Journal of Uncertain Systems
Vol.2, No.4, pp.299-312, 2008
Online at: www.jus.org.uk

Computing the Mean Chance Distributions of Fuzzy Random Variables*

Rui Qin† Fang-Fang Hao
College of Mathematics and Computer Science, Hebei University, Baoding 071002, China

Received 10 November 2007; Accepted 6 February 2008

Abstract

Fuzzy random variable is a combination of fuzzy variable and random variable, and can characterize both fuzziness and randomness in the real world. The mean chance of a fuzzy random event is an important concept in fuzzy random optimization, just like the probability of a stochastic event in stochastic optimization and the credibility of a fuzzy event in fuzzy optimization. In fuzzy random programming, the constraints are always represented by the mean chance function of fuzzy random events, so it is difficult to solve them directly. In this paper, we deduce the formulas for the mean chance of fuzzy random events in some special cases such as the fuzzy random variables are trapezoidal fuzzy random variables for normal fuzzy random variables. The obtained formulas are very useful in fuzzy random optimization problems when we design algorithm to solve them.
©2008 World Academic Press, UK. All rights reserved.
Keywords: fuzzy random variable, mean chance, credibility theory, fuzzy random optimization

1 Introduction

In the real world problems such as in a manufacturing system or a production process, we may only obtain data by statistic due to the complexity or the big scale of the problem. In such situations, the data can not be precisely known. To solve these problems, we may use probability to deal with the uncertainty involved in it. In fact, probability is a very useful and important tool to solve stochastic programming problems.

On the other hand, with the development of fuzzy set theory [23], many researchers realized that there exists another uncertainty called fuzziness besides randomness in the world [18]. Fuzziness exists because sometimes we can only obtain data from experts, and the data are usually given by experts according to their knowledge or experience. However, different experts may have different ideas and even if the data were given by one expert, it contains many objective factors, this leads to the existence of the fuzziness in the data. In a fuzzy environment, to deal with the fuzziness, we may use credibility measure.

However, in a more complex system, there exists randomness and fuzziness in the data simultaneously. In this case, we can not only consider its randomness or fuzziness and ignore the other uncertainty because this will give us a meaningless solution. To consider the twofold uncertainty in the problem, we can represent it by fuzzy random variables, which is a fundamental concept in fuzzy random theory and was initially introduced by Kwakernaak [3] to depict the phenomena containing both fuzziness and randomness. In the literature, fuzzy random variable was studied by many researchers aiming at different purposes [1][2][12][19]. With fuzzy random variables, fuzzy random optimization theory has been developed rapidly and more and more optimization models have been well developed in recent years [4][5][10][17][22]. In fuzzy random programming, we may use mean chance to measure a fuzzy random event, just like probability in stochastic programming and credibility in fuzzy programming. For example, a two-stage fuzzy random nimimum-risk problem was established by using mean chance function as the objective [11]; and the convergence modes in mean chance theory has been developed recently in [15], which facilitate us to design algorithms to solve fuzzy random optimization problems.

In fuzzy random optimization problems, the constraints are often modeled as the mean chance of fuzzy random events. Due to the complexity of the computation of mean chance, it is always difficult to solve the programming problems. However, in some special cases, such as for the trapezoidal fuzzy random event and

*This work is supported by the National Natural Science Foundation of China (No.70571021), the Program for One Hundred Excellent and Innovative Talents in Colleges and Universities of Hebei Province, and the Natural Science Foundation of Hebei Province (No.A2008000563.)
†Corresponding author. Email: qinrui07@163.com (R. Qin).

normal fuzzy random event, the mean chance function can be turned into its equivalent form. In this paper, we will consider several special cases, including the randomness is characterized by normal distribution, exponential distribution and uniform distribution. We also give some examples to illustrate the obtained results. With these formulas, we can solve these problems more precisely.

This paper is organized as follows. In Section 2, we recall some basic concepts in fuzzy random theory. In Section 3 we deduce some formulas about the mean chance distributions related to trapezoidal fuzzy random variables. In Section 4 we study some formulas about the mean chance distributions related to normal fuzzy random variables. In Section 5 we draw our conclusions.

2 Preliminaries

Given a universe Γ, an ample field $\mathcal{A}$ on Γ is a class of subsets of Γ that is closed under arbitrary union, intersection, and complement in Γ [20].

Let $\text{Pos} : \mathcal{A} \mapsto [0,1]$ be a set function on the ample field $\mathcal{A}$. Pos is said to be a possibility measure [20] if it satisfies the following conditions:

1) $\text{Pos}(\emptyset) = 0$, and $\text{Pos}(\Gamma) = 1$;

2) For any subclass $\{A_i | i \in I\}$ of $\mathcal{A}$ (finite, countable or uncountable),

$$\text{Pos}(\bigcup_{i\in I} A_i) = \sup_{i\in I} \text{Pos}(A_i).$$

Based on possibility measure, the credibility measure [9] is defined as

$$\text{Cr}(A) = \frac{1}{2}\left(1 + \text{Pos}(A) - \text{Pos}(A^c)\right), \quad A \in \mathcal{A}, \tag{1}$$

where A^c is the complement of A.

The triplet $(\Gamma, \mathcal{A}, \text{Cr})$ is called a credibility space [7]. A credibility measure has the following properties:

(1) $\text{Cr}(\emptyset)$=0, and $\text{Cr}(\Gamma) = 1$.

(2) Monotonicity: $\text{Cr}(A) \le \text{Cr}(B)$ for all $A, B \subset \Gamma$ with $A \subset B$.

(3) Self-duality: $\text{Cr}(A) + \text{Cr}(A^c) = 1$ for all $A \subset \Gamma$.

(4) Subadditivity: $\text{Cr}(A \cup B) \le \text{Cr}(A) + \text{Cr}(B)$ for all $A, B \subset \Gamma$.

An n-dimensional fuzzy vector is defined as a function from a credibility space $(\Gamma, \mathcal{A}, \text{Cr})$ to the set of n-dimensional vectors. If $n = 1$, then it is called a fuzzy variable [9]. More properties about credibility measure can be found in [21].

If $\xi = (r_1, r_2, r_3, r_4)$ is a trapezoidal fuzzy variable, then we have

$$\text{Cr}\{\xi \ge r\} = \begin{cases} 1, & \text{if } r \le r_1 \\ \frac{2r_2 - r_1 - r}{2(r_2 - r_1)}, & if\, r_1 \le r < r_2 \\ \frac{1}{2}, & if\, r_2 \le r < r_3 \\ \frac{r_4 - r}{2(r_4 - r_3)}, & if\, r_3 \le r < r_4 \\ 0, & \text{if } r_4 < r. \end{cases}$$

If ξ is a normal fuzzy variable with parameter (a, b) and its possibility distribution function is $\mu_\xi(x) = \exp(-\frac{(x-a)^2}{2b^2})$, $a \in R$, $b > 0$. Then we have

$$\text{Cr}\{\xi \ge r\} = \begin{cases} 1 - \frac{1}{2}\exp(-\frac{(r-a)^2}{2b^2}), & if\, a \ge r \\ \frac{1}{2}\exp(-\frac{(r-a)^2}{2b^2}), & if\, a < r. \end{cases}$$

Let $(\Omega, \Sigma, \Pr)$ be a probability space, and F_n^v a collection of n-ary fuzzy vectors. A map $\xi = (\xi_1, \cdots, \xi_n)^T$: $\Omega \to F_n^v$ is said to be an n-ary fuzzy random vector if for any Borel subset B of R^n, the function

$$\mathrm{Cr}\{\gamma \in \Gamma \mid \xi_\omega(\gamma) \in B\}$$

is measurable with respect to ω. As $n = 1$, ξ is called a fuzzy random variable [12].

In a word, a fuzzy random variable is a random variable with fuzzy values. For example, if

$$\xi = \begin{cases} (-2, 0, 2), & \text{with probability } \frac{1}{3} \\ (-1, 0, 1), & \text{with probability } \frac{1}{6} \\ (8, 9, 10), & \text{with probability } \frac{1}{2}, \end{cases}$$

then ξ is a discrete triangular fuzzy random variable. If

$$\xi = (X - 2, X - 1, X + 1, X + 2), X \sim \mathcal{N}(1, 2),$$

then ξ is a continuous trapezoidal fuzzy random variable.

Let ξ be a trapezoidal fuzzy random variable. Then for each ω, $\xi(\omega) = (X(\omega), X(\omega)+a, X(\omega)+b, X(\omega)+c)$ is a trapezoidal fuzzy variable with the following possibility distribution function

$$\mu_{\xi(\omega)}(x) = \begin{cases} \frac{x - X(\omega)}{a}, & if\, X(\omega) \le x < X(\omega) + a \\ 1, & if\, X(\omega) + a \le x < X(\omega) + b \\ \frac{-x + X(\omega) + c}{c - b}, & if\, X(\omega) + b \le x < X(\omega) + c \\ 0, & \text{otherwise}, \end{cases}$$

where $c > b > a > 0$, and X is a random variable.

Let ξ be a normal fuzzy random variable. Then for each ω, $\xi(\omega)$ is a normal fuzzy variable with the following possibility distribution function

$$\mu_{\xi(\omega)}(x) = \exp\left(-\frac{(x - X(\omega))^2}{2b^2}\right),$$

where $b > 0$, and X is a random variable.

Let ξ be an n-ary fuzzy random vector, and B a Borel subset of $\Re^n$. Then the mean chance of a fuzzy random event $\{\xi \in B\}$ is defined as [13]

$$\mathrm{Ch}\{\xi \in B\} = \int_\Omega \mathrm{Cr}\{\gamma \in \Gamma \mid \xi_\omega(\gamma) \in B\} \Pr(\mathrm{d}\omega).$$

3 Mean Chance Distributions for Trapezoidal Fuzzy Random Variables

Theorem 3.1 *Let ξ be a continuous trapezoidal fuzzy random variable such that $\xi(\omega) = (X(\omega), X(\omega) + a, X(\omega) + b, X(\omega) + c)$ with $c > b > a > 0$ for each ω, and X a random variable.*
(1) If $X \sim \mathcal{N}(\mu, \sigma^2)$ with probability distribution Φ, then we have

$$\begin{aligned} \mathrm{Ch}\{\xi \ge r\} = & -\tfrac{\sigma}{2a\sqrt{2\pi}}\left(\exp\left(-\tfrac{(r-\mu)^2}{2\sigma^2}\right) - \exp\left(-\tfrac{(a+\mu-r)^2}{2\sigma^2}\right)\right) \\ & -\tfrac{\sigma}{2(c-b)\sqrt{2\pi}}\left(\exp\left(-\tfrac{(b+\mu-r)^2}{2\sigma^2}\right) - \exp\left(-\tfrac{(c+\mu-r)^2}{2\sigma^2}\right)\right) \\ & +\tfrac{\mu-r}{2a}\Phi(\tfrac{r-\mu}{\sigma}) - \tfrac{\mu+a-r}{2a}\Phi(\tfrac{r-a-\mu}{\sigma}) + \tfrac{\mu+b-r}{2(c-b)}\Phi(\tfrac{r-b-\mu}{\sigma}) - \tfrac{\mu+c-r}{2(c-b)}\Phi(\tfrac{r-c-\mu}{\sigma}) + 1. \end{aligned}$$

(2) If $X \sim \mathcal{EXP}(\lambda)$, $\lambda > 0$, then we have

$$\mathrm{Ch}\{\xi \geq r\} = \begin{cases} \frac{1}{2a\lambda}(\exp(-\lambda(r-a) - \exp(-\lambda r)) \\ \quad + \frac{1}{2\lambda(c-b)}[\exp(-\lambda(r-c)) - \exp(-\lambda(r-b))], & \text{if } r > c \\ -\frac{1}{2(c-b)\lambda}\exp(-\lambda(r-b)) + \frac{1}{2a\lambda}(\exp(-\lambda(r-a)) - \exp(-r\lambda)) + \frac{1+c\lambda-r\lambda}{2\lambda(c-b)}, & \text{if } b < r \leq c \\ -\frac{1}{2a\lambda}\exp(-\lambda r) + \frac{1}{2a\lambda}\exp(-\lambda(r-a)) + \frac{1}{2}, & \text{if } a < r \leq b \\ -\frac{1}{2a\lambda}\exp(-r\lambda) + \frac{1}{2a\lambda}(1 + 2a\lambda - r\lambda), & \text{if } 0 < r \leq a \\ 1, & \text{if } r \leq 0. \end{cases}$$

(3) If $X \sim \mathcal{U}(l_1, l_2)$, $l_1 < l_2$, then we have

$$\mathrm{Ch}\{\xi \geq r\} = \begin{cases} \frac{(l_2-r+c)^2}{4(c-b)(l_2-l_1)}, & \text{if } l_1 \leq r-c < l_2 \leq r-b \\ \frac{c+b+2l_2-2r}{4(l_2-l_1)}, & \text{if } l_1 \leq r-c,\ r-b < l_2 \leq r-a \\ \frac{(l_2-r)^2+a(a+4l_2-4r+c+b)}{4a(l_2-l_1)}, & \text{if } l_1 \leq r-c,\ r-a < l_2 \leq r \\ \frac{a+b+c+4(l_2-r)}{4(l_2-l_1)}, & \text{if } l_1 \leq r-c,\ r < l_2 \\ \frac{l_1+l_2+2c-2r}{4(c-b)}, & \text{if } r-c < l_1 < l_2 \leq r-b \\ \frac{(r-b)(2c-b-r)-l_1(l_1+2c-2r)+2(c-b)(l_2-r+b)}{4(c-b)(l_2-l_1)}, & \text{if } r-c < l_1 \leq r-b,\ r-b < l_2 \leq r-a \\ \frac{(r-b)(2c-b-r)-l_1(l_1+2c-2r)}{4(c-b)(l_2-l_1)} + \frac{b-a}{2(l_2-l_1)} \\ \quad + \frac{(r-a)(r-3a)+l_2(l_2+4a-2r)}{4a(l_2-l_1)}, & \text{if } r-c < l_1 \leq r-b,\ r-a < l_2 \leq r \\ \frac{(r-b)(2c-b-r)-l_1(l_1+2c-2r)}{4(c-b)(l_2-l_1)} + \frac{2b+a}{4(l_2-l_1)} + \frac{l_2-r}{l_2-l_1}, & \text{if } r-c < l_1 \leq r-b,\ r < l_2 \\ \frac{1}{2}, & \text{if } r-b < l_1 < l_2 \leq r-a \\ \frac{(r-a)^2-2al_1+l_2(l_2+4a-2r)}{4a(l_2-l_1)}, & \text{if } r-b < l_1 \leq r-a,\ r-a < l_2 \leq r \\ \frac{a+4l_2-2l_1-2r}{4(l_2-l_1)}, & \text{if } r-b < l_1 \leq r-a,\ r < l_2 \\ \frac{4a-2r+l_1+l_2}{4a}, & \text{if } r-a < l_1 < l_2 \leq r \\ \frac{(r-l_1)(4a-r+l_1)+4a(l_2-r)}{4a(l_2-l_1)}, & \text{if } r-a < l_1 < l_2 \leq r \\ 1, & \text{if } r < l_1 < l_2. \end{cases}$$

Proof: According to the credibility measure of fuzzy event, for each ω, we have

$$\mathrm{Cr}\{\xi(\omega) \geq r\} = \begin{cases} 1, & \text{if } X(\omega) \geq r \\ \frac{X(\omega)+2a-r}{2a}, & \text{if } r-a \leq X(\omega) < r \\ \frac{1}{2}, & \text{if } r-b \leq X(\omega) < r-a \\ \frac{X(\omega)+c-r}{2(c-b)}, & \text{if } r-c \leq X(\omega) < r-b \\ 0, & \text{if } X(\omega) < r-c. \end{cases}$$

(1) If $X(\omega) \geq r$, then

$$M_1 = \int_r^{+\infty} \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right) dx = 1 - \Phi\left(\frac{r-\mu}{\sigma}\right).$$

If $r - a \le X(\omega) < r$, then

$$\begin{aligned}
M_2 &= \int_{r-a}^{r} \frac{x+2a-r}{2a} \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right) dx \\
&= \int_{r-a}^{r} \frac{(x-\mu)+(\mu+2a-r)}{2a} \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right) dx \\
&= \int_{r-a}^{r} \frac{(x-\mu)}{2a} \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right) dx + \int_{r-a}^{r} \frac{(\mu+2a-r)}{2a} \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right) dx \\
&= \int_{r-a}^{r} \frac{\sigma}{2a\sqrt{2\pi}} \frac{x-\mu}{\sigma^2} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right) dx + \frac{\mu+2a-r}{2a} \int_{\frac{r-a-\mu}{\sigma}}^{\frac{r-\mu}{\sigma}} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right) d\left(\frac{x-\mu}{\sigma}\right) \\
&= -\frac{\sigma}{2a\sqrt{2\pi}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right) \Big|_{r-a}^{r} + \frac{\mu+2a-r}{2a}\left[\Phi\left(\frac{r-\mu}{\sigma}\right) - \Phi\left(\frac{r-a-\mu}{\sigma}\right)\right] \\
&= -\frac{\sigma}{2a\sqrt{2\pi}} \left(\exp\left(-\frac{(r-\mu)^2}{2\sigma^2}\right) - \exp\left(-\frac{(r-a-\mu)^2}{2\sigma^2}\right)\right) + \frac{\mu+2a-r}{2a}\left[\Phi\left(\frac{r-\mu}{\sigma}\right) - \Phi\left(\frac{r-a-\mu}{\sigma}\right)\right].
\end{aligned}$$

If $r - b \le X(\omega) < r - a$, then

$$M_3 = \int_{r-b}^{r-a} \frac{1}{2} \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right) dx = \frac{1}{2}\left(\Phi\left(\frac{r-a-\mu}{\sigma}\right) - \Phi\left(\frac{r-b-\mu}{\sigma}\right)\right).$$

If $r - c \le X(\omega) < r - b$, then

$$\begin{aligned}
M_4 &= \int_{r-c}^{r-b} \frac{x+c-r}{2(c-b)} \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right) dx \\
&= \int_{r-c}^{r-b} \frac{(x-\mu)+(\mu+c-r)}{2(c-b)} \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right) dx \\
&= \int_{r-c}^{r-b} \frac{(x-\mu)}{2(c-b)} \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right) dx + \int_{r-c}^{r-b} \frac{(\mu+c-r)}{2(c-b)} \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right) dx \\
&= \int_{r-c}^{r-b} \frac{\sigma}{2(c-b)\sqrt{2\pi}} \frac{x-\mu}{\sigma^2} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right) dx + \frac{\mu+c-r}{2(c-b)} \int_{\frac{r-c-\mu}{\sigma}}^{\frac{r-b-\mu}{\sigma}} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right) d\left(\frac{x-\mu}{\sigma}\right) \\
&= -\frac{\sigma}{2(c-b)\sqrt{2\pi}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right) \Big|_{r-c}^{r-b} + \frac{\mu+c-r}{2(c-b)}\left(\Phi\left(\frac{r-b-\mu}{\sigma}\right) - \Phi\left(\frac{r-c-\mu}{\sigma}\right)\right) \\
&= -\frac{\sigma}{2(c-b)\sqrt{2\pi}} \left(\exp\left(-\frac{(r-b-\mu)^2}{2\sigma^2}\right) - \exp\left(-\frac{(r-c-\mu)^2}{2\sigma^2}\right)\right) + \frac{\mu+c-r}{2(c-b)}\left(\Phi\left(\frac{r-b-\mu}{\sigma}\right) - \Phi\left(\frac{r-c-\mu}{\sigma}\right)\right).
\end{aligned}$$

Therefore, we have

$$\begin{aligned}
\mathrm{Ch}\{\xi \ge r\} &= M_1 + M_2 + M_3 \\
&= -\frac{\sigma}{2a\sqrt{2\pi}}\left(\exp\left(-\frac{(r-\mu)^2}{2\sigma^2}\right) - \exp\left(-\frac{(a+\mu-r)^2}{2\sigma^2}\right)\right) \\
&\quad -\frac{\sigma}{2(c-b)\sqrt{2\pi}}\left(\exp\left(-\frac{(b+\mu-r)^2}{2\sigma^2}\right) - \exp\left(-\frac{(c+\mu-r)^2}{2\sigma^2}\right)\right) \\
&\quad +\frac{\mu-r}{2a}\Phi\left(\frac{r-\mu}{\sigma}\right) - \frac{\mu+a-r}{2a}\Phi\left(\frac{r-a-\mu}{\sigma}\right) + \frac{\mu+b-r}{2(c-b)}\Phi\left(\frac{r-b-\mu}{\sigma}\right) - \frac{\mu+c-r}{2(c-b)}\Phi\left(\frac{r-c-\mu}{\sigma}\right) + 1.
\end{aligned}$$

(2) Case 1: $r > c$.
If $X(\omega) \ge r$, then we have

$$M_1 = \int_r^{+\infty} \lambda \exp(-\lambda x) dx = \exp(-\lambda r).$$

If $r - a \le X(\omega) < r$, then we have

$$\begin{aligned}
M_2 &= \int_{r-a}^{r} \frac{x+2a-r}{2a} \lambda \exp(-\lambda x) dx \\
&= \frac{1}{2a}\int_{r-a}^{r} \lambda x \ \exp(-\lambda x) dx + \frac{2a-r}{2a}\int_{r-a}^{r} \lambda \exp(-\lambda x) dx \\
&= -\left(\frac{1}{2a}x + \frac{1}{2a\lambda}\right)\exp(-\lambda x)\Big|_{r-a}^{r} - \frac{2a-r}{2a}\exp(-\lambda x)\Big|_{r-a}^{r} \\
&= \frac{a\lambda+1}{2a\lambda}\exp(-\lambda(r-a)) - \frac{2a\lambda+1}{2a\lambda}\exp(-\lambda r).
\end{aligned}$$

If $r - b < X(\omega) \le r - a$, then we have

$$\begin{aligned}
M_3 &= \int_{r-b}^{r-a} \frac{1}{2}\lambda \exp(-\lambda x) dx \\
&= -\exp(-\lambda x)\Big|_{r-b}^{r-a} \\
&= \frac{1}{2}\exp(-\lambda(r-b)) - \frac{1}{2}\exp(-\lambda(r-a)).
\end{aligned}$$

If $r-c \le X(\omega) < r-b$, then we have

$$\begin{aligned}M_4 &= \int_{r-c}^{r-b} \tfrac{x+c-r}{2(c-b)}\lambda\exp(-\lambda x)dx\\ &= \tfrac{1}{2(c-b)}\int_{r-c}^{r-b}\lambda x \quad \exp(-\lambda x)dx + \tfrac{c-r}{2(c-b)}\int_{r-c}^{r-b}\lambda\exp(-\lambda x)dx\\ &= -(\tfrac{1}{2(c-b)}x + \tfrac{1}{2(c-b)\lambda})\exp(-\lambda x)|_{r-c}^{r-b} - \tfrac{c-r}{2(c-b)}\exp(-\lambda x)|_{r-c}^{r-b}\\ &= (\tfrac{1}{2(c-b)}(r-c) + \tfrac{1}{2(c-b)\lambda})\exp(-\lambda(r-c)) - (\tfrac{1}{2(c-b)}(r-b)+\tfrac{1}{2(c-b)\lambda})\exp(-\lambda(r-b))\\ &\quad -\tfrac{c-r}{2(c-b)}(\exp(-\lambda(r-b)) - \exp(-\lambda(r-c)))\\ &= \tfrac{1}{2(c-b)\lambda}\exp(-\lambda(r-c)) - (\tfrac{1}{2}+\tfrac{1}{2(c-b)\lambda})\exp(-\lambda(r-b)).\end{aligned}$$

Combining the above, we have

$$\begin{aligned}\text{Ch}\{\xi\ge r\} &= M_1+M_2+M_3+M_4\\ &= \tfrac{1}{2a\lambda}(\exp(-\lambda(r-a) - \exp(-\lambda r))) + \tfrac{1}{2\lambda(c-b)}[\exp(-\lambda(r-c)) - \exp(-\lambda(r-b))].\end{aligned}$$

Case 2: $b < r \le c$.
If $X(\omega) \ge r$, then we have

$$M_1 = \int_r^{+\infty}\lambda\exp(-\lambda x)dx = \exp(-\lambda r).$$

If $r-a \le X(\omega) < r$, then we have

$$\begin{aligned}M_2 &= \int_{r-a}^{r}\tfrac{x+2a-r}{2a}\lambda\exp(-\lambda x)dx\\ &= \tfrac{1}{2a}\int_{r-a}^{r}\lambda x \quad \exp(-\lambda x)dx + \tfrac{2a-r}{2a}\int_{r-a}^{r}\lambda\exp(-\lambda x)dx\\ &= -(\tfrac{1}{2a}x+\tfrac{1}{2a\lambda})\exp(-\lambda x)|_{r-a}^{r} - \tfrac{2a-r}{2a}\exp(-\lambda x)|_{r-a}^{r}\\ &= \tfrac{a\lambda+1}{2a\lambda}\exp(-\lambda(r-a)) - \tfrac{2a\lambda+1}{2a\lambda}\exp(-\lambda r).\end{aligned}$$

If $r-b < X(\omega) \le r-a$, then we have

$$\begin{aligned}M_3 &= \int_{r-b}^{r-a}\tfrac{1}{2}\lambda\exp(-\lambda x)dx\\ &= -\exp(-\lambda x)|_{r-b}^{r-a}\\ &= \tfrac{1}{2}\exp(-\lambda(r-b)) - \tfrac{1}{2}\exp(-\lambda(r-a)).\end{aligned}$$

If $r-c \le X(\omega) < r-b$, then we have

$$\begin{aligned}M_4 &= \int_0^{r-b}\tfrac{x+c-r}{2(c-b)}\lambda\exp(-\lambda x)dx\\ &= \tfrac{1}{2(c-b)}\int_0^{r-b}\lambda x \quad \exp(-\lambda x)dx + \tfrac{c-r}{2(c-b)}\int_0^{r-b}\lambda\exp(-\lambda x)dx\\ &= -(\tfrac{1}{2(c-b)}x+\tfrac{1}{2(c-b)\lambda})\exp(-\lambda x)|_0^{r-b} - \tfrac{c-r}{2(c-b)}\exp(-\lambda x)|_0^{r-b}\\ &= \tfrac{1}{2(c-b)\lambda} - (\tfrac{1}{2(c-b)}(r-b)+\tfrac{1}{2(c-b)\lambda})\exp(-\lambda(r-b)) - \tfrac{c-r}{2(c-b)}\exp(-\lambda(r-b))\\ &= \tfrac{1}{2(c-b)\lambda} + \tfrac{c-r}{2(c-b)} - (\tfrac{1}{2}+\tfrac{1}{2(c-b)\lambda})\exp(-\lambda(r-b)).\end{aligned}$$

Combining the above, we have

$$\begin{aligned}\text{Ch}\{\xi\ge r\} &= M_1+M_2+M_3+M_4\\ &= -\tfrac{1}{2(c-b)\lambda}\exp(-\lambda(r-b)) + \tfrac{1}{2a\lambda}(\exp(-\lambda(r-a)) - \exp(-r\lambda)) + \tfrac{1+c\lambda-r\lambda}{2\lambda(c-b)}.\end{aligned}$$

Case 3: $a < r \le b$.
If $X(\omega) \ge r$, then we have

$$M_1 = \int_r^{+\infty}\lambda\exp(-\lambda x)dx = \exp(-\lambda r).$$

If $r-a \le X(\omega) < r$, then we have

$$\begin{aligned}M_2 &= \int_{r-a}^{r}\tfrac{x+2a-r}{2a}\lambda\exp(-\lambda x)dx\\ &= \tfrac{1}{2a}\int_{r-a}^{r}\lambda x \quad \exp(-\lambda x)dx + \tfrac{2a-r}{2a}\int_{r-a}^{r}\lambda\exp(-\lambda x)dx\\ &= -(\tfrac{1}{2a}x+\tfrac{1}{2a\lambda})\exp(-\lambda x)|_{r-a}^{r} - \tfrac{2a-r}{2a}\exp(-\lambda x)|_{r-a}^{r}\\ &= \tfrac{a\lambda+1}{2a\lambda}\exp(-\lambda(r-a)) - \tfrac{2a\lambda+1}{2a\lambda}\exp(-\lambda r).\end{aligned}$$

If $r-b<X(\omega)\le r-a$, then we have

$$\begin{aligned}M_3 &= \int_0^{r-a}\tfrac{1}{2}\lambda\exp(-\lambda x)dx\\ &= \tfrac{1}{2}(-\exp(-\lambda x))|_0^{r-a}\\ &= \tfrac{1}{2}-\tfrac{1}{2}\exp(-\lambda(r-a)).\end{aligned}$$

Combining the above, we have

$$\begin{aligned}\mathrm{Ch}\{\xi\ge r\} &= M_1+M_2+M_3\\ &= -\tfrac{1}{2a\lambda}\exp(-\lambda r)+\tfrac{1}{2a\lambda}\exp(-\lambda(r-a))+\tfrac{1}{2}.\end{aligned}$$

Case 4: $0<r\le a$.
If $X(\omega)\ge r$, then we have

$$M_1=\int_r^{+\infty}\lambda\exp(-\lambda x)dx=\exp(-\lambda r).$$

If $r-a\le X(\omega)<r$, then we have

$$\begin{aligned}M_2 &= \int_0^r \tfrac{x+2a-r}{2a}\lambda\exp(-\lambda x)dx\\ &= \tfrac{1}{2a}\int_0^r\lambda x\ \ \exp(-\lambda x)dx+\tfrac{2a-r}{2a}\int_0^r\lambda\exp(-\lambda x)dx\\ &= -(\tfrac{1}{2a}x+\tfrac{1}{2a\lambda})\exp(-\lambda x)|_0^r-\tfrac{2a-r}{2a}\exp(-\lambda x)|_0^r\\ &= -(1+\tfrac{1}{2a\lambda})\exp(-\lambda r)+\tfrac{2a\lambda+1-r\lambda}{2a\lambda}.\end{aligned}$$

Combining the above, we have

$$\begin{aligned}\mathrm{Ch}\{\xi\ge r\} &= M_1+M_2\\ &= -\tfrac{1}{2a\lambda}\exp(-r\lambda)+\tfrac{1}{2a\lambda}(1+2a\lambda-r\lambda).\end{aligned}$$

Case 5: $r\le 0$.
If $X(\omega)\ge r$, then we have

$$M=M_1=\int_0^{+\infty}\lambda\exp(-\lambda x)dx=1.$$

(3) Case 1: $l_1\le r-c<l_2\le r-b$.

$$\begin{aligned}\mathrm{Ch}\{\xi\ge r\} &= \int_{r-c}^{l_2}\tfrac{x+c-r}{2(c-b)}\tfrac{1}{l_2-l_1}dx\\ &= \tfrac{1}{l_2-l_1}\left(\tfrac{x^2}{4(c-b)}+\tfrac{(c-r)x}{2(c-b)}\right)\Big|_{r-c}^{l_2}\\ &= \tfrac{1}{l_2-l_1}\left(\tfrac{l_2^2}{4(c-b)}+\tfrac{(c-r)l_2}{2(c-b)}-\tfrac{(r-c)^2}{4(c-b)}+\tfrac{(c-r)(r-c)}{2(c-b)}\right)\\ &= \tfrac{(l_2+c-r)^2}{4(c-b)(l_2-l_1)}.\end{aligned}$$

Case 2: $l_1\le r-c$, $r-b<l_2\le r-a$.

$$\begin{aligned}\mathrm{Ch}\{\xi\ge r\} &= \int_{r-b}^{l_2}\tfrac{1}{2}\tfrac{1}{l_2-l_1}dx+\int_{r-c}^{r-b}\tfrac{x+c-r}{2(c-b)}\tfrac{1}{l_2-l_1}dx\\ &= \tfrac{l_2-r+b}{2(l_2-l_1)}+\left(\tfrac{x^2}{4(c-b)}\tfrac{1}{l_2-l_1}+\tfrac{1}{l_2-l_1}\tfrac{(c-r)x}{2(c-b)}\right)\Big|_{r-c}^{r-b}\\ &= \tfrac{l_2-r+b}{2(l_2-l_1)}+\tfrac{c-b}{4(l_2-l_1)}\\ &= \tfrac{b+c+2l_2-2r}{4(l_2-l_1)}.\end{aligned}$$

Case 3: $l_1\le r-c$, $r-a<l_2\le r$.

$$\begin{aligned}\mathrm{Ch}\{\xi\ge r\} &= \int_{r-b}^{r-a}\tfrac{1}{2}\tfrac{1}{l_2-l_1}dx+\int_{r-c}^{r-b}\tfrac{x+c-r}{2(c-b)}\tfrac{1}{l_2-l_1}dx+\int_{r-a}^{l_2}\tfrac{x+2a-r}{2a}\tfrac{1}{l_2-l_1}dx\\ &= \tfrac{r-a-r+b}{2(l_2-l_1)}+\left(\tfrac{x^2}{4(c-b)}\tfrac{1}{l_2-l_1}+\tfrac{1}{l_2-l_1}\tfrac{(c-r)x}{2(c-b)}\right)\Big|_{r-c}^{r-b}+\left(\tfrac{x^2}{4a(l_2-l_1)}+\tfrac{1}{l_2-l_1}\tfrac{(2a-r)x}{2a}\right)\Big|_{r-a}^{l_2}\\ &= \tfrac{b-a}{2(l_2-l_1)}+\left(\tfrac{(r-c)^2}{4(c-b)}\tfrac{1}{l_2-l_1}+\tfrac{1}{l_2-l_1}\tfrac{(c-r)(r-c)}{2(c-b)}\right)-\left(\tfrac{(r-b)^2}{4(c-b)}\tfrac{1}{l_2-l_1}+\tfrac{1}{l_2-l_1}\tfrac{(c-r)(r-b)}{2(c-b)}\right)\\ &\quad+\left(\tfrac{l_2^2}{4a(l_2-l_1)}+\tfrac{1}{l_2-l_1}\tfrac{(2a-r)l_2}{2a}\right)-\left(\tfrac{(r-a)^2}{4a(l_2-l_1)}+\tfrac{1}{l_2-l_1}\tfrac{(2a-r)(r-a)}{2a}\right)\\ &= \tfrac{b-a}{2(l_2-l_1)}+\tfrac{c-b}{4(l_2-l_1)}+\tfrac{(l_2-r)^2+4a(l_2-r)+3a^2}{4a(l_2-l_1)}\\ &= \tfrac{(l_2-r)^2+a(4l_2-4r+a+b+c)}{4a(l_2-l_1)}.\end{aligned}$$

Case 4: $l_1 \le r-c$, $r < l_2$.

$$\begin{aligned}
\mathrm{Ch}\{\xi \ge r\} &= \textstyle\int_{r-b}^{r-a} \frac{1}{2}\frac{1}{l_2-l_1}dx + \int_{r-c}^{r-b} \frac{x+c-r}{2(c-b)}\frac{1}{l_2-l_1}dx + \int_{r-a}^{r} \frac{x+2a-r}{2a}\frac{1}{l_2-l_1}dx + \int_{r}^{l_2} \frac{1}{l_2-l_1}dx \\
&= \tfrac{r-a-r+b}{2(l_2-l_1)} + \left(\tfrac{x^2}{4(c-b)}\tfrac{1}{l_2-l_1} + \tfrac{1}{l_2-l_1}\tfrac{(c-r)x}{2(c-b)}\right)\Big|_{r-c}^{r-b} + \left(\tfrac{x^2}{4a(l_2-l_1)} + \tfrac{1}{l_2-l_1}\tfrac{(2a-r)x}{2a}\right)\Big|_{r-a}^{r} + \tfrac{l_2-r}{l_2-l_1} \\
&= \tfrac{b-a}{2(l_2-l_1)} + \left(\tfrac{(r-c)^2}{4(c-b)}\tfrac{1}{l_2-l_1} + \tfrac{1}{l_2-l_1}\tfrac{(c-r)(r-c)}{2(c-b)}\right) - \left(\tfrac{(r-b)^2}{4(c-b)}\tfrac{1}{l_2-l_1} + \tfrac{1}{l_2-l_1}\tfrac{(c-r)(r-b)}{2(c-b)}\right) \\
&\quad + \left(\tfrac{r^2}{4a(l_2-l_1)} + \tfrac{1}{l_2-l_1}\tfrac{(2a-r)r}{2a}\right) - \left(\tfrac{(r-a)^2}{4a(l_2-l_1)} + \tfrac{1}{l_2-l_1}\tfrac{(2a-r)(r-a)}{2a}\right) + \tfrac{l_2-r}{l_2-l_1} \\
&= \tfrac{b-a}{2(l_2-l_1)} + \tfrac{c-b}{4(l_2-l_1)} + \tfrac{3a}{4(l_2-l_1)} + \tfrac{l_2-r}{l_2-l_1} \\
&= \tfrac{a+b+c+4(l_2-r)}{4(l_2-l_1)}.
\end{aligned}$$

Case 5: $r-c < l_1 < l_2 \le r-b$.

$$\begin{aligned}
\mathrm{Ch}\{\xi \ge r\} &= \textstyle\int_{l_1}^{l_2} \frac{x+c-r}{2(c-b)}\frac{1}{l_2-l_1}dx \\
&= \left(\tfrac{x^2}{4(c-b)}\tfrac{1}{l_2-l_1} + \tfrac{1}{l_2-l_1}\tfrac{(c-r)x}{2(c-b)}\right)\Big|_{l_1}^{l_2} \\
&= \left(\tfrac{l_2^2}{4(c-b)}\tfrac{1}{l_2-l_1} + \tfrac{1}{l_2-l_1}\tfrac{(c-r)l_2}{2(c-b)}\right) - \left(\tfrac{l_1^2}{4(c-b)}\tfrac{1}{l_2-l_1} + \tfrac{1}{l_2-l_1}\tfrac{(c-r)l_1}{2(c-b)}\right) \\
&= \tfrac{l_1+l_2+2(c-r)}{4(c-b)}.
\end{aligned}$$

Case 6: $r-c < l_1 \le r-b < l_2 \le r-a$.

$$\begin{aligned}
\mathrm{Ch}\{\xi \ge r\} &= \textstyle\int_{l_1}^{r-b} \frac{x+c-r}{2(c-b)}\frac{1}{l_2-l_1}dx + \int_{r-b}^{l_2} \frac{1}{2}\frac{1}{l_2-l_1}dx \\
&= \left(\tfrac{x^2}{4(c-b)}\tfrac{1}{l_2-l_1} + \tfrac{1}{l_2-l_1}\tfrac{(c-r)x}{2(c-b)}\right)\Big|_{l_1}^{r-b} + \tfrac{l_2-r+b}{2(l_2-l_1)} \\
&= \left(\tfrac{(r-b)^2}{4(c-b)}\tfrac{1}{l_2-l_1} + \tfrac{1}{l_2-l_1}\tfrac{(c-r)(r-b)}{2(c-b)}\right) - \left(\tfrac{l_1^2}{4(c-b)}\tfrac{1}{l_2-l_1} + \tfrac{1}{l_2-l_1}\tfrac{(c-r)l_1}{2(c-b)}\right) + \tfrac{l_2-r+b}{2(l_2-l_1)} \\
&= \tfrac{(r-b)(2c-b-r)+l_1(2r-2c-l_1)}{4(c-b)(l_2-l_1)} + \tfrac{l_2-r+b}{2(l_2-l_1)} \\
&= \tfrac{(r-b)(2c-b-r)-l_1(l_1+2c-2r)+2(c-b)(l_2-r+b)}{4(c-b)(l_2-l_1)}.
\end{aligned}$$

Case 7: $r-c < l_1 \le r-b$, $r-a < l_2 \le r$.

$$\begin{aligned}
\mathrm{Ch}\{\xi \ge r\} &= \textstyle\int_{l_1}^{r-b} \frac{x+c-r}{2(c-b)}\frac{1}{l_2-l_1}dx + \int_{r-b}^{r-a} \frac{1}{2}\frac{1}{l_2-l_1}dx + \int_{r-a}^{l_2} \frac{x+2a-r}{2a}\frac{1}{l_2-l_1}dx \\
&= \left(\tfrac{x^2}{4(c-b)}\tfrac{1}{l_2-l_1} + \tfrac{1}{l_2-l_1}\tfrac{(c-r)x}{2(c-b)}\right)\Big|_{l_1}^{r-b} + \tfrac{r-a-r+b}{2(l_2-l_1)} + \left(\tfrac{x^2}{4a(l_2-l_1)} + \tfrac{1}{l_2-l_1}\tfrac{(2a-r)x}{2a}\right)\Big|_{r-a}^{l_2} \\
&= \left(\tfrac{(r-b)^2}{4(c-b)}\tfrac{1}{l_2-l_1} + \tfrac{1}{l_2-l_1}\tfrac{(c-r)(r-b)}{2(c-b)}\right) - \left(\tfrac{l_1^2}{4(c-b)}\tfrac{1}{l_2-l_1} + \tfrac{1}{l_2-l_1}\tfrac{(c-r)l_1}{2(c-b)}\right) + \tfrac{b-a}{2(l_2-l_1)} \\
&\quad + \left(\tfrac{l_2^2}{4a(l_2-l_1)} + \tfrac{1}{l_2-l_1}\tfrac{(2a-r)l_2}{2a}\right) - \left(\tfrac{(r-a)^2}{4a(l_2-l_1)} + \tfrac{1}{l_2-l_1}\tfrac{(2a-r)(r-a)}{2a}\right) \\
&= \tfrac{(r-b)(2c-b-r)+l_1(2r-2c-l_1)}{4(c-b)(l_2-l_1)} + \tfrac{b-a}{2(l_2-l_1)} + \tfrac{(l_2-r)^2+4a(l_2-r)+3a^2}{4a(l_2-l_1)} \\
&= \tfrac{(r-b)(2c-b-r)-l_1(l_1+2c-2r)}{4(c-b)(l_2-l_1)} + \tfrac{b-a}{2(l_2-l_1)} + \tfrac{(r-a)(r-3a)+l_2(l_2+4a-2r)}{4a(l_2-l_1)}.
\end{aligned}$$

Case 8: $r-c < l_1 \le r-b$, $r < l_2$.

$$\begin{aligned}
\mathrm{Ch}\{\xi \ge r\} &= \textstyle\int_{l_1}^{r-b} \frac{x+c-r}{2(c-b)}\frac{1}{l_2-l_1}dx + \int_{r-b}^{r-a} \frac{1}{2}\frac{1}{l_2-l_1}dx + \int_{r-a}^{r} \frac{x+2a-r}{2a}\frac{1}{l_2-l_1}dx + \int_{r}^{l_2} \frac{1}{l_2-l_1}dx \\
&= \left(\tfrac{x^2}{4(c-b)}\tfrac{1}{l_2-l_1} + \tfrac{1}{l_2-l_1}\tfrac{(c-r)x}{2(c-b)}\right)\Big|_{l_1}^{r-b} + \tfrac{r-a-r+b}{2(l_2-l_1)} + \left(\tfrac{x^2}{4a(l_2-l_1)} + \tfrac{1}{l_2-l_1}\tfrac{(2a-r)x}{2a}\right)\Big|_{r-a}^{r} + \tfrac{l_2-r}{l_2-l_1} \\
&= \left(\tfrac{(r-b)^2}{4(c-b)}\tfrac{1}{l_2-l_1} + \tfrac{1}{l_2-l_1}\tfrac{(c-r)(r-b)}{2(c-b)}\right) - \left(\tfrac{l_1^2}{4(c-b)}\tfrac{1}{l_2-l_1} + \tfrac{1}{l_2-l_1}\tfrac{(c-r)l_1}{2(c-b)}\right) + \tfrac{b-a}{2(l_2-l_1)} \\
&\quad + \left(\tfrac{r^2}{4a(l_2-l_1)} + \tfrac{1}{l_2-l_1}\tfrac{(2a-r)r}{2a}\right) - \left(\tfrac{(r-a)^2}{4a(l_2-l_1)} + \tfrac{1}{l_2-l_1}\tfrac{(2a-r)(r-a)}{2a}\right) + \tfrac{l_2-r}{l_2-l_1} \\
&= \tfrac{(r-b)(2c-b-r)+l_1(2r-2c-l_1)}{4(c-b)(l_2-l_1)} + \tfrac{b-a}{2(l_2-l_1)} + \tfrac{3a}{4(l_2-l_1)} + \tfrac{l_2-r}{l_2-l_1} \\
&= \tfrac{(r-b)(2c-b-r)-l_1(l_1+2c-2r)}{4(c-b)(l_2-l_1)} + \tfrac{2b+a}{4(l_2-l_1)} + \tfrac{l_2-r}{l_2-l_1}.
\end{aligned}$$

Case 9: $r-b < l_1 < l_2 \le r-a$.

$$\mathrm{Ch}\{\xi \ge r\} = \int_{l_1}^{l_2} \frac{1}{2}\frac{1}{l_2-l_1}dx = \frac{1}{2}.$$

Case 10: $r-b<l_1\le r-a<l_2\le r$.

$$\begin{aligned}\mathrm{Ch}\{\xi\ge r\} &= \int_{l_1}^{r-a}\tfrac{1}{2}\tfrac{1}{l_2-l_1}dx+\int_{r-a}^{l_2}\tfrac{x+2a-r}{2a}\tfrac{1}{l_2-l_1}dx\\ &= \tfrac{l_1-r+a}{2(l_2-l_1)}+\left(\tfrac{x^2}{4a(l_2-l_1)}+\tfrac{1}{l_2-l_1}\tfrac{(2a-r)x}{2a}\right)\Big|_{r-a}^{l_2}\\ &= \tfrac{l_1-r+a}{2(l_2-l_1)}+\left(\tfrac{l_2^2}{4a(l_2-l_1)}+\tfrac{1}{l_2-l_1}\tfrac{(2a-r)l_2}{2a}\right)-\left(\tfrac{(r-a)^2}{4a(l_2-l_1)}+\tfrac{1}{l_2-l_1}\tfrac{(2a-r)(r-a)}{2a}\right)\\ &= \tfrac{(r-a)^2-2al_1+l_2(l_2+4a-2r)}{4a(l_2-l_1)}.\end{aligned}$$

Case 11: $r-b<l_1\le r-a$, $r<l_2$.

$$\begin{aligned}\mathrm{Ch}\{\xi\ge r\} &= \int_{l_1}^{r-a}\tfrac{1}{2}\tfrac{1}{l_2-l_1}dx+\int_{r-a}^{r}\tfrac{x+2a-r}{2a}\tfrac{1}{l_2-l_1}dx+\int_{r}^{l_2}\tfrac{1}{l_2-l_1}dx\\ &= \tfrac{l_1-r+a}{2(l_2-l_1)}+\left(\tfrac{x^2}{4a(l_2-l_1)}+\tfrac{1}{l_2-l_1}\tfrac{(2a-r)x}{2a}\right)\Big|_{r-a}^{r}+\tfrac{l_2-r}{l_2-l_1}\\ &= \tfrac{l_1-r+a}{2(l_2-l_1)}+\left(\tfrac{r^2}{4a(l_2-l_1)}+\tfrac{1}{l_2-l_1}\tfrac{(2a-r)r}{2a}\right)-\left(\tfrac{(r-a)^2}{4a(l_2-l_1)}+\tfrac{1}{l_2-l_1}\tfrac{(2a-r)(r-a)}{2a}\right)+\tfrac{l_2-r}{l_2-l_1}\\ &= \tfrac{l_1-r+a}{2(l_2-l_1)}+\tfrac{3a}{4(l_2-l_1)}+\tfrac{l_2-r}{l_2-l_1}\\ &= \tfrac{a+4l_2-2l_1-2r}{4(l_2-l_1)}.\end{aligned}$$

Case 12: $r-a<l_1<l_2\le r$.

$$\begin{aligned}\mathrm{Ch}\{\xi\ge r\} &= \int_{l_1}^{l_2}\tfrac{x+2a-r}{2a}\tfrac{1}{l_2-l_1}dx\\ &= \left(\tfrac{x^2}{4a(l_2-l_1)}+\tfrac{1}{l_2-l_1}\tfrac{(2a-r)x}{2a}\right)\Big|_{l_1}^{l_2}\\ &= \left(\tfrac{l_2^2}{4a(l_2-l_1)}+\tfrac{1}{l_2-l_1}\tfrac{(2a-r)l_2}{2a}\right)-\left(\tfrac{(l_1^2}{4a(l_2-l_1)}+\tfrac{1}{l_2-l_1}\tfrac{(2a-r)l_1}{2a}\right)\\ &= \tfrac{l_1+l_2+4a-2r}{4a}.\end{aligned}$$

Case 13: $r-a<l_1\le r<l_2$.

$$\begin{aligned}\mathrm{Ch}\{\xi\ge r\} &= \int_{l_1}^{r}\tfrac{x+2a-r}{2a}\tfrac{1}{l_2-l_1}dx+\int_{r}^{l_2}\tfrac{1}{l_2-l_1}dx\\ &= \left(\tfrac{x^2}{4a(l_2-l_1)}+\tfrac{1}{l_2-l_1}\tfrac{(2a-r)x}{2a}\right)\Big|_{l_1}^{r}+\tfrac{l_2-r}{l_2-l_1}\\ &= \left(\tfrac{r^2}{4a(l_2-l_1)}+\tfrac{1}{l_2-l_1}\tfrac{(2a-r)r}{2a}\right)-\left(\tfrac{(l_1^2}{4a(l_2-l_1)}+\tfrac{1}{l_2-l_1}\tfrac{(2a-r)l_1}{2a}\right)+\tfrac{l_2-r}{l_2-l_1}\\ &= \tfrac{4ar-4al_1-(r-l_1)^2-2r}{4a(l_2-l_1)}+\tfrac{l_2-r}{l_2-l_1}\\ &= \tfrac{(r-l_1)(4a-r+l_1)+4a(l_2-r)}{4a(l_2-l_1)}.\end{aligned}$$

Case 14: $r<l_1<l_2$.

$$\mathrm{Ch}\{\xi\ge r\}=\int_{l_1}^{l_2}\frac{1}{l_2-l_1}dx=1.$$

The proof is complete.

Theorem 3.2 *Let $\xi_1,\cdots,\xi_n$ be fuzzy random variables such that for each ω, $\xi_i(\omega)=(X_i(\omega),X_i(\omega)+a_i,X_i(\omega)+b_i),X_i(\omega)+c_i)$, $i=1,\cdots,n$ are mutually independent fuzzy variables. If $X_i\sim\mathcal{N}(\mu_i,\sigma_i^2)$ with probability distribution Φ, and $c_i>b_i>a_i$, $i=1,\cdots,n$ are positive numbers. Then we have*

$$\begin{aligned}\mathrm{Ch}\{\textstyle\sum_{i=1}^n x_i\xi_i\ge r\} &= -\tfrac{\sqrt{\sum_{i=1}^n x_i^2\sigma_i^2}}{2\sum_{i=1}^n x_ia_i\sqrt{2\pi}}\left(\exp\left(-\tfrac{(\sum_{i=1}^n x_i\mu_i-r)^2}{2\sum_{i=1}^n x_i^2\sigma_i^2}\right)-\exp\left(-\tfrac{(\sum_{i=1}^n x_i(a_i+\mu_i)-r)^2}{2\sum_{i=1}^n x_i^2\sigma_i^2}\right)\right)\\ &\quad-\tfrac{\sqrt{\sum_{i=1}^n x_i^2\sigma_i^2}}{2\sum_{i=1}^n x_i(c_i-b_i)\sqrt{2\pi}}\left(\exp\left(-\tfrac{(\sum_{i=1}^n x_i(b_i+\mu_i)-r)^2}{2\sum_{i=1}^n x_i^2\sigma_i^2}\right)-\exp\left(-\tfrac{(\sum_{i=1}^n x_i(c_i+\mu_i)-r)^2}{2\sum_{i=1}^n x_i^2\sigma_i^2}\right)\right)\\ &\quad+\tfrac{\sum_{i=1}^n x_i\mu_i-r}{2\sum_{i=1}^n x_ia_i}\Phi\left(\tfrac{r-\sum_{i=1}^n x_i\mu_i}{\sqrt{\sum_{i=1}^n x_i^2\sigma_i^2}}\right)-\tfrac{\sum_{i=1}^n x_i(\mu_i+a_i)-r}{2\sum_{i=1}^n x_ia_i}\Phi\left(\tfrac{r-\sum_{i=1}^n x_i(a_i+\mu_i)}{\sqrt{\sum_{i=1}^n x_i^2\sigma_i^2}}\right)\\ &\quad+\tfrac{\sum_{i=1}^n x_i(\mu_i+b_i)-r}{2\sum_{i=1}^n x_i(c_i-b_i)}\Phi\left(\tfrac{r-\sum_{i=1}^n x_i(b_i+\mu_i)}{\sqrt{\sum_{i=1}^n x_i^2\sigma_i^2}}\right)+\tfrac{\sum_{i=1}^n x_i(\mu_i+c_i)-r}{2\sum_{i=1}^n x_i(c_i-b_i)}\Phi\left(\tfrac{r-\sum_{i=1}^n x_i(c_i+\mu_i)}{\sqrt{\sum_{i=1}^n x_i^2\sigma_i^2}}\right).\end{aligned}$$

Proof: Let $\xi=\sum_{i=1}^n x_i\xi_i$, $Z=\sum_{i=1}^n x_iX_i$. Then for each ω, we have

$$\xi(\omega)=\left(Z(\omega),Z(\omega)+\sum_{i=1}^n x_ia_i,Z(\omega)+\sum_{i=1}^n x_ib_iZ(\omega)+\sum_{i=1}^n x_ic_i,\right),$$

where

$$Z \sim \mathcal{N}\left(\sum_{i=1}^{n} x_i\mu_i, \sum_{i=1}^{n} x_i^2\sigma_i^2\right).$$

It follows from Theorem 3.1 that the theorem is valid. The proof is complete.

Example 3.1 *Let ξ be a trapezoidal fuzzy random variables such that for each ω, $\xi(\omega) = (X(\omega), X(\omega)+2, X(\omega)+4, X(\omega)+6)$.*
(1) If $X \sim \mathcal{N}(2,1)$ with probability distribution Φ, then according to Theorem 3.1, it is easy to derive that

$$\begin{aligned}\mathrm{Ch}\{\xi \geq 3\} &= -\tfrac{1}{4\sqrt{2\pi}}(\exp(-\tfrac{1}{2}) - \exp(-\tfrac{1}{2})) - \tfrac{1}{4\sqrt{2\pi}}(\exp(-\tfrac{9}{2}) - \exp(-\tfrac{25}{2}))\\ &\quad -\tfrac{1}{4}\Phi(1) - \tfrac{1}{4}\Phi(-1) + \tfrac{3}{4}\Phi(-3) - \tfrac{5}{4}\Phi(-5) + 1\\ &= -\tfrac{1}{4\sqrt{2\pi}}(\exp(-\tfrac{9}{2}) - \exp(-\tfrac{25}{2})) - \tfrac{1}{4} + \tfrac{3}{4} - \tfrac{3}{4}\Phi(3) - \tfrac{5}{4} + \tfrac{5}{4}\Phi(5) + 1\\ &= -\tfrac{1}{4\sqrt{2\pi}}(\exp(-\tfrac{9}{2}) - \exp(-\tfrac{25}{2})) - \tfrac{1}{4}(3\Phi(3) - 5\Phi(5) - 1).\end{aligned}$$

(2) If $X \sim \mathcal{EXP}(2)$, then according to Theorem 3.1, it is easy to derive that

$$\mathrm{Ch}\{\xi \geq 3\} = -\tfrac{1}{8}\exp(-6 + \tfrac{1}{8}\exp(-2) + \tfrac{1}{2},$$

and

$$\begin{aligned}\mathrm{Ch}\{\xi \geq 7\} &= \tfrac{1}{8}(\exp(-10) - \exp(-14)) + \tfrac{1}{8}(\exp(-2) - \exp(-6))\\ &= \tfrac{1}{8}(\exp(-10) - \exp(-14) + \exp(-2) - \exp(-6)).\end{aligned}$$

(3) If $X \sim \mathcal{U}(1,3)$, then according to Theorem 3.1, it is easy to derive that

$$\mathrm{Ch}\{\xi \geq 4\} = \tfrac{4-4+3(3+8-8)}{16} = \tfrac{9}{16},$$

and

$$\mathrm{Ch}\{\xi \geq 6\} = \tfrac{2(12-4-6)-(1+12-12)+4(3-6+4)}{16} = \tfrac{7}{16}.$$

Example 3.2 *Let ξ_1, ξ_2 be two independent trapezoidal fuzzy random variables such that for each ω, $\xi_1(\omega) = (X(\omega), X(\omega)+0.5, X(\omega)+1, X(\omega)+1.5)$ and $\xi_2(\omega) = (Y(\omega), Y(\omega)+1, Y(\omega)+2, Y(\omega)+3)$, where $X \sim \mathcal{N}(2,1)$, $Y \sim \mathcal{N}(1,1)$ with probability distribution Φ. Since ξ_1 and ξ_2 are mutually independent, for each ω, according to Theorem 3.2, we have*

$$\begin{aligned}\mathrm{Ch}\{2\xi_1 + 3\xi_2 \geq 6\} &= -\tfrac{\sqrt{13}}{8\sqrt{\pi}}\left(\exp\left(-\tfrac{1}{26}\right) - \exp\left(-\tfrac{9}{26}\right)\right) - \tfrac{\sqrt{13}}{8\sqrt{\pi}}\left(\exp\left(-\tfrac{49}{26}\right) - \exp\left(-\tfrac{121}{26}\right)\right)\\ &\quad -\tfrac{1}{8}\Phi(\tfrac{1}{\sqrt{13}}) - \tfrac{3}{8}\Phi(-\tfrac{3}{\sqrt{13}}) + \tfrac{7}{8}\Phi(-\tfrac{7}{\sqrt{13}}) - \tfrac{11}{8}\Phi(-\tfrac{11}{\sqrt{13}}) + 1\\ &= -\tfrac{\sqrt{13}}{8\sqrt{\pi}}\left(\exp\left(-\tfrac{1}{26}\right) - \exp\left(-\tfrac{9}{26}\right) + \exp\left(-\tfrac{49}{26}\right) - \exp\left(-\tfrac{121}{26}\right)\right)\\ &\quad -\tfrac{1}{8}\Phi(\tfrac{1}{\sqrt{13}}) - \tfrac{3}{8} + \tfrac{3}{8}\Phi(-\tfrac{3}{\sqrt{13}}) + \tfrac{7}{8} - \tfrac{7}{8}\Phi(-\tfrac{7}{\sqrt{13}}) - \tfrac{11}{8} + \tfrac{11}{8}\Phi(-\tfrac{11}{\sqrt{13}}) + 1\\ &= -\tfrac{\sqrt{13}}{8\sqrt{\pi}}\left(\exp\left(-\tfrac{1}{26}\right) - \exp\left(-\tfrac{9}{26}\right) + \exp\left(-\tfrac{49}{26}\right) - \exp\left(-\tfrac{121}{26}\right)\right)\\ &\quad -\tfrac{1}{8}\left(\Phi(\tfrac{1}{\sqrt{13}}) - 3\Phi(\tfrac{3}{\sqrt{13}}) - 11\Phi(\tfrac{11}{\sqrt{13}}) - 1\right).\end{aligned}$$

4 Mean Chance Distributions for Normal Fuzzy Random Variables

Theorem 4.1 *Suppose that ξ is a continuous normal fuzzy random variable such that for each ω, $\xi(\omega)$ is a normal fuzzy variable with the following possibility distribution function*

$$\mu_{\xi(\omega)}(x) = \exp\left(-\frac{(x - X(\omega))^2}{2b^2}\right),$$

where $b > 0$, and X is a random variable.
(1) If $X \sim \mathcal{N}(\mu, \sigma^2)$ with probability distribution Φ, then we have

$$\mathrm{Ch}\{\xi \geq r\} = 1 - \Phi(\tfrac{r-\mu}{\sigma}) + \tfrac{b}{2\sqrt{b^2+\sigma^2}}\exp\left(-\tfrac{(\mu-r)^2}{2(b^2+\sigma^2)}\right)\left(2\Phi\left(\tfrac{r\sqrt{b^2+\sigma^2}}{b\sigma} - \tfrac{r\sigma^2+\mu b^2}{b\sigma\sqrt{b^2+\sigma^2}}\right) - 1\right).$$

(2) If $X \sim \mathcal{EXP}(\lambda)$, $\lambda > 0$, *then we have*

$$\mathrm{Ch}\{\xi \geq r\} = \begin{cases} -\exp(-\lambda r) - \frac{\sqrt{2\pi}}{2} b\lambda \exp(-\frac{2r\lambda - \lambda^2 b^2}{2})(1 - \Phi(\lambda b)), & \text{if } r \leq 0 \\ -\exp(-\lambda r) + \frac{\sqrt{2\pi}}{2} b\lambda \exp(-\frac{2r\lambda - \lambda^2 b^2}{2})(2\Phi(\lambda b) - \Phi(\frac{b^2\lambda - r}{b}) - 1), & \text{if } r > 0. \end{cases}$$

(3) If $X \sim \mathcal{U}(l_1, l_2)$, $l_1 < l_2$, *then we have*

$$\mathrm{Ch}\{\xi \geq r\} = \begin{cases} 1 - \frac{b\sqrt{2\pi}}{l_2 - l_1}(\Phi(\frac{l_2 - r}{b}) - \Phi(\frac{l_1 - r}{b})), & \text{if } r \leq l_1 \\ \frac{l_2 + l_1 - 2r}{l_2 - l_1} - \frac{b\sqrt{2\pi}}{l_2 - l_1}(\Phi(\frac{l_2 - r}{b}) - \Phi(\frac{l_1 - r}{b})), & \text{if } l_1 < r \leq l_2 \\ 1 - \frac{b\sqrt{2\pi}}{l_2 - l_1}(\Phi(\frac{l_2 - r}{b}) - \Phi(\frac{l_1 - r}{b})), & \text{if } r > l_2. \end{cases}$$

Proof: According to the credibility measure of fuzzy event, for each ω, we have

$$\mathrm{Cr}\{\xi(\omega) \geq r\} = \begin{cases} 1 - \frac{1}{2}\exp\left(-\frac{(r - X(\omega))^2}{2b^2}\right), & \text{if } X(\omega) \geq r \\ \frac{1}{2}\exp\left(-\frac{(r - X(\omega))^2}{2b^2}\right), & \text{if } X(\omega) < r. \end{cases}$$

(1) If $X(\omega) \geq r$, then

$$\begin{aligned} M_1 &= \int_r^{+\infty} \left(1 - \frac{1}{2}\exp(-\frac{(r-x)^2}{2b^2})\right) \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right) dx \\ &= \int_r^{+\infty} \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right) dx - \int_r^{+\infty} \frac{1}{2\sqrt{2\pi}\sigma} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2} - \frac{(r-x)^2}{2b^2}\right) dx \\ &= 1 - \Phi(\frac{r-\mu}{\sigma}) - \frac{1}{2\sigma}\exp\left(-\frac{(\mu-r)^2}{2(b^2+\sigma^2)}\right) \int_r^{+\infty} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}\left(\frac{\sqrt{b^2+\sigma^2}(x - \frac{r\sigma^2 + b^2\mu}{b^2 + 2\sigma^2})}{\sigma b}\right)^2\right) dx \\ &= 1 - \Phi(\frac{r-\mu}{\sigma}) - \frac{b}{2\sqrt{b^2+\sigma^2}} \exp\left(-\frac{(\mu-r)^2}{2(b^2+\sigma^2)}\right)\left(1 - \Phi\left(\frac{r\sqrt{b^2+\sigma^2}}{b\sigma} - \frac{r\sigma^2 + \mu b^2}{b\sigma\sqrt{b^2+\sigma^2}}\right)\right). \end{aligned}$$

If $X(\omega) < r$, then

$$\begin{aligned} M_2 &= \int_{-\infty}^r \frac{1}{2}\exp(-\frac{(r-x)^2}{2b^2}) \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right) dx \\ &= \int_{-\infty}^r \frac{1}{2\sqrt{2\pi}\sigma} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2} - \frac{(r-x)^2}{b^2}\right) dx \\ &= \frac{1}{2\sigma}\exp\left(-\frac{(\mu-r)^2}{2(b^2+\sigma^2)}\right) \int_{-\infty}^r \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}\left(\frac{\sqrt{b^2+\sigma^2}(x - \frac{r\sigma^2 + b^2\mu}{b^2+\sigma^2})}{\sigma b}\right)^2\right) dx \\ &= \frac{b}{2\sqrt{b^2+\sigma^2}} \exp\left(-\frac{(\mu-r)^2}{2(b^2+\sigma^2)}\right) \Phi\left(\frac{r\sqrt{b^2+\sigma^2}}{b\sigma} - \frac{r\sigma^2+\mu b^2}{b\sigma\sqrt{b^2+\sigma^2}}\right). \end{aligned}$$

Combining the above, we have

$$\begin{aligned} \mathrm{Ch}\{\xi \geq r\} &= M_1 + M_2 \\ &= 1 - \Phi(\frac{r-\mu}{\sigma}) + \frac{b}{2\sqrt{b^2+\sigma^2}} \exp\left(-\frac{(\mu-r)^2}{2(b^2+\sigma^2)}\right)\left(2\Phi\left(\frac{r\sqrt{b^2+\sigma^2}}{b\sigma} - \frac{r\sigma^2+\mu b^2}{b\sigma\sqrt{b^2+\sigma^2}}\right) - 1\right). \end{aligned}$$

(2) Case 1: $r \leq 0$
If $X(\omega) \geq r$, then

$$\begin{aligned} M_1 &= \int_r^{+\infty} \left(1 - \frac{1}{2}\exp(-\frac{(r-x)^2}{2b^2})\right) \lambda \exp(-\lambda x) dx \\ &= \int_r^{+\infty} \lambda \exp(-\lambda x) dx - \int_r^{+\infty} \frac{1}{2}\lambda \exp(-\frac{(r-x)^2}{2b^2} - \lambda x) dx \\ &= -\exp(-\lambda x)|_r^{+\infty} - \int_r^{+\infty} \frac{1}{2}\lambda \exp(-\frac{2r\lambda - \lambda^2 b^2}{2}) \exp(-\frac{(x - r + b^2\lambda)^2}{2b^2}) dx \\ &= -\exp(-\lambda r) - \frac{\sqrt{2\pi}}{2} b\lambda \exp(-\frac{2r\lambda - \lambda^2 b^2}{2})(1 - \Phi(\lambda b)). \end{aligned}$$

Thus

$$\mathrm{Ch}\{\xi \geq r\} = M_1 = -\exp(-\lambda r) - \frac{\sqrt{2\pi}}{2} b\lambda \exp(-\frac{2r\lambda - \lambda^2 b^2}{2})(1 - \Phi(\lambda b)).$$

Case 2: $r > 0$.
If $X(\omega) \geq r$, then

$$\begin{aligned} M_1 &= \int_r^{+\infty} \left(1 - \tfrac{1}{2}\exp(-\tfrac{(r-x)^2}{2b^2})\right) \lambda\exp(-\lambda x)dx \\ &= \int_r^{+\infty} \lambda\exp(-\lambda x)dx - \int_r^{+\infty} \tfrac{1}{2}\lambda\exp(-\tfrac{(r-x)^2}{2b^2} - \lambda x)dx \\ &= -\exp(-\lambda x)|_r^{+\infty} - \int_r^{+\infty} \tfrac{1}{2}\lambda\exp(-\tfrac{2r\lambda-\lambda^2b^2}{2})\exp(-\tfrac{(x-r+b^2\lambda)^2}{2b^2})dx \\ &= -\exp(-\lambda r) - \tfrac{\sqrt{2\pi}}{2}b\lambda\exp(-\tfrac{2r\lambda-\lambda^2b^2}{2})(1-\Phi(\lambda b)). \end{aligned}$$

If $X(\omega) < r$, then

$$\begin{aligned} M_2 &= \int_0^r \tfrac{1}{2}\exp(-\tfrac{(r-x)^2}{2b^2})\lambda\exp(-\lambda x)dx \\ &= \tfrac{\sqrt{2\pi}}{2}\lambda b\exp(-\tfrac{2r\lambda-\lambda^2b^2}{2})(\Phi(\lambda b) - \Phi(\tfrac{b^2\lambda-r}{b})). \end{aligned}$$

Combining the above, we have

$$\begin{aligned} \mathrm{Ch}\{\xi \geq r\} &= M_1 + M_2 \\ &= -\exp(-\lambda r) + \tfrac{\sqrt{2\pi}}{2}b\lambda\exp(-\tfrac{2r\lambda-\lambda^2b^2}{2})(2\Phi(\lambda b) - \Phi(\tfrac{b^2\lambda-r}{b}) - 1). \end{aligned}$$

(3) Case 1: $r \leq l_1$
If $X(\omega) \geq r$, then

$$\begin{aligned} M_1 &= \int_{l_1}^{l_2} \tfrac{1}{l_2-l_1}(1-\exp(-\tfrac{(r-x)^2}{2b^2}))dx \\ &= 1 - \tfrac{b\sqrt{2\pi}}{l_2-l_1}(\Phi(\tfrac{l_2-r}{b}) - \Phi(\tfrac{l_1-r}{b})). \end{aligned}$$

Therefore, we have

$$\mathrm{Ch}\{\xi \geq r\} = M_1 = 1 - \tfrac{b\sqrt{2\pi}}{l_2-l_1}(\Phi(\tfrac{l_2-r}{b}) - \Phi(\tfrac{l_1-r}{b})).$$

Case 2: $l_1 < r \leq l_2$
If $X(\omega) \geq r$, then

$$\begin{aligned} M_1 &= \int_r^{l_2} \tfrac{1}{l_2-l_1}(1-\exp(-\tfrac{(r-x)^2}{2b^2}))dx \\ &= \tfrac{l_2-r}{l_2-l_1} - \tfrac{b\sqrt{2\pi}}{l_2-l_1}(\Phi(\tfrac{l_2-r}{b}) - \tfrac{1}{2}). \end{aligned}$$

If $X(\omega) < r$, then

$$\begin{aligned} M_2 &= \int_{l_1}^r \tfrac{1}{l_2-l_1}\exp(-\tfrac{(r-x)^2}{2b^2})dx \\ &= \tfrac{l_1-r}{l_2-l_1} - \tfrac{b\sqrt{2\pi}}{l_2-l_1}(\tfrac{1}{2} - \Phi(\tfrac{l_1-r}{b})). \end{aligned}$$

Therefore, we have

$$\begin{aligned} \mathrm{Ch}\{\xi \geq r\} &= M_1 + M_2 \\ &= \tfrac{l_2+l_1-2r}{l_2-l_1} - \tfrac{b\sqrt{2\pi}}{l_2-l_1}(\Phi(\tfrac{l_2-r}{b}) - \Phi(\tfrac{l_1-r}{b})). \end{aligned}$$

Case 3: $r > l_2$
If $X(\omega) < r$, then

$$\begin{aligned} M_2 &= \int_{l_1}^{l_2} \tfrac{1}{l_2-l_1}\exp(-\tfrac{(r-x)^2}{2b^2})dx \\ &= 1 - \tfrac{b\sqrt{2\pi}}{l_2-l_1}(\Phi(\tfrac{l_2-r}{b}) - \Phi(\tfrac{l_1-r}{b})). \end{aligned}$$

Therefore, we have

$$\mathrm{Ch}\{\xi \geq r\} = M_2 = 1 - \tfrac{b\sqrt{2\pi}}{l_2-l_1}(\Phi(\tfrac{l_2-r}{b}) - \Phi(\tfrac{l_1-r}{b})).$$

The proof of the theorem is complete.

Theorem 4.2 *Let $\xi_1, \cdots, \xi_n$ be fuzzy random variables. For each ω, the possibility distribution function of $\xi_i(\omega)$ is*

$$\mu_{\xi_i(\omega)}(x) = \exp\left(-\frac{(x - X_i(\omega))^2}{2b_i^2}\right),$$

and $\xi_i(\omega)$ $i = 1, \cdots, n$, are mutually independent fuzzy variables. If $X_i \sim \mathcal{N}(\mu_i, \sigma_i^2)$ with probability distribution Φ, b_i is a positive number, $i = 1, \cdots, n$. Then we have

$$\begin{aligned} \mathrm{Ch}\{\textstyle\sum_{i=1}^n x_i\xi_i \geq r\} = 1 - \Phi\left(\frac{r-\sum_{i=1}^n x_i\mu_i}{\sqrt{\sum_{i=1}^n x_i^2\sigma_i^2}}\right) + \frac{\sqrt{\sum_{i=1}^n x_i^2b_i^2}}{2\sqrt{\sum_{i=1}^n(x_i^2b_i^2+x_i^2\sigma_i^2)}}\exp\left(-\frac{(\sum_{i=1}^n x_i\mu_i - r)^2}{\sum_{i=1}^n(x_i^2b_i^2+x_i^2\sigma_i^2)}\right) \\ \left(2\Phi\left(\frac{r\sqrt{\sum_{i=1}^n(x_i^2b_i^2+x_i^2\sigma_i^2)}}{\sqrt{\sum_{i=1}^n x_i^2b^2\sum_{i=1}^n x_i^2\sigma^2}} - \frac{\sum_{i=1}^n x_i^2b_i^2\sum_{i=1}^n x_i\mu_i + r\sum_{i=1}^n x_i^2\sigma_i^2}{b\sigma\sqrt{\sum_{i=1}^n(x_i^2b_i^2+x_i^2\sigma_i^2)}}\right) + 1\right). \end{aligned}$$

Proof: Let $\xi = \sum_{i=1}^n x_i\xi_i$, $Z = \sum_{i=1}^n x_i X_i$. Then for each ω, we have

$$\mu_{\xi(\omega)}(x) = \exp\left(-\frac{(x - Z(\omega))^2}{2\sum_{i=1}^n x_i^2 b_i^2}\right),$$

where

$$Z \sim \mathcal{N}\left(\sum_{i=1}^n x_i\mu_i, \sum_{i=1}^n x_i^2\sigma_i^2\right).$$

It follows from Theorem 4.1 that the theorem holds true. The proof is complete.

Example 4.1 *Let ξ be a normal fuzzy random variable such that for each ω, $\mu_{\xi(\omega)}(x) = \exp\left(-\frac{(x-X(\omega))^2}{4}\right)$. (1) If $X \sim \mathcal{N}(1,1)$ with probability distribution Φ, then according to the first part of Theorem 4.1, it is easy to derive that*

$$\mathrm{Ch}\{\xi \geq 2\} = 1 - \Phi(1) + \frac{1}{\sqrt{5}}\exp\left(-\frac{1}{10}\right)\left(2\Phi\left(\frac{2\sqrt{5}}{2} - \frac{6}{2\sqrt{5}}\right) - 1\right).$$

(2) If $X \sim \mathcal{EXP}(2)$, then according to the second part of Theorem 4.1, it is easy to derive that

$$\mathrm{Ch}\{\xi \geq 2\} = -\exp(-4) + 2\sqrt{2\pi}\exp(4)\left(2\Phi(4) - \Phi(3)\right).$$

(3) If $X \sim \mathcal{U}(1,4)$, then according to the third part of Theorem 4.1, it is easy to derive that

$$\mathrm{Ch}\{\xi \geq 2\} = \frac{1}{3} - \frac{2\sqrt{2\pi}}{3}\left(\Phi(1) + \Phi\left(\frac{1}{2}\right) - 1\right).$$

Example 4.2 *Let ξ_1, ξ_2 be two independent normal fuzzy random variables such that for each ω, $\mu_{\xi_1(\omega)}(x) = \exp\left(-\frac{(x-X(\omega))^2}{4}\right)$, and $\mu_{\xi_2(\omega)}(x) = \exp\left(-\frac{(x-Y(\omega))^2}{9}\right)$, where $X \sim \mathcal{N}(2,1)$, $Y \sim \mathcal{N}(1,2)$ with probability distribution Φ. Then according to Theorem 4.2, it is easy to derive that*

$$\mathrm{Ch}\{2\xi_1 + \xi_2 \geq 0\} = 1 + \Phi\left(\frac{5}{2\sqrt{2}}\right) - \frac{5}{\sqrt{41}}\exp\left(-\frac{25}{41}\right)\Phi\left(\frac{25}{2\sqrt{82}}\right).$$

5 Conclusions

In fuzzy random optimization problems, the uncertain parameters are usually represented by fuzzy random variables, and the constraints are always containing mean chance functions. It is difficult to compute mean chance of a fuzzy event due to the two fold uncertainty of the event. In this paper, we deduced some formulas for computing the mean chance distribution of fuzzy random variables in some special cases, in which the mean chance distribution can be turned into their equivalent forms. In our future work, we will apply the formulas to fuzzy random optimization to discuss the computation of the mean chance of fuzzy random constraints.

References

[1] Feng, X., and Y.K. Liu, Measurability criteria for fuzzy random vectors, *Fuzzy Optim. Decis. Making*, vol.5, no.3, pp.245–253, 2006.

[2] Kruse, R., and K.D. Meye, *Statistics with Vague Data*, D. Reidel Publishing Company, Dordrecht, 1987.

[3] Kwakernaak, H., Fuzzy random variables–I: Definitions and theorems, *Inform. Sci.*, vol.15, no.1, pp.1–29, 1978.

[4] Liu, B., Fuzzy random chance-constrainted programming, *IEEE Trans. Fuzzy Syst.*, vol.9, no.5, pp.713–720, 2001.

[5] Liu, B., Fuzzy random dependent-chance programming, *IEEE Trans. Fuzzy Syst.*, vol.9, no.5, pp.721–726, 2001.

[6] Liu, B., *Uncertainty Theory: An Introduction to Its Axiomatic Foundations*, Springer-Verlag, Berlin, 2004.

[7] Liu, B., A survey of credibility theory, *Fuzzy Optim. Decis. Making*, vol.5, no.4, pp.387–408, 2006.

[8] Liu, B., and K. Iwamura, Fuzzy programming with fuzzy decisions and fuzzy simulation-based genetic algorithm, *Fuzzy Sets Syst.*, vol.122, no.2, pp.69–77, 2001.

[9] Liu, B., Y.K. Liu, Expected value of fuzzy variable and fuzzy expected value models, *IEEE Trans. Fuzzy Syst.*, vol.10, no.4, pp. 445–450, 2002.

[10] Liu, Y.K., The approximation method for two-stage fuzzy random programming with recourse, *IEEE Trans. Fuzzy Syst.*, vol.15, no.6, pp.1197–1208, 2007.

[11] Liu, Y.K., The convergent results about approximating fuzzy random minimum risk problems, *Appl. Math. Comput.*, vol.205, no.2, pp.608–621, 2008.

[12] Liu, Y.K., and B. Liu, Fuzzy random variable: A scalar expected value operator, *Fuzzy Optim. Decis. Making*, vol.2, no.2, pp.143–160, 2003.

[13] Liu, Y.K., and B. Liu, On minimum-risk problems in fuzzy random decision systems, *Comput. Oper. Res.*, vol.32, no.2, pp.257–283, 2005.

[14] Liu, Y.K., and J. Gao, The independence of fuzzy variables with applications to fuzzy random optimization, *Int. J. Uncertainty Fuzziness Knowl.-Based Syst.*, vol.15, suppl.2, pp.1–20, 2007.

[15] Liu, Y.K., Liu, Z.Q., and J. Gao, The modes of convergence in the approximation of fuzzy random optimization problems, *Soft Comput.*, vol.13, no.2, pp.117–125, 2009.

[16] Liu, Y.K., and S.M. Wang, *Theory of Fuzzy Random Optimization*, China Agricultural University Press, Beijing, China, 2006.

[17] Luhandjula, M.K., and M.M. Gupta, On fuzzy stochastic optimization. *Fuzzy Sets Syst.*, vol.81, pp.47–55, 1996.

[18] Luhandjula, M.K., Fuzzy mathematical programming: theory, applications and extension, *Journal of Uncertain Systems*, vol.1, no.2, pp.124–136, 2007.

[19] Puri, M.L., D.A. Ralescu, Fuzzy random variables, *J. Math. Anal. Appl.*, vol.114, no.2, pp.409–422, 1986.

[20] Wang, P., Fuzzy contactability and fuzzy variables, *Fuzzy Sets Syst.*, vol.8, pp.81–92, 1982.

[21] Wang, S., Y. Liu, and X.D. Dai, On the continuity and absolute continuity of credibility functions, *Journal of Uncertain Systems*, vol.1, no.3, pp.185–200, 2007.

[22] Wang, S., and J. Watada, T-independence condition for fuzzy random vector based on continuous triangular norms, *Journal of Uncertain Systems*, vol.2, no.2, pp.155–160, 2008.

[23] Zadeh, L.A., Fuzzy sets as a basis for a theory of possibility, *Fuzzy Sets and Syst.*, vol.1, no.1, pp.3–28, 1978.