_i\} \tag{2}$$

for any sets $B_1, B_2, \cdots, B_m$ of real numbers.

Definition 2.2 (Liu and Liu [27]) *Let ξ be a fuzzy variable. The expected value of ξ is defined as*

$$\mathrm{E}[\xi] = \int_0^{\infty}\mathrm{Cr}\{\xi \ge r\}\mathrm{d}r - \int_{-\infty}^{0}\mathrm{Cr}\{\xi \le r\}\mathrm{d}r, \tag{3}$$

provided that at least one of the two integrals is finite.

Definition 2.3 (Liu and Liu [28]) *Let ξ be a fuzzy variable, and $\alpha \in (0,1]$. Then*

$$\xi_{\sup}(\alpha) = \sup\{r \mid \mathrm{Cr}\{\xi \ge r\} \ge \alpha\} \tag{4}$$

is called the critical value at confidence level α.

Let ξ and η be independent fuzzy variables. Then for any real numbers a, b and $\alpha \in (0,1]$, we have $\mathrm{E}[a\xi + b\eta] = a\mathrm{E}[\xi] + b\mathrm{E}[\eta]$ and $(a\xi + b\eta)_{\sup}(\alpha) = a\xi_{\sup}(\alpha) + b\eta_{\sup}(\alpha)$. That is, both the expected value operator and the critical value operator have linear property under assumption of independence. Let r be a real number. Based on the credibility theory, we may have three fuzzy ranking criteria as follows:

1. Expected value criterion: $\xi < \eta$ iff $E[\xi] < E[\eta]$;
2. Critical value criterion: $\xi < \eta$ iff $\xi_{\sup}(\alpha) < \eta_{\sup}(\alpha)$ for some predetermined confidence level $\alpha \in (0,1]$;
3. Credibility criterion: $\xi < \eta$ iff $\mathrm{Cr}\{\xi \geq r\} < \mathrm{Cr}\{\eta \geq r\}$ for some predetermined level r.

A fuzzy variable ξ is bounded if there exist two numbers a and b such that $\mathrm{Cr}\{a \leq \xi \leq b\} = 1$. A fuzzy variable is called continuous if its membership function is continuous. In this paper, we assume all the fuzzy variables involved are bounded and continuous.

3 Credibilistic Game

Games may fall into three forms when considering the interaction of players: strategic form, extensive form and coalitional form. Strategic form specifies all possible strategies of each player, along with the payoffs that result from the strategy choices of players. Its difference from extensive form is that all the players' decisions are made simultaneously and once for all. While the extensive form details not only the information available to and motivation of players, but also the various stages of the interaction and conditions for a player to move. Lastly, coalitional game focuses the social interactions in which players can make agreements with each other.

In this section, we consider a two-player zero-sum strategic game with fuzzy information. In this game, two players compete by choosing *mixed strategies* over their own *pure strategy sets* and get their fuzzy *payoffs* with zero sum. That is, the two players are in pure opposition to each other. Let $M = \{1, 2, \cdots, m\}$ be the pure strategy set of one player I, and let $N = \{1, 2, \cdots, n\}$ be the pure strategy set of the other player J. Denote the sets of all mixed strategies available for players I and J by

$$S_I = \left\{ (x_1, x_2, \cdots, x_m)^T \in \Re_+^m \mid \sum_{i=1}^{m} x_i = 1 \right\},$$

$$S_J = \left\{ (y_1, y_2, \cdots, y_n)^T \in \Re_+^m \mid \sum_{j=1}^{n} y_j = 1 \right\}.$$

Then a mixed strategic game with fuzzy payoffs is any Γ of the form

$$\Gamma = \langle \{I, J\}, S_I \times S_J, \tilde{A} \rangle$$

where $\tilde{A} = (\xi_{ij})_{m \times n}$ is the fuzzy payoff matrix.

Traditionally, $\tilde{A}$ is a payoff matrix with crisp entries. If players I and J have choosed one mixed strategy from their own strategy set, say $\boldsymbol{x}$ and $\boldsymbol{y}$, respectively, then the strategy profile $(\boldsymbol{x}, \boldsymbol{y})$ determines the expected payoff $\boldsymbol{x}^T \tilde{A} \boldsymbol{y}$ of player I, and $-\boldsymbol{x}^T A \boldsymbol{y}$ of player J [37]. But $\tilde{A}$ is a fuzzy payoff matrix, which resulting that the expected payoff $\boldsymbol{x}^T \tilde{A} \boldsymbol{y}$ is also a fuzzy variable. Then for a given strategy $\boldsymbol{y}$ of player J, how does player I evaluate his strategy $\boldsymbol{x}$. That is, what is the optimal decision criteria of the players. In the following, we employ three decision criteria from credibility theory to give fuzzy models for players competing in the fuzzy-payoff games. Then based on them, we present three credibilistic equilibria for different decision situations. Moreover, we give their existence theorems.

First, assume that the players' optimal decision criteria are to maximize the expected value of their fuzzy payoffs. Then a maxminimizer of player I is the solution of the fuzzy expected value model:

$$\max_{\boldsymbol{x}\in S_I}\min_{\boldsymbol{y}\in S_J} \mathrm{E}\left[\boldsymbol{x}^T\tilde{A}\boldsymbol{y}\right], \tag{5}$$

and a maxminimizer of player J is the solution of the fuzzy expected value model:

$$\min_{\boldsymbol{y}\in S_J}\max_{\boldsymbol{x}\in S_I} \mathrm{E}\left[\boldsymbol{x}^T\tilde{A}\boldsymbol{y}\right]. \tag{6}$$

Definition 3.1 *An array $(\boldsymbol{x}^*,\boldsymbol{y}^*)$ is called an expected credibilistic equilibrium if it satisfies*

$$\mathrm{E}\left[\boldsymbol{x}^T\tilde{A}\boldsymbol{y}^*\right] \le \mathrm{E}\left[\boldsymbol{x}^{*T}\tilde{B}\boldsymbol{y}^*\right] \le \mathrm{E}\left[\boldsymbol{x}^{*T}\tilde{A}\boldsymbol{y}\right].$$

Second, assume that player I optimal decision criteria is to maximize the critical value of his fuzzy payoff $\boldsymbol{x}^T\tilde{A}\boldsymbol{y}$ at given confidence level α. Then a maxminimizer of player I is the solution of the fuzzy chance-constrained programming model:

$$\max_{\boldsymbol{x}\in S_I}\min_{\boldsymbol{y}\in S_J}\max_{u} \mathrm{Cr}\left\{\boldsymbol{x}^T\tilde{A}\boldsymbol{y}\ge u\right\}\ge\alpha. \tag{7}$$

And a maxminimizer of player J is the solution of the fuzzy chance-constrained programming model:

$$\min_{\boldsymbol{y}\in S_J}\max_{\boldsymbol{x}\in S_I}\max_{u} \mathrm{Cr}\left\{\boldsymbol{x}^T\tilde{A}\boldsymbol{y}\ge u\right\}\ge\alpha. \tag{8}$$

Definition 3.2 *An array $(\boldsymbol{x}^*,\boldsymbol{y}^*)$ is called an α-credibilistic equilibrium, if it satisfies*

$$\begin{aligned}&\max\left\{u\mid \mathrm{Cr}\left\{\boldsymbol{x}^T\tilde{A}\boldsymbol{y}^*\ge u\right\}\ge\alpha\right\}\\ &\quad\le \max\left\{u\mid \mathrm{Cr}\left\{\boldsymbol{x}^{*T}\tilde{A}\boldsymbol{y}^*\ge u\right\}\ge\alpha\right\}\\ &\qquad\le \max\left\{u\mid \mathrm{Cr}\left\{\boldsymbol{x}^{*T}\tilde{A}\boldsymbol{y}\ge u\right\}\ge\alpha\right\}.\end{aligned}$$

Thirdly, we assume that the player I has specified a prospective payoff level u, and wants to maximize the credibility of his fuzzy total payoff's being greater than u. Then a maxminimizer of player I is the solution of the following fuzzy dependent-chance programming model:

$$\max_{\boldsymbol{x}\in S_I}\min_{\boldsymbol{y}\in S_J} \mathrm{Cr}\left\{\boldsymbol{x}^T\tilde{A}\boldsymbol{y}\ge u\right\}. \tag{9}$$

And a maxminimizer of player J is the solution of the following fuzzy dependent-chance programming model:

$$\min_{\boldsymbol{y}\in S_J}\max_{\boldsymbol{x}\in S_I} \mathrm{Cr}\left\{\boldsymbol{x}^T\tilde{B}\boldsymbol{y}\ge u\right\}. \tag{10}$$

Definition 3.3 *An array $(\boldsymbol{x}^*,\boldsymbol{y}^*)$ is called a most credibilistic equilibrium, if it satisfies*

$$\mathrm{Cr}\left\{\boldsymbol{x}^T\tilde{A}\boldsymbol{y}^*\ge u\right\}\le \mathrm{Cr}\left\{\boldsymbol{x}^{*T}\tilde{A}\boldsymbol{y}^*\ge u\right\}\le \mathrm{Cr}\left\{\boldsymbol{x}^{*T}\tilde{B}\boldsymbol{y}\ge u\right\}.$$

Once the equilibria are defined, one may be concerned with their existence in the fuzzy-payoff game. The following theorems will give answers.

Theorem 3.1 *Let all entries* ξ_{ij} *in the payoff matrix be independent fuzzy variables. Then there exists at least one expected credibilistic equilibrium in the game* $\Gamma = \langle \{I, J\}, S_I \times S_J, \tilde{A} \rangle$.

Theorem 3.2 *Let all entries* ξ_{ij} *in the payoff matrix be independent fuzzy variables, and* $\alpha \in (0, 1]$. *Then there exists at least one* α*-credibilistic equilibrium in the game* $\Gamma = \langle \{I, J\}, S_I \times S_J, \tilde{A} \rangle$.

Remark 3.1 *It follows from the linearity of the expected value and critical value operator that the proof of the first two existence theorems can be converted to that of the classical two-player zero-sum game [34]. We omit them here.*

Theorem 3.3 *Let all entries* ξ_{ij} *in the payoff matrix be independent fuzzy variables and* u *be a real number. Then there exists at least one most credibilistic equilibrium in the game* $\Gamma = \langle \{I, J\}, S_I \times S_J, \tilde{A} \rangle$.

Remark 3.2 *The proof of the third existence theorem is similar to that of the classical one [34]. We omit it here.*

Conclusion

In this paper, we proposed a spectrum of credibilistic game. That is, the incomplete information in games was interpreted as fuzzy variables and processed via credibilistic approach. As an example, we discussed a fuzzy-payoff zero-sum game by defining three types of credibilistic equilibria and presenting their existence theorems. However, this is only a leadoff of credibilistic game with fuzzy information. Further works may focus on the different types of incomplete information, and interpret them as fuzzy variables. Then, new results may be gotten by the credibilistic approach.

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