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On a Continuous Lattice Approach to Modeling of Coarse Data in System Analysis

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Abstract

We emphasize the need to model rigorously uncertain coarse data in systems such as decision-support systems and intelligent machines. We exemplify the framework of continuous lattices by formulating human perception-based information as coarsening schemes using fuzzy partitions. Specifically, the formal concept of random fuzzy sets is treated as random elements with values in the space of upper semicontinuous functions which is endowed with a continuous lattice structure. As a result, random fuzzy (closed) sets on locally compact, Hausdorfff and second countable spaces are special random elements taking values in separable metric spaces.
Keywords: Choquet theorem, coarse data , continuous lattices, random sets, random fuzzy sets

1 Introduction

In building systems from empirical data we often face coarse data in various forms. For example, in biostatistics or medical statistics, data are censored, grouped or missing; in bioinformatics, data are not directly observable (e.g. in the problem of identification of DNA sequences where hidden Markov processes are usually used to model the observable data); in intelligent control, linguistic control rules are imprecise due to the fuzziness in our natural language; in social systems, information is uncertain due to randomness of occurences of events as well as to the fuzziness in the meaning representation of terms. The most general form of coarse data (i.e. data with low quality) seems to be random fuzzy sets.

In order to carry out inference procedures based upon random fuzzy sets it is necessary to define rigorously this concept. The theory of random closed sets has been rigorously defined by Matheron (1975) in which the counter-part of the Lebesgue-Stieltjes theorem is the Choquet theorem serving as a way to specify distribution laws, and hence suggesting models for random set observations. Extending ordinary (closed) sets to fuzzy sets via indicator functions of sets leads to fuzzy sets whose membership functions are upper semicontinuous. Thus, we are led to topologize the space of upper semicontinuous functions in order to define random elements with values in it. A familiar approach to topologize a functional space in probability theory is to search for some plausible metric on it. For example, the space $C[0,1]$ of sample paths of a Brownian motion is equiped with the supremun norm; the space $D[0,1]$ of functions that are right continuous with left limits of jump processes is equiped with the Skorohov' s metric (see e.g. Billingsley, 1968). Attempts to follow this approach when facing $USC(X)$, the space of real-valued (or with values in $[0,1]$), upper semicontinuous functions,

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defined on a locally compact, Hausdorff and second countable (LCHS) space X (like $\mathbb{R}^d$), only led to the restriction of $USC(X)$ to some subsets of it (see e.g. Li et al, 2002). Inspired by research on upper semicontinuous random functions (e.g. Norberg, 1989), we will make it explicitly here that the whole space $USC(X)$ is metrizable in the Lawson topology of the continuous lattice $USC(X)$ (for background on continuous lattices, see Gierz et al, 2003). While a version of Choquet theorem can be obtained in the context of continuous lattices, we point out that the Choquet theorem is also obtained as a direct derivation of that in the theory of random closed sets using hypographs.

2 Random Closed Sets

As random elements, random closed sets on LCHS spaces are generalizations of random vectors. The theory of random closed sets was developed fully in Matheron (1975) as follows. Let X be a LCHS space. We denote by $\mathcal{F}(X), \mathcal{O}(X), \mathcal{K}(X)$ or simply $\mathcal{F}, \mathcal{O}, \mathcal{K}$ the spaces of closed, open and compact subsets of X, respectively. The so-called hit-or-miss topology on $\mathcal{F}$ is generated by the base consisting of

$$\mathcal{F}^K_{G_1,\ldots,G_n} = \mathcal{F}^K \cap \mathcal{F}_{G_1} \cap \mathcal{F}_{G_2} \cap \ldots \cap \mathcal{F}_{G_n}, \text{ for } n \in \mathbb{N},\ K \in \mathcal{K},\ G_i \in \mathcal{O}$$
$$\text{where } \mathcal{F}_A = \{F \in \mathcal{F}\ ,\ F \cap A \neq \varnothing\}\ ,\ \mathcal{F}^A = \{F \in \mathcal{F}\ ,\ F \cap A = \varnothing\}.$$

With this topology, $\mathcal{F}$ is a compact, Hausdorff and second countable topological space and hence metrizable. When $X = \mathbb{R}^d$, a compatible metric on $\mathcal{F}$ is the stereographical metric (see Rockaffelar and Wets, 1984) . For general LCHS spaces, concrete metrics are obtained similarly by using Alexandroff compactification (see Wang and Wei, 2007).

By a random closed set we mean a random element S, defined on some probability space $(\Omega, \mathcal{A}, P)$, with values in the measurable space $(\mathcal{F}, \sigma(\mathcal{F}))$, where $\sigma(\mathcal{F})$ is the Borel σ-field generated by the hit-or-miss topology. The probability law of S is the probability measure P_S on $\sigma(\mathcal{F})$ given by $P_S = PS^{-1}$ as usual. As in the case of random vectors (i.e., random elements with values in $\mathbb{R}^d$), where the one-to-one correspondence between probability measures on $\sigma(\mathbb{R}^d)$ and distributions functions (the Lebesgue-Stieltjes theorem) serves as a practical tool to propose models for random vectors, there is an one-to-one correspondence between probability measures on $\sigma(\mathcal{F})$ and capacity functionals (the Choquet theorem).

3 Random Fuzzy Sets

Recall that a fuzzy subset of X is a mapping f from X to the unit interval $[0,1]$. For such mappings to generalize closed sets, they have to be upper semicontinuous (usc) so that their level-sets $A_\alpha(f) = \{x \in X : f(x) \geq \alpha\}$, $\alpha \in [0,1]$, are closed. Thus, formally, by a random fuzzy set, we mean a fuzzy subset whose membership function is usc. To be rigorous, we need to topologize the space $USC(X)$ (from now on we suppose functions takes values in $[0,1]$).

If $f : X \to [0,1]$, then f can be identified with the level-sets $A_\alpha(f)$ for $\alpha \in \mathbb{Q}_1 = \mathbb{Q} \cap [0,1]$ (rationals in $[0,1]$). Thus, the mapping $\psi : USC(X) \to \prod \mathcal{F}_\alpha$, the countable cartesian product of identical copies $\mathcal{F}_\alpha$ of $\mathcal{F}$, sending f to $(A_\alpha(f)\ , \alpha \in \mathbb{Q}_1)$, is an embedding. Thus one hopes to induce a topology on $USC(X)$ from the product topology of $\prod \mathcal{F}_\alpha$ which is a compact and second countable space. Unfortunately, the induced topology does not make $USC(X)$ a compact space. For a counter example, see Nguyen et al (2006).

As mentioned in the introduction, the above embedding process is only satisfactory if one restricts the space $USC(X)$ to a subset consisting of usc functions with compact supports.

4 The Space of Closed Sets as a Continuous Lattice

It is known that there is a canonical Hausdorff and compact topology (called the Lawson topology) on every continuous lattice, and the space of closed sets of a Hausdorff and locally compact space is a continuous lattice (see Gierz et al, 2003). We give here essential background details for LCHS space X establishing that the space $\mathcal{F}$ is a compact, Hausdorff and second countable space whose Lawson topology coincides with the hit-or-miss topology. As we will see in the next section, the space $USC(X)$ is also a continuous lattice, and hence its Lawson topology is a natural topology to consider.

Recall that if $(L, \leq)$ is a poset, then x is said to be *way below* y, denoted as $x \ll y$, iff for all directed sets $D \subseteq L$ for which $\sup D$ exists, the relation $y \leq \sup D$ always implies $\exists d \in D$ such that $x \leq d$. Note that in a complete lattice, $x \ll y$ iff for any $A \subseteq L$, $y \leq \sup A$ implies the existence of a finite subset $B \subseteq A$ such that $x \leq \sup B$.

A lattice $(L, \leq)$ is called a continuous lattice if L is complete and satisfies the *axiom of approximation:* $x = \sup \Downarrow x$ where $\Downarrow x = \{u \in L : u \ll x\}$ for all $x \in L$.

Note that $(\mathcal{F}(X), \subseteq)$ is a complete lattice but not continuous in general, where $\wedge\{F_i : i \in I\} = \cap\{F_i : i \in I\}$, and $\vee\{F_i : i \in I\} =$ the closure of $\cup\{F_i : i \in I\}$.

To see that, take $X = \mathbb{R}$, we notice that if $A \ll \mathbb{R}$, then for any subset B of A i.e $B \subseteq A$, we also have $B \ll \mathbb{R}$ (just use the equivalent condition of the way-below relation). Then any singleton closed set, e.g $\{0\}$, is not way-below $\mathbb{R}$. Indeed, $\bigvee_{n\in\mathbb{N}}\{(-\infty, -1/n] \cup [1/n, \infty)\} = \mathbb{R}$, but we can not find any finite subset A of $\{(-\infty, -1/n] \cup [1/n, \infty)\}_{n\in\mathbb{N}}$ such that $\{0\} \subseteq \bigvee A$. Therefore, the only closed set that is way-below $\mathbb{R}$ is the empty set. Then $\sup\{A \in \mathcal{F}(\mathbb{R}) : A \ll \mathbb{R}\} = \sup\{\emptyset\} = \emptyset \neq \mathbb{R}$.

However, for locally compact X, $(\mathcal{F}(X), \supseteq)$ is a continuous lattice. Indeed, for any $F \in \mathcal{F}(X)$, it is enough to show that $F \leq \sup\{A \in \mathcal{F}(X) : A \ll F\}$, i.e $F \supseteq \cap\{A \in \mathcal{F}(X) : A \ll F\}$ or $F^c \subseteq \cup\{A^c \in \mathcal{F}(X) : A \ll F\}$. For any $x \in F^c$: *open*; since X is locally compact, there exists a compact set $Q_x \subseteq F^c$ such that its interior W_x containing x. Let $A = W_x^c \in \mathcal{F}(X)$, then $A^c = W_x \subseteq Q_x \subseteq F^c$. It follows that $A \ll F$; and therefore, $x \in \cup\{A^c \in \mathcal{F}(X) : A \ll F\}$ by the following lemma.

Lemma 1 *Let X be a topological space, and let $L = \mathcal{F}(X)^{op} = (\mathcal{F}(X), \supseteq)$.*

(i) If $A, B \in L$, and if there is a compact set $Q \subseteq X$ such that $A^c \subseteq Q \subseteq B^c$, then $A \ll B$.

(ii) Suppose now that X is locally compact. Then $A \ll B$ implies the existence of a compact set $Q \subseteq X$ such that $A^c \subseteq Q \subseteq B^c$ (i.e (i) holds conversely).

Proof. (i) For any $\{F_i\}_{i\in I} \subseteq \mathcal{F}(X)$ such that $B \leq \sup\{F_i\}_{i\in I}$, i.e., $B \supseteq \cap_{i\in I}\{F_i\}$ equivalently, $B^c \subseteq \cup_{i\in I} F_i^c$. Since $Q \subseteq B^c$, we get $Q \subseteq \cup_{i\in I} F_i^c$. But Q is compact, so there exists a finite subset of $\{F_i^c\}_{i\in I}$, say, $\{F_i^c\}_{i=1}^n$ such that $Q \subseteq \bigcup\limits_{i=1}^n F_i^c$. Hence, $A^c \subseteq \bigcup\limits_{i=1}^n F_i^c$, i.e $A \supseteq \bigcap\limits_{i=1}^n F_i$ or $A \leq \sup\{F_i : i = 1, ..., n\}$. Therefore, $A \ll B$.

(ii) Since X is locally compact, each point $b \in B^c$ has a compact neighborhood $Q_b \subseteq B^c$ such that its interior W_b contains b. Then

$$B^c = \cup\{W_b : b \in B^c\} \text{ or } B = \cap\{W_b^c : b \in B^c\} = \sup\{W_b^c : b \in B^c\}.$$

By assumption $A \ll B$, we have $A \leq \sup\{W_{b_1}^c, ..., W_{b_n}^c\}$ where $b_i \in B^c, i = 1, ..., n$. This means

$$A \supseteq \bigcap_{i=1}^n W_{b_i}^c, \text{ or equivalently } A^c \subseteq \bigcup_{i=1}^n W_{b_i} \subseteq \bigcup_{i=1}^n Q_{b_i} \subseteq B^c.$$

Then the set $Q = \bigcup\limits_{i=1}^n Q_{b_i}$ is the required compact set. ■

Recall that the Lawson topology on a continuous lattice L, denoted by $\Lambda(L)$, has a subbase of the form $\Uparrow x = \{y \in L : x \ll y\}$ or $L\backslash \uparrow x = \{y \in L : x \nleq y\}$ where $x \in L$.

Note: the sets of the form $\Uparrow x = \{y \in L : x \ll y\}$, $x \in L$ form a base for a topology that we call the Scott topology. The space $(L, \Lambda(L))$ is written ΛL.

Lemma 2 *Let X be a locally compact topological space. Then for any compact set K contained in an open set A, i.e. $K \subseteq A \subseteq X$, there exists an open set B and a compact set $K^{'}$ such that*

$K \subseteq B \subseteq K^{'} \subseteq A$.

Proposition 3 *Let L be the continuous lattice $\mathcal{F}(X)^{op}$. Then the Scott topology of $\mathcal{F}(X)^{op}$ has as a base the sets of the form*

$$\{F \in \mathcal{F}(X) : F \cap K = \emptyset\}, \text{ where } K \in \mathcal{K}.$$

Proof. First, we need to show that for any compact subset $K \subseteq X$, $\{F \in \mathcal{F}(X) : F \cap K = \emptyset\}$ is an open subset of the Scott topology. We will show that

$$\{F \in \mathcal{F}(X) : F \cap K = \emptyset\} = \bigcup_{G_i \subseteq K^c G_i \in \mathcal{F}(X)} \Uparrow G_i = \bigcup_{\substack{G_i \subseteq K^c \\ G_i \in \mathcal{F}(X)}} \{F \in \mathcal{F}(X) : G_i \ll F\}.$$

For any $G_i \subseteq K^c$(i.e $K \subseteq G_i^c$), it easy to show that $\Uparrow G_i \subseteq \{F \in \mathcal{F}(X) : F \cap K = \emptyset\}$. Indeed, for any $F \in \mathcal{F}(X)$ such that $G_i \ll F$. By Lemma 1 (ii), there is a compact set K_i with $G_i^c \subseteq K_i \subseteq F^c$, then we have $K \subseteq F^c$, i.e., $F \cap K = \emptyset$.

Conversely, for any $F \in \mathcal{F}(X)$ such that $F \cap K = \emptyset$, i.e., $K \subseteq F^c$: open. By the Lemma 2, there exists an open set B and a compact set K' such that $K \subseteq B \subseteq K^{'} \subseteq F^c$.

Let $G = B^c$ (or $B = G^c$), then we have $K \subseteq G^c \subseteq K^{'} \subseteq F^c$ which implies $G \ll F$ and $G \subseteq K^c$. Therefore,

$$F \in \bigcup_{G_i \subseteq K^c G_i \in \mathcal{F}(X)} \{F \in \mathcal{F}(X) : G_i \ll F\}.$$

Second, for any Scott open set U, U can be written as $\bigcup\{\Uparrow A : A \in U\}$. Now for any $F_0 \in U$, i.e $F_0 \in \Uparrow A$ for some $A \in U$, we have $A \ll F_0$. This means that there is a compact set K such that $F_0 \cap K = \emptyset$ and $A \cup K = X$. Hence, $F_0 \in \{F \in \mathcal{F}(X) : F \cap K = \emptyset\}$. Now for any $F \in \{F \in \mathcal{F}(X) : F \cap K = \emptyset\}$, then we have $F \cap K = \emptyset$ and $A \cup K = X$ which imply $A \ll F$. Hence, $F \in \Uparrow A \subseteq U$. ■

The following result shows that the hit-or-miss topology coincides with the Lawson topology on $(\mathcal{F}(X), \supseteq)$.

Proposition 4 *The Lawson topology of $\mathcal{F}(X)^{op}$, denoted by τ_F, has a subbase consisting of sets of the form*

$$\{F \in \mathcal{F}(X) : F \cap K = \emptyset\} \textit{ and } \{F \in \mathcal{F}(X) : F \cap U \neq \emptyset\}, \textit{ where } K \in \mathcal{K} \textit{ and } U \in \mathcal{G}.$$

Proof. We only need to verify that for any $A \in \mathcal{F}(X)$, $\mathcal{F}(X)\backslash \uparrow A = \{F \in \mathcal{F}(X) : F \cap A^c \neq \emptyset\}$ and then we just let $U = A^c$.

Indeed, since $\uparrow A = \{F \in \mathcal{F}(X) : A \leq F\} = \{F \in \mathcal{F}(X) : F \subseteq A\}$, we have

$$\mathcal{F}(X)\backslash \uparrow A = \{F \in \mathcal{F}(X) : F \nsubseteq A\} = \{F \in \mathcal{F}(X) : F \cap A^c \neq \emptyset\}.$$

In fact, $\{F \in \mathcal{F}(X) : F \cap K = \emptyset\}_{K \in \mathcal{K}}$ is closed under finite intersection, so the Lawson topology of $\mathcal{F}(X)^{op}$ has as a base the sets of the form $\{F \in \mathcal{F}(X) : F \cap K = \emptyset \text{ and } F \cap U_i \neq \emptyset,\ i = 1, ..., n\}$, where $K \in \mathcal{K}$ and $U_i \in \mathcal{G}$. ■

Proposition 5 *For a LCHS space* $(X, \mathcal{G})$*, the space* $\mathcal{F}(X)^{op}$ *is compact, Hausdorff and second countable (and hence metrizable).*

Proof. By the standard result, namely, the Lawson topology of any continuous lattice is compact and Hausdorff, we only need to show that $\mathcal{F}(X)$ is second countable.

Recall that if X is locally compact Hausdorff and second countable, then

(i) it admits a countable base of opens $\{U_n\}$ with compact closure (i.e. relative compact);

(ii) any compact set $K \subset X$ admits a countable fundamental system of open neighborhoods $G_1, G_2, ...$, and it is possible to suppose that the sequence $\{G_n\}$ is decreasing in X. In other words, for any open set $G \supset K$, the inclusion $G_n \subset G$ holds for n large; In particular, $K = \cap G_n$;

(iii) for any closed set F and any compact set K such that $F \cap K = \emptyset$, there exist two disjoint open sets G and $G^{'}$ with $G \supset F$ and $G^{'} \supset K$. Notice that X is normal, and that any compact subset K of a Hausdorff space is closed;

(iv) for any open set G, there exists an increasing sequence of relative compact and open sets $\{B_n\}$ with $\bar{B}_n \subset B_{n+1}$and $G = \cup B_n = \cup \bar{B}_{n+1}$. In particular, any compact set $K \subset G$ is contained in B_n for n large;

(v) there exists a countable family $\mathcal{B}$ of relative compact and open sets in X such that each open set G is the union of the sets $B \in \mathcal{B}$ satisfying $\bar{B} \subset G$.

Let $\mathcal{B}$ be a countable base of the topology $\mathcal{G}$ on X such that each $B \in \mathcal{B}$ is a relative compact set, and each $G \in \mathcal{G}$, $G = \bigcup\limits_{i \in I} B_i, B_i \in \mathcal{B}$ and $\bar{B}_i \subseteq G$. Let τ_B be the class of subsets $\mathcal{F}^{\bar{D}_1 \cup ... \cup \bar{D}_k}_{B_1,...,B_m}$of $\mathcal{F}$, m and k integers $\geq 0, B_1, ..., B_m, D_1, ..., D_k \in \mathcal{B}$.

The class τ_B is countable, and it suffices to show that τ_B is a base for τ_F.

Notice that each $\mathcal{F}^{\bar{D}_1 \cup ... \cup \bar{D}_k}_{B_1,...,B_m}$ is an open set of $(\mathcal{F}, \tau_F)$.

Let $F \in \mathcal{F}$ be a closed set, and $\mathcal{F}^K_{U_1,...,U_n}$ $(K \in \mathcal{K}, U_1, ..., U_n \in \mathcal{G})$ an open neighborhood of F in $(\mathcal{F}, \tau_F)$. We need to find $\mathcal{V} \in \tau_B$ such that $F \in \mathcal{V} \subseteq \mathcal{F}^K_{U_1,...,U_n}$

Case 1: $F = \emptyset$. Then $n = 0$,and we may choose $B \in \mathcal{B}$ such that $K \subseteq \bar{B}$. Let $\mathcal{V} = \mathcal{F}^{\bar{B}} \in \tau_B$, then $F \in \mathcal{F}^{\bar{B}} \subseteq \mathcal{F}^K$.

Case 2: $F \neq \emptyset$.For each $i = 1, ..., n$, we choose a point $x_i \in F \cap U_i$ and an open set $B_i \in \mathcal{B}$ such that $x_i \in B_i \subset \bar{B}_i \subset U_i \cap K^c$.

Since $F \cup (\bigcup\limits_{i=1}^{n} \bar{B}_i)$ and K are disjoint, we can find two disjoint open sets $G_1 = \bigcup D_j$ (where $D_j \in \mathcal{B}$) and G_2 such that $(F \cup (\bigcup\limits_{i=1}^{n} \bar{B}_i)) \subseteq G_2$ and $K \subseteq G_1$.

By the compactness of the set K, we have $K \subseteq \bigcup\limits_{j=1}^{k} D_j \subseteq G_1 \subseteq G_2^c$, $D_1, ..., D_k \in \mathcal{B}$; by definition of the closure, $\overline{\bigcup\limits_{j=1}^{k} D_j} \equiv \bigcup\limits_{j=1}^{k} \bar{D}_j \subseteq G_2^c$ and so $(\bigcup\limits_{j=1}^{k} \bar{D}_j\ ) \cap G_2 = \emptyset$. Thus, $F \in \mathcal{F}^{\bar{D}_1 \cup ... \cup \bar{D}_k}_{B_1,...,B_n}$ and $\mathcal{F}^{\bar{D}_1 \cup ... \cup \bar{D}_k}_{B_1,...,B_n} \subseteq \mathcal{F}^K_{U_1,...,U_n}$. ■

5 Upper Semicontinuous Functions as a Continuous Lattice

In this section, X is LCHS and $USC(X)$ is the space of all usc functions from X to $[0, 1]$.

Proposition 6 $USC(X)$ *is a complete lattice but not continuous with the poinwise order* $\leq$

$$f \leq g, \text{ i.e., } f(x) \leq g(x)\ \forall x \in X,$$

where $\underset{j \in J}{\wedge} f_j = \inf\limits_{j \in J} f_j(\ f_j \in USC(X), j \in J)$.

Proof. Let $f = \inf_{j \in J} f_j$, to show $USC(X)$ is a complete lattice it is enough to show that $f \in USC(X)$ i.e for any $r \in [0,1]$, $\{x : f(x) < r\}$ is open.

Indeed, $\{x : f(x) < r\} = \underset{j \in J}{\cup} \{x : f_j(x) < r\}$ and since each $\{x : f_j(x) < r\}$ is open, $\{x : f(x) < r\}$ is also open. To see $(USC(X), \leq)$ is not a continuous lattice, consider the following example. Take $f(x) = 1, \forall x \in X$. We will see that the only usc function that is way-below f is the zero function. Indeed, to show the statement above it is enough to show that any $r1_{\{\text{single point}\}}$, for example $\frac{1}{2}1_{\{0\}}$, is not way-below f.

Choose $f_n(x) = 1_{(-\infty,-1/n)\cup(1/n,\infty)}$, then $\vee f_n = 1$, but for any $\underset{finite}{\vee} f_n(0) = 0$. Hence, $\frac{1}{2}1_{\{0\}} \not\leq \underset{finite}{\vee} f_n$ which implies $\frac{1}{2}1_{\{0\}}$ is not way-below f.

However, let $X = \mathbb{R}$, and $F \subset USC(X)$, where $F = \{1_{[1/n,\infty)} : n \in \mathbb{N}\}$, while each $1_{[1/n,\infty)}$ is usc, but $\vee F : x \to \sup\{1_{[1/n,\infty)}(x) : n \in \mathbb{N}\} = 1_{(0,\infty)}(x)$, with $1_{(0,\infty)} \notin USC(X)$. Thus the pointwise supremum is not closed in L.

Hence, to find the $\vee F$ in $USC(X)$, observe the following:

Fact 1: If $\{M_\alpha : \alpha \in I\}$ be a family of non-empty closed sets in X such that $M_\alpha \supseteq M_\beta$ for all $\alpha < \beta$, then $f(x) = \sup\{\alpha \in I : x \in M_\alpha\}$ is an upper semicontinuous function.

Fact 2: For any function $f : X \to I$, we have $f(x) = \sup\{\alpha \in I : x \in M_\alpha\}$where $M_\alpha = \{x \in X : f(x) \geq \alpha\}$ is the α-level set of f.

Now for $F = \{f_j\}_{j \in J} \subseteq USC(X)$, we define $f(x) = \sup\{\alpha \in I : x \in M_\alpha\}$, where $M_\alpha = \overline{\underset{j \in J}{U} \{y : f_j(y) \geq \alpha\}}$

We will verify that $f = \vee F$. Indeed, first, by Fact 1, $f \in USC(X)$.

Second, $\forall x \in X$, we have $x \in \overline{\underset{j \in J}{U} \{y : f_j(y) \geq f_i(x)\}} \equiv M_{f_i(x)}$ for any $i \in J$, so $f(x) \geq f_i(x) \implies f \geq f_i$, $\forall i \in J$. For any $g \geq f_i$, $\forall i \in J, i.e.,$ $g(x) \geq f_i(x)$, $\forall x \in X$, we need to show that $g(x) \geq f(x)$. By the Fact 2, $g(x) = \sup\{\alpha \in I : x \in M_\alpha^g\}$, where $M_\alpha^g = \{y \in X : g(y) \geq \alpha\}$ is the α-level set of g, we only need to verify that

$$\sup\{\alpha \in I : x \in M_\alpha^g\} \geq \sup\{\alpha \in I : x \in M_\alpha\}.$$

For any α, it is easy to show that $M_\alpha^g \supseteq M_\alpha$. Indeed, since M_α^g is closed, it is enough to show that $M_\alpha^g \supseteq \underset{j \in J}{U} \{y : f_j(y) \geq \alpha\}$. For any $y \in \underset{j \in J}{U} \{y : f_j(y) \geq \alpha\}$, i.e $f_j(y) \geq \alpha$ for some j. Therefore, $g(y) \geq \alpha$ i.e $y \in M_\alpha^g$. ■

Note: we can use this representation to see that

$$\underset{n \in \mathbb{N}}{\vee} 1_{(-\infty,-1/n)\cup(1/n,\infty)}(x) = 1 \text{ for any } x \in \mathbb{R}.$$

Lemma 7 *For any $f \in USC(X)$, $f = \inf_{r,K \text{ (compact)}} \{g_{r,K} : f(y) < r, \forall y \in K\}$, where $g_{r,K}(x) = r$ if $x \in \overset{\circ}{K}$ and $= 1$ otherwise.*

Proof. For any $x \in X$,

* Case 1: $f(x) = 1$, then $g_{r,K}(x) = 1$ for all such (r, K). (if $x \notin \overset{\circ}{K} \implies g_{r,K}(x) = 1$, and if $x \in \overset{\circ}{K}$ there is no such an "r" such that $f(y) < r, \forall y \in K$)

* Case 2: $f(x) < 1$.

+ If $x \in \overset{\circ}{K}$, then $f(x) < r = g_{r,K}(x)$.

+ If $x \notin \overset{\circ}{K}$, then $f(x) < 1 = g_{r,K}(x) \implies f(x)$ is a lower bound of $\{g_{r,K}(x)\}$.

Now for any $\epsilon > 0$, we need to show that $f(x) + \epsilon$ is no longer a lower bound of $\{g_{r,K}(x)$ i.e., we need to find r_0, K_0 such that

$$f(y) < r_0,\ \forall y \in K_0 \text{ and } f(x) + \epsilon > g_{r_0,K_0}(x).$$

Indeed, if we take $r_0 = f(x) + \epsilon/2$, then $x \in \{y \in X : f(y) < r_0 = f(x) + \epsilon/2\}$: open.

By local compactness, there is a K_0 (compact) $\subseteq \{y : f(y) < r_0\}$ such that $x \in \overset{\circ}{K}_0$. Thus, $g_{r_0,K_0}(x) = r_0 < f(x) + \epsilon$. ∎

Proposition 8 *$L = (USC(X), \leq^{op})$ is a continuous lattice, where $f \leq^{op} g$ iff $f(x) \geq g(x),\ \forall x \in X$.*

Proof. L is a complete lattice with $\underset{j\in J}{\vee} f_j = \inf f_j$ and $\underset{j\in J}{\wedge} f_j = h$ where $h(x) = \sup\{\alpha \in I : x \in M_\alpha\}$, with $M_\alpha = \overline{\underset{j\in J}{U}\{y : f_j(y) \geq \alpha\}}$.

To show the continuity of L, we need to show that for any $f \in L,\ f \leq^{op} \vee\{g : g \ll f\}$, i.e., $\inf\{g : g \ll f\} \leq f = \inf_{r,K}\{g_{r,K} : f(y) < r, \forall y \in K\}$. So, we only need to show that $g_{r,K} \ll f$ for any r, K such that $f(y) < r, \forall y \in K$. For any $F_{directed} \subseteq L$, say $F = \{f_j\}_{j\in J}$, such that $f \leq^{op} \vee F$, i.e $f \geq \inf F$ pointwise. We need to find $h \in F$ such that $g_{r,K} \leq^{op} h$ i.e $g_{r,K} \geq h$ pointwise. With such a $(r, K) : r > f \geq \inf F$ pointwise on K, we will show that $\exists h \in F$ such that $r > h$ on K, i.e $r > h(x),\ \forall x \in K$. For any $h \in F$, let $K_h = \{x \in K : r \leq h(x)\}$. This set is closed because of upper semicontinuity of h.

Claim: $\underset{h\in F}{\cap} K_h = \emptyset$. Suppose there is an $x \in \underset{h\in F}{\cap} K_h$, then $r \leq h(x),\ \forall h \in F,\ \forall x \in K$. This implies $r \leq \inf F$ on K, which contradicts $r > \inf F$ on K. So, $\underset{h\in F}{\cap} K_h = \emptyset$. Since K_h is closed and $K_h \subseteq K$: compact, there is a finite intersection $\bigcap_{i=1}^{n} K_{h_i} = \emptyset$.

Note:

$$\begin{aligned} K_{h_1} \cap K_{h_2} &= \{x \in K : r \leq h_1(x) \text{ and } r \leq h_2(x)\} \\ &= \{x \in K : r \leq \inf(h_1(x), h_2(x))\} \\ &= K_{\inf(h_1,h_2)}. \end{aligned}$$

Since F is directed, $\bigvee_{i=1}^{n} h_i = \inf_{i=1,\ldots,n} h_i \in F$. Let $h = \inf_{i=1,\ldots,n} h_i$, then we have $K_h = \emptyset$, i.e $\forall x \in K, r > h(x)$. Therefore, $\forall x \in X$, if $x \in K,\ h(x) < r \leq g_{r,K}(x)$ and if $x \notin K,\ h(x) \leq 1 = g_{r,K}(x)$. Hence, $h(x) \leq g_{r,K}(x),\ \ \forall x \in X$. ∎

Remark. For any $f, g \in L$, then $g \ll f$ implies $\forall x \in X,\ \exists r, K$ such that $x \in \overset{\circ}{K}$ and $f(y) < r \leq g(y),\ \forall y \in K$.

Proposition 9 *For any $r \in (0, 1]$ and K (compact) $\subseteq X$, we have*

$$\{f \in L : f(y) < r,\ \forall y \in K\} = \bigcup_{\overset{\circ}{K}_i \supseteq K} \{f \in L : g_{r,K_i} \ll f\}$$

where g_{r,K_i} is defined as above.

Proof. ($\subseteq$): For any $f \in L$ such that $f(y) < r, \forall y \in K$. Let $O = \{x : f(x) < r\}$, then $K \subseteq O$: open. There are U : open and K': compact in X such that $K \subseteq U \subseteq K' \subseteq O$ (which implies $U \subseteq \overset{\circ}{K'}$). Since $K' \subseteq O$, we get $g_{r,K'} \ll f$.

($\supseteq$) For any $f \in L$ such that $g_{r,K_i} \ll f$, where $\overset{\circ}{K}_i \supseteq K$, then $\forall y \in K$, there are r_y and K_y such that $y \in \overset{\circ}{K}_y$ and $f(z) < r_y \leq g_{r,K_i}(z),\ \forall z \in K_y$.

Particularly, take $z = y$, we get $f(y) < r_y \leq g_{r,K_i}(y) = r$. Hence, $f(y) < r,\ \forall y \in K$. ∎

Proposition 10 *The Scott topology $\tau(L)$ has as a subbase the sets $\{f : f(y) < r,\ \forall y \in K\}$, where $r \in (0,1]$ and K (compact) $\subseteq X$. In other words, the Scott topology $\tau(L)$ has as a base the sets $\bigcap_{i=1}^{n}\{f : f(y) < r_i,\ \forall y \in K_i\}$, where $r_i \in (0,1]$, K_i (compact) $\subseteq X$, and $n \in \mathbb{N}$.*

Proof. Recall: $\tau(L)$ has as a base the sets of the form $\{f : g \ll f\}$,where $g \in L$, and for any Scott open set O, we can write $O = \underset{g\in O}{\cup}\{f : g \ll f\}$. By the proposition above, any set of the form $\{f : f(y) < r,\ \forall y \in K\}$ is Scott open (notice: $g_{r,K_i} \in L$). Now for any $h \in O$: Scott open set, then we need to find $\{r_i\}_{i=1}^{n}$ and $\{K_i\}_{i=1}^{n}$ such that $h \in \bigcap_{i=1}^{n}\{f : f(y) < r_i,\ \forall y \in K_i\} \subseteq O$. Since O can be written as $\underset{g\in O}{\cup}\{f : g \ll f\}$ and $h \in O$, we have $g \ll h$ for some $g \in O$. Since $h = \inf_{r,K}\{g_{r,K} : h(y) < r, \forall y \in K\} = \vee\{g_{r,K} : h(y) < r, \forall y \in K\}$, we get $g \leq^{op} \underset{i=1,..,n}{\vee}\{g_{r_i,K_i} : h(y) < r_i, \forall y \in K_i\} \ = \inf_{i=1,..,n}\{g_{r_i,K_i} : h(y) < r_i, \forall y \in K_i\}$. Hence, $h \in \bigcap_{i=1}^{n}\{f : f(y) < r_i,\ \forall y \in K_i\}$. Now for any $f \in L$ such that $f(y) < r_i, \forall y \in K_i$, and $\forall i = 1, ..., n$, then $g_{r_i,K_i} \ll f,\ \forall i = 1, ..., n$. This implies $\underset{i=1,..,n}{\vee}\{g_{r_i,K_i} : h(y) < r_i, \forall y \in K_i\} \ll f$, so $g \ll f$. Hence, $\bigcap_{i=1}^{n}\{f : f(y) < r_i,\ \forall y \in K_i\} \subseteq O$. ■

Corollary 11 *The Lawson topology $\Lambda(L)$ has as a subbase the sets $\{f : f(y) < r,\ \forall y \in K\}$, where $r \in (0,1]$ and K (compact) $\subseteq X$ together with the sets $\{f : \exists x \in X$ such that $g(x) < f(x)\}$, where $g \in L$.*

Proof. By definition, the Lawson topology $\Lambda(L)$ has as a subbase the sets $\{f : g \ll f\}$, where $g \in L$ together with the sets $\{f : g \not\leq^{op} f\}$, where $g \in L$.Notice that $\{f : g \not\leq^{op} f\} = \{f : g \not\geq f$ on $X\}$, i.e $\{f : g(x) < f(x)$ for some $x \in X\}$. ■

Now if we view $\mathcal{F}(X)$ as $1_{\mathcal{F}(X)}$ which is the set of all indicator functions of closed sets, then since closed sets are closed under arbitrary intersection, $(1_{\mathcal{F}(X)}, \leq^{op}) \simeq (\mathcal{F}(X), \subseteq^{op})$ is a complete sublattice of $(USC(X), \leq^{op})$. In view of topological spaces, $1_{\mathcal{F}(X)}$ is a subspace of $(USC(X), \Lambda(L))$, so we hope that the induced topology of $\Lambda(L)$ coincides with the Lawson topology $\Lambda(\mathcal{F}(X))$ on $\mathcal{F}(X)$. It is indeed the case, as shown below.

An element in the base of $\Lambda(L)$ has the form

$$(\bigcap_{i=1}^{n}\{f : f(y) < r_i, \forall y \in K_i\}) \cap (\bigcap_{j=1}^{m}\{f : g_j(x) < f(x) \text{ for some } x \in X\}), \text{ where}$$

$$r_i \in (0,1], K_i: \text{ compact, and } g_j \in USC(X)$$

$$\text{or } \{f : f(y) < r_i, \forall y \in K_i, \forall i = 1..., n \text{ and } g_j(x) < f(x) \text{ for some } x \in X, \forall j = 1, ..., m\}(*)$$

If we restrict $USC(X)$ on $1_{\mathcal{F}(X)}$, then all functions f, g and g_j can be written as $1_F, 1_G$, and 1_{G_j} respectively, where F, G, and $G_j \in \mathcal{F}(X)$.

Then (*) is rewritten as

$$\{1_F : 1_F(y) < r_i, \forall y \in K_i, \forall i = 1..n \text{ and } 1_{G_j}(x) < 1_F(x) \text{ for some } x \in X, \forall j = 1, .., m\}, \text{ where}$$
$$r_i \in (0,1], K_i : \text{compact, and } G_j \in \mathcal{F}(X).$$

However, indicator functions just take values $0, 1$, so the set above is the same as the set

$$\{1_F : 1_F(y) = 0, \forall y \in K_i, \forall i = 1..n \text{ and } 1_F(x) = 1 \text{ and } 1_{G_j}(x) = 0 \text{ for some } x \in X, \forall j = 1,..,m\},$$
where K_i : compact, and $G_j \in \mathcal{F}(X)$., i.e.,
$$\{1_F : 1_F(y) = 0, \forall y \in K = \underset{i=1,\ldots,n}{U} K_i \text{ and } 1_F(x) = 1 \text{ and } 1_{G_j^c}(x) = 1 \text{ for some } x \in X, \forall j = 1,..,m\},$$
where $K = \underset{i=1,\ldots,n}{U} K_i$: compact, and $G_j^c \in \mathcal{O}(X)$.

This exactly means that the set above is the set of all indicator functions of closed sets F such that $F \cap K = \emptyset$ and $F \cap G_j^c \neq \emptyset$.

Hence, those elements form the hit-or-miss or the Lawson topology on $\mathcal{F}(X)$.

Remark. It is known that $(L, \sigma(L))$ is second countable iff $(L, \Lambda(L))$ is second countable (Gierz et al, 2003).

Thus, to show that $(L, \Lambda(L))$ is second countable, it suffices to show that $(L, \sigma(L))$ has a countable base.

Notation: $\mathcal{F}^{\,r,K} \triangleq \{f : f(y) < r, \forall y \in K\}$

Recall that: (1) $(L, \sigma(L))$ has as a base the sets of the form

$$\bigcap_{i=1}^{n} \{f : f(y) < r_i, \forall y \in K_i\},$$

where $r_i \in (0, 1]$, K_i (compact) $\subseteq X$, and $n \in \mathbb{N}$, or $\bigcap_{i=1}^{n} \mathcal{F}^{\,r_i,K_i}$

(2) X : LCHS implies

a) there is a countable base $\mathcal{B}$ of $(X, \mathcal{G}(X))$ such that $\forall B \in \mathcal{B}, \bar{B}$ is compact and $\forall O$: *open*, $O = \underset{j \in J}{\cup} B_j$ with $\bar{B}_j \subseteq O$.

b) X is normal.

We claim that $(L, \sigma(L))$ has as a base the sets of the form

$$\bigcap_{i=1}^{n} \mathcal{F}^{\,q_i, \cup_{j=1}^{m_i} \bar{B}_{i_j}}, \quad \text{where } q_i \in \mathbb{Q} \cap (0, 1] \text{ and } B_{i_j} \in \mathcal{B}.$$

Proof. For any $f \in \bigcap_{i=1}^{n} \mathcal{F}^{\,r_i,K_i}$, then $f(y) < r_i,\ \forall y \in K_i,\ \forall i = 1, ..., n$. Let $A_i = \{x \in X : f(x) \geq r_i\}$: closed, and K_i is also closed; and $A_i \cap K_i = \emptyset$. Hence, there is an open set $G_i \subseteq X$ such that $G_i = \cup B_{i_j}$, $\bar{B}_{i_j} \subseteq G_i$, and $K_i \subseteq G_i$, $A_i \cap G_i = \emptyset$.

By the compactness of K_i, $K \subseteq \bigcup_{j=1}^{m_j} B_{i_j} \subseteq \bigcup_{j=1}^{m_j} \bar{B}_{i_j} \subseteq G_i$. Therefore, $A_i \cap (\bigcup_{j=1}^{m_j} \bar{B}_{i_j}) = \emptyset$. $\Longrightarrow \forall y \in \bigcup_{j=1}^{m_j} \bar{B}_{i_j}$, $y \notin A_i$, i.e $f(y) < r_i$. Hence, $f(y) < r_i$, $\forall y \in \bigcup_{j=1}^{m_j} \bar{B}_{i_j}$.

Since f is upper semicontinuous, f attains a maximum on $\bigcup_{j=1}^{m_j} \bar{B}_{i_j}$, so we can find a $q_i \in \mathbb{Q} \cap (0, 1]$ such that $f(y) < q_i < r_i$, $\forall y \in \bigcup_{j=1}^{m_j} \bar{B}_{i_j}$. This means $f \in \bigcap_{i=1}^{n} \mathcal{F}^{\,q_i, \cup_{j=1}^{m_i} \bar{B}_{i_j}}$. It is easy to check that $\bigcap_{i=1}^{n} \mathcal{F}^{\,q_i, \cup_{j=1}^{m_i} \bar{B}_{i_j}} \subseteq \bigcap_{i=1}^{n} \mathcal{F}^{\,r_i,K_i}$. ∎

Remark. In view of the above results, by a *random fuzzy (closed) set* on a LCSHS space X, we mean a random element with values in the measurable space $(USC(X), \sigma(\Lambda))$, where $\sigma(\Lambda)$ is the Borel σ-field associated with the Lawson topology of the continuous lattice $USC(X)$ (with reverse order $\geq$). With the Lawson topology, $USC(X)$ is a compact, Hausdorff and second countable (hence metrizable). This falls neatly in the framework of separable metric spaces in probability theory.

6 A metric compatible with the Lawson topology of USC(X)

We already know that by using the embedding of $\mathcal{F}(X)$ into the hyperspace 2^{wX}, where wX is the one-point compactification of X and 2^{wX} is the set of all non-empty closed sets of wX, we can have a metric compatible with the Lawson topology or the Matheron topology of $\mathcal{F}(X)$ (see Wang and Wei, 2007). Moreover, we also know that there is a bijection $i : USC(X) \longrightarrow HYP(X)$, where $i(f) = Hyp(f) = \{(x, \alpha) \in X \times [0, 1] : f(x) \geq \alpha\}$, and $HYP(X)$ is the set of all hypographs of u.s.c functions on X. This set is just a closed subspace of $\mathcal{F}(X \times [0, 1])$ which again has a metric d which is compatible with the Lawson or hit-or-miss topology of $\mathcal{F}(X \times [0, 1])$ (see Nguyen et al, 2007). Thus the induced metric on the subspace $HYP(X)$ is also compatible with the induced hit-or-miss topology on $HYP(X)$.It remains to show that $(HYP(X), d)$ is homeomorphic to $(USC(X), \Lambda(L))$, where $L = (USC(X), \leq^{op})$.

Proposition 12 *The map* $i : (USC(X), \Lambda(L)) \longrightarrow (HYP(X), d)$*, where* $i(f) = F \equiv Hyp(f) = \{(x, \alpha) \in X \times [0, 1] : f(x) \geq \alpha\}$*, is a homeomorphism.*

Proof. Since $(USC(X), \Lambda(L))$ is compact and $(HYP(X), d)$ is Hausdorff, to prove the theorem we only need to show that the map i is continuous, i.e., $\forall f_n, f \in L$ such that $f_n \to f$, then we have to show that $F_n \to F$, where $F_n = Hyp(f_n)$, $F = Hyp(f)$, and

a) $f_n \to f$ means

i) whenever $f \in \{g \in USC(X) : g(y) < r, \forall y \in K_X\}$ for some $r \in (0, 1]$ and K_X : compact in X, then $f_n \in \{g \in USC(X) : g(y) < r, \forall y \in K_X\}$ for n large;

ii) whenever $f \in \{h \in USC(X) : g(x) < h(x)$ for some $x \in X\}$ where $g \in USC(X)$, then $f_n \in \{h \in USC(X) : g(x) < h(x)$ for some $x \in X\}$ for n large. And

b) $F_n \to F$ means

i) for any G open in $X \times [0, 1]$, if $F \cap G \neq \emptyset$, then $F_n \cap G \neq \emptyset$ for n large;

ii) for any K compact in $X \times [0, 1]$, if $F \cap K = \emptyset$, then $F_n \cap K = \emptyset$ for n large.

Suppose $F \cap G \neq \emptyset$, then $\exists(x_0, \alpha)$ such that $f(x_0) \geq \alpha$ and $(x_0, \alpha) \in G$.

Case 1: $\alpha = 0$, we have$(x_0, \alpha) \in G$, and $f_n(x_0) \geq 0 = \alpha$. This implies $F_n \cap G \neq \emptyset$.

Case 2: $\alpha > 0$. Since $(x_0, \alpha) \in G$ which is open in $X \times [0, 1]$, $(x_0, \alpha) \in O \times U$, where O is open in X and U is open in $[0, 1]$. Since $0 < \alpha \in U$, we can find $\beta \in U$ such that $\beta < \alpha$. Define function g as by $g(x) = \beta$ if $x \in O$ and $g(x) = 1$ otherwise. Then $g \in USC(X)$, and $f(x_0) \geq \alpha > \beta$, so $f \in \{h \in USC(X) : g(x) < h(x)$ for some $x \in X\}$. Since $f_n \to f$, there is a M such that $f_n \in \{h \in USC(X) : g(x) < h(x)$ for some $x \in X\}$, $\forall n > M$. By definition of $g(x)$, we can conclude that $\exists x_n \in O$ such that $f_n(x_n) > \beta$. i.e., $(x_n, \beta) \in G$ and $(x_n, \beta) \in F_n$. Hence, $G \cap F_n \neq \emptyset$, $\forall n > M$.

Second, suppose $F \cap K_X \times [a, b] = \emptyset$, where K_X is compact in X and $[a, b] \subseteq [0, 1]$. This implies $\forall(x, \alpha) \in F \implies (x, \alpha) \notin K_X \times [a, b]$. We have $(x, f(x)) \in F$, so $(x, f(x)) \notin K_X \times [a, b], \forall x \in X$. Hence, particularly $\forall x \in K_X, f(x) \notin [a, b]$, which implying either $f(x) < a$ or $f(x) > b$. The later inequality, however, never holds. Indeed, suppose $f(x) > b$, then $(x, b) \in F \cap K_X \times [a, b]$ which contradicts the assumption $F \cap K_X \times [a, b] = \emptyset$. Therefore, we must have $f(x) < a, \forall x \in K_X$. Since $f_n \to f$ in L, and $f(x) < a$, $\forall x \in K_X$, there is a N such that $\forall n > N$, $f_n(x) < a$, $\forall x \in K_X$

Claim: $F_n \cap K_X \times [a, b] = \emptyset, \forall n > N$.

Indeed, suppose $\exists(x, \alpha) \in F_n \cap K_X \times [a, b]$, i.e. $f_n(x) \geq \alpha \geq a$ and $x \in K$. These conditions contradict $f_n(x) < a$, $\forall x \in K_X$. Hence, $F_n \cap K_X \times [a, b] = \emptyset, \forall n > N$. ■

Therefore, we can define a metric on $(USC(X), \Lambda(L))$ by

$$\delta : USC(X) \times USC(X) \to \mathbb{R}^+,\ \delta(f, g) = d(F, G),$$

where $F = Hyp(f)$, and $G = Hyp(g)$. This metric δ is compatible with the Lawson topology $\Lambda(L)$.

Remark. A Choquet theorem for $USC(X)$ can be obtained by embedding $USC(X)$ into the space of closed sets of the LCHS space $X \times [0,1]$ via hypographs and then use Choquet theorem for random closed sets (Matheron, 1975) of the LCHS space $X \times [0,1]$. This embedding makes $USC(X)$ a closed subset of $\mathcal{F}(X \times [0,1])$ with respect to its hit-or-miss topology. For details, see Nguyen, Wang and Wei (2007).

A version of Choquet theorem in the context of continuous lattices (Norberg, 1989) is the following.

If L is a continuous lattice, and Σ_L is the Borel algebra of the Lawson topology on L, then for any probability space (L, Σ_L, P) the function $S : L \to [0,1]$ defined by $S(a) = P(a \uparrow)$ satisfies

i) $a_n \uparrow a$ implies $S(a_n) \downarrow S(a)$,

ii) $S(0) = 1$,

iii) S is "nti-alternating of infinite order", i.e

$$S_1(x; x_1) = S(x \vee x_1) - S(x) \leq 0,$$
$$S_2(x; x_1, x_2) = -S(x \vee x_1 \vee x_2) + S(x \vee x_1) + S(x \vee x_2) - S(x) \leq 0...$$
$$S_n(x; x_1, ..., x_n) = S_{n-1}(x; x_1, ..., x_{n-1}) - S(x \vee x_n; x_1, ..., x_{n-1}) \leq 0.$$

Conversely, each such $S : L \to [0,1]$ comes from a (unique) probability measure on Σ_L.

This theorem generalizes Lebesgue-Stieltjes theorem for random vectors as illustrated by the following simple situations.

Example 1: $X = \{a\}$ with discrete topology then $(USC(X), \Lambda_L) \cong ([0,1], \geq)$. Then $S(x) = P(x \uparrow) = P(y \leq x)$, and $[0,1] \cong \mathbb{R} \cup \{\pm\infty\}$.

Recalling that $USC(X) = \{f : X \to [0,1]$ such that f is u.s.c$\}$. If X is LCHS, then $L \equiv (USC(X), \geq)$ is a continuous lattice with the Lawson topology $\Lambda(L)$. The identification map $Id : (USC(X), \Lambda_L) \to ([0,1], \geq)$ defined by $Id(f) = f(a)$ is an isomorphism.

Example 2: $X = \{a, b\}$ with discrete topology, then $USC(X) \cong [0,1] \times [0,1]$, and $S(x_0, y_0) = P((x_0, y_0) \uparrow) = P(x \leq x_0, y \leq y_0)$.

Again applying Norberg's theorem, we get the Lebesgue -Stieltjes' theorem for $\mathbb{R}^2$.

In general, if $X = \{x_1, ..., x_n\}$, then $USC(X) \cong [0,1]^n$.

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