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Classical Belief Conditioning and its Generalization to DSm Theory

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Abstract

Brief introductions to both Dempster-Shafer and DSm theories are presented. Classical belief conditioning is recalled and generalized to DSm hyper-power sets. Relation of generalization of classic conditioning rules to belief conditioning defined in DSmT is discussed.

Keywords: belief functions, Dempster-Shafer theory, belief conditioning, DSm theory, overlapping elements, hyper-power set, DSm model

1 Introduction

Belief functions are one of the widely used formalisms for uncertainty representation and processing, that enable representation of incomplete and uncertain knowledge, belief updating and combination of evidence. They were originally introduced as a principal notion of Dempster-Shafer Theory or the Mathematical Theory of Evidence [12]. Dempster-Shafer theory is often considered as a generalization of the Bayesian theory of subjective probability.

In the Dempster-Shafer theory, Dempster's rule of combination is used for combination of beliefs. Under strict probabilistic assumptions its results are probabilistically interpretable. Nevertheless these assumptions are rarely fulfilled in real applications and even then there are not rare examples where results of the Dempster's rule are counter intuitive.

Hence series of modifications of the Dempster's rule were suggested and alternative approaches were created, e.g. see [4, 8, 16, 17].

Dempster's rule of conditioning [12], which is strictly related to Dempster's rule of combination, is used for conditioning, when there appears a sure evidence or knowledge that the true element (state) must be in a determined subset of a frame of discernment (state space). There are also some alternative approaches to conditioning, but they are not as numerous as alternative combination rules are.

A new approach to belief functions performs the Dezert-Smarandache (or Demspter-Shafer modified) theory (DSmT) with its DSm rule of combination. There are two main differences: 1) mutual exclusivity of elements of a frame of discernment is not assumed in general; mathematically this means that belief functions are not defined on a power set of a frame, but on a so called hyper-power set, i.e. on the Dedekind lattice defined by the frame; 2) a new combination mechanism which overcomes problems with conflict among the combined beliefs and which also enables a dynamic fusion of beliefs.

In addition to DSm combination mechanism, a series of belief conditioning rules (BCR) has been also defined in [14].

As the classical Shafer's frame of discernment may be considered a special case of a so called hybrid DSm model, the DSm rule of combination has been compared to the classical rules of combination in the original publications about DSmT [7, 13], and recently also in [5]. To be able to make a similar comparison of the classic ways of conditioning with those suggested within DSmT, a generalization of conditioning of the classic belief functions to generalized ones defined on DSm hyper-power sets is necessary.

This paper briefly recalls basic notions from the classic Dempster-Shafer theory, including Demspter's rule of conditioning and its alternative (Section II). The DSmT is analogically presented later in Section III. A part of this section is also devoted to DSm models, which enable greater variability of the theory. In Sections IV and V, formal generalizations of the classic Demspter's rule of conditioning and of alternative belief focusing rule are defined and presented. A brief comparison of the generalized rules of conditioning with two of DSm

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belief conditioning rules is sketched in Section VI. Finally, some related approaches are mentioned in Section VII.

2 Primer on Dempster-Shafer Theory

2.1 Basic Notions

All the classic definitions assume an exhaustive finite *frame of discernment* $\Theta = \{\theta_1, \cdots, \theta_n\}$, whose elements are mutually exclusive.

A *basic belief assignment (bba)* is a mapping $m : \mathcal{P}(\Theta) \longrightarrow [0, 1]$, such that

$$\sum_{A \subseteq \Theta} m(A) = 1,$$

the values of bba are called *basic belief masses (bbm)*. The value $m(A)$ is called a *basic belief mass (bbm) of* A[1]. A *belief function (BF)* is a mapping $Bel : \mathcal{P}(\Theta) \longrightarrow [0, 1]$,

$$Bel(A) = \sum_{\emptyset \neq X \subseteq A} m(X).$$

Let us further recall a *plausibility function* $Pl : \mathcal{P}(\Theta) \longrightarrow [0, 1]$,

$$Pl(A) = \sum_{\emptyset \neq X \cap A} m(X).$$

Belief function Bel, plausibility function Pl and bba m always uniquely correspond to each other. $\mathcal{P}(\Theta)$ is often denoted also by 2^{Θ}.

Frame Θ is interpreted as a set of possibilities (or possible worlds), where exactly one of them corresponds to the truth. For each subset A of Θ, the value $Bel(A)$ can be then interpreted as one's degree of belief, that the truth lies in A; whereas value $Pl(A)$ corresponds to the possibility, that the truth (or the actual world) lies in A.

A *focal element* is a subset X of the frame of discernment Θ, such that $m(X) > 0$. If a focal element is a one-element subset of Θ, we are referring to a *singleton*. If all the focal elements are *singletons*, then we speak about a *Bayesian belief function*; in fact, this is a probability distribution on Θ.

2.2 Belief Combination

Let us start with the classic definition of Dempster's rule. *Dempster's (conjunctive) rule of combination* $\oplus$ is given as

$$(m_1 \oplus m_2)(A) = \sum_{X,Y \subseteq \Theta,\, X \cap Y = A} K m_1(X) m_2(Y) \quad for \quad A \neq \emptyset,$$

where

$$K = \frac{1}{1-\kappa}, \kappa = \sum_{X,Y \subseteq \Theta, X \cap Y = \emptyset} m_1(X) m_2(Y),$$

and

$$(m_1 \oplus m_2)(\emptyset) = 0,$$

see [12]; putting $K = 1$ and

$$(m_1 \oplus m_2)(\emptyset) = \kappa,$$

we obtain the *non-normalized conjunctive rule of combination* $\circledcirc$, see e.g. [16].

Yager's rule of combination $\circledY$, see [17], is given as

$$(m_1 \circledY m_2)(A) = \sum_{X,Y \subseteq \Theta,\, X \cap Y = A} m_1(X)\, m_2(Y)$$

[1] $m(\emptyset) = 0$ is often assumed in accordance with Shafer's definition [12]. A classical counter example is Smets' Transferable Belief Model (TBM) which admits positive $m(\emptyset)$ as it assumes $m(\emptyset) \geq 0$.

for $\emptyset \neq A \subset \Theta, (m_1 \textcircled{Y} m_2)(\emptyset) = 0$, and

$$(m_1 \textcircled{Y} m_2)(\Theta) = m_1(\Theta)m_2(\Theta) + \sum_{X,Y\subseteq\Theta,\, X\cap Y=\emptyset} m_1(X)m_2(Y).$$

Dubois-Prade's rule of combination $\textcircled{DP}$ is given as

$$(m_1 \textcircled{DP} m_2)(A) = \sum_{X,Y\subseteq\Theta,\, X\cap Y=A} m_1(X)m_2(Y) + \sum_{X,Y\subseteq\Theta,\, X\cap Y=\emptyset,\, X\cup Y=A} m_1(X)m_2(Y)$$

for $\emptyset \neq A \subseteq \Theta, (m_1 \textcircled{DP} m_2)(\emptyset) = 0$, see [8].

2.3 Belief Conditioning

Now let us assume, that we have a sure evidence that the truth (or the actual world) definitely lies in subset B of frame Θ.

There are several equivalent forms of *Dempster's rule of conditioning.* The original introduced by Shafer[2] in [12] uses plausibility measure:

$$Pl(A|B) = \frac{Pl(A\cap B)}{Pl(B)},$$

an expression which uses basic belief assignment is the following

$$m(A|B) = \frac{1}{1-k} \sum_{X\cap B=A} m(X),$$

where

$$k = \sum_{X\cap B=\emptyset} m(X);$$

the rule is applicable whenever $Pl(B) > 0$, i.e., whenever there exists some $X \cap B \neq \emptyset$ such that $m(X) > 0$. Let us also mention the 'belief form' of the rule:

$$Bel(A|B) = \frac{Bel(A\cup \overline{B}) - Bel(\overline{B})}{1 - Bel(\overline{B})},$$

where $\overline{B}$ is a complement of B in Θ, thus $\overline{B} = \Theta \setminus B$.

Note, that Dempster's rule of conditioning is defined without the use of Demspter's rule of combination. Nevertheless, it has the following interesting properties: it commutes with Dempster's rule of combination, i.e.,

$$m_1(\_|B) \oplus m_2(\_|B) = (m_1 \oplus m_2)(\_|B),$$

and there exists a basic belief assignment m_B such that

$$m(A|B) = (m_1 \oplus m_B)(A),$$

there is

$$m_B(B) = 1, m_B(X) = 0$$

for $X \neq B$. These properties are admired by some researchers, but criticized by others.

There is another belief conditioning rule, which is also called *belief focusing*[3]:

$$m(A||B) = \frac{m(A)}{Bel(B)} = \frac{m(A)}{\sum_{X\subseteq B} m(X)}$$

for $A \subseteq B$,

$$m(A||B) = 0$$

[2] Dezert & Smarandache call this rule Shafer's conditioning rule (or SCR) in [15].

[3] We use a notation $m(\_||\_)$ to distinguish it from Dempster's conditioning rule $m(\_|\_)$.

for $A \not\subseteq B$. This rule is applicable whenever $Bel(B) > 0$, i.e., whenever there exist some $\emptyset \neq X \subseteq B$ such that $m(X) > 0$, see [9]. Similarly to the case of Dempster's conditioning, there are also another equivalent forms of the focusing rule:

$$Bel(A||B) = \frac{Bel(A \cap B)}{Bel(B)}$$

and

$$Pl(A||B) = \frac{Pl(A \cup \overline{B}) - Pl(\overline{B})}{1 - Pl(\overline{B})}.$$

It is easy to verify that for Bayesian belief functions both the conditioning rules coincide, i.e., the following holds true

$$Bel(A||B) = Bel(A|B).$$

3 Basics of DSm Theory

Because DSmT is a new theory, which is in permanent dynamic evolution, we have to note, that this text corresponds to its state described by formulas and text presented in the basic publication on DSmT — in the DSmT book Vol. 1 [13]. Rapid development of the theory is demonstrated by appearance of the second volume of the book [14]. For the new advances in DSmT, including belief conditioning, see the second volume. Both volumes include theoretic contributions to DSmT and also presentation of DSmT applications.

3.1 Dedekind Lattice, Basic DSm Notions

Dezert-Smarandache Theory (DSmT) or Dempster-Shafer modified Theory by Dezert and Smarandache [7, 13] allows mutually overlapping elements of a frame of discernment. Thus, a frame of discernment is a finite exhaustive set of elements $\Theta = \{\theta_1, \theta_2, ..., \theta_n\}$, but not necessarily exclusive in DSmT. As an example, we can introduce a three-element set of colours $\{Red, Green, Blue\}$ from the DSmT homepage[4]. DSmT allows that an object can have 2 or 3 colours at the same time: e.g., it can be both red and blue, or red and green and blue at the same time, this corresponds to a composition of the colours from the 3 basic ones.

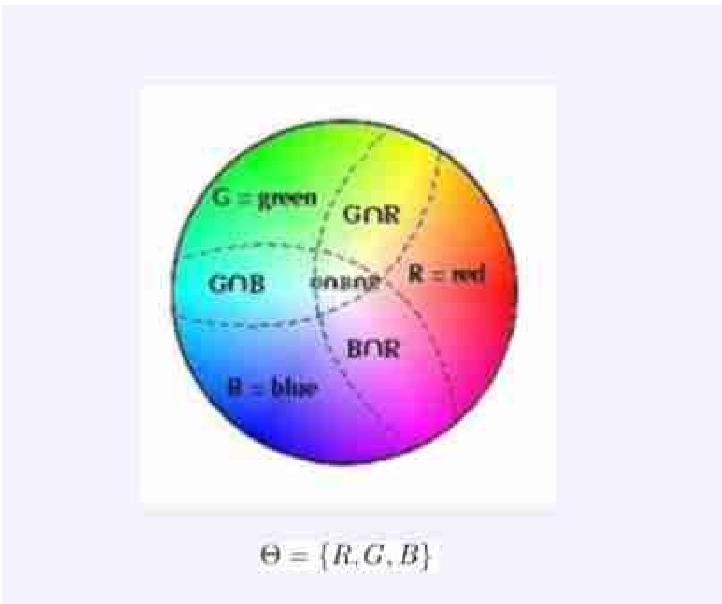

Figure 1: Three colour example of a hyper-power set

[4] www.gallup.unm.edu/∼smarandache/DSmT.htm

DSmT uses basic belief assignments and belief functions defined analogically to the classic Dempster-Shafer theory, but they are defined on a so-called hyper-power set or Dedekind lattice instead of the classic power set of the frame of discernment. To be distinguished from the classic definitions, they are called generalized basic belief assignments and generalized belief functions.

The *Dedekind lattice*, more frequently called *hyper-power set* D^Θ in DSmT, is defined as the set of all composite propositions built from elements of Θ with union and intersection operators $\cup$ and $\cap$ such that $\emptyset, \theta_1, \theta_2, ..., \theta_n \in D^\Theta$, and if $A, B \in D^\Theta$ then also $A \cup B \in D^\Theta$ and $A \cap B \in D^\Theta$, no other elements belong to D^Θ ($\theta_i \cap \theta_j \neq \emptyset$ in general, $\theta_i \cap \theta_j = \emptyset$ iff $\theta_i = \emptyset$ or $\theta_j = \emptyset$).

Thus the hyper-power set D^Θ of Θ is closed under $\cup$ and $\cap$ and $\theta_i \cap \theta_j \neq \emptyset$ in general. Whereas the classic power set 2^Θ of Θ is also closed under $\cup$, $\cap$, and moreover, it is closed under complement, but $\theta_i \cap \theta_j = \emptyset$ for every $i \neq j$.

Examples of hyper-power sets. Let $\Theta = \{\theta_1, \theta_2\}$, we have $D^\Theta = \{\emptyset, \theta_1 \cap \theta_2, \theta_1, \theta_2, \theta_1 \cup \theta_2\}$, i.e. $|D^\Theta| = 5$. Let $\Theta = \{\theta_1, \theta_2, \theta_3\}$ now, we have $D^\Theta = \{\alpha_0, \alpha_1, ...\alpha_{18}\}$, where $\alpha_0 = \emptyset, \alpha_1 = \theta_1 \cap \theta_2 \cap \theta_3, \alpha_2 = \theta_1 \cap \theta_2, \alpha_3 = \theta_1 \cap \theta_3, ..., \alpha_{17} = \theta_2 \cup \theta_3, \alpha_{18} = \theta_1 \cup \theta_2 \cup \theta_3$, i.e., $|D^\Theta| = 19$ for $|\Theta| = 3$, e.g., $\theta_1 = Red$, $\theta_2 = Green$, $\theta_3 = Blue$.

As the elements of the hyper-power sets are expressed in a form of formulas using $\theta_i, \cap, \cup$, we can equivalently express them in unique canonical forms: *conjunctive normal form* (CNF), i.e. intersection of unions, and *disjunctive normal form* (DNF), i.e. union of intersections.

A *generalized basic belief assignment (gbba)* m is a mapping $m : D^\Theta \longrightarrow [0, 1]$, such that

$$\sum_{A \in D^\Theta} m(A) = 1$$

and $m(\emptyset) = 0$. The quantity $m(A)$ is called the *generalized basic belief mass (gbbm) of A*. A *generalized belief function (gBF) Bel* is a mapping $Bel : D^\Theta \longrightarrow [0, 1]$, such that

$$Bel(A) = \sum_{X \subseteq A, X \in D^\Theta} m(X),$$

generalized belief function Bel uniquely corresponds to gbba m and vice-versa. A *generalized plausibility Pl* is a mapping $Pl : D^\Theta \longrightarrow [0, 1]$, such that

$$Pl(A) = \sum_{X \cap A \neq \emptyset, X \in D^\Theta} m(X),$$

thus $Pl(A) \equiv 1$ on D^Θ.

3.2 DSm Models

If we assume a Dedekind lattice (hyper-power set) according to the above definition without any other assumptions, i.e., all elements of an exhaustive frame of discernment can mutually overlap, we refer to the *free DSm model* $\mathcal{M}^f(\Theta)$, i.e., to the DSm model free of constraints.

It is possible, in general, to add exclusivity or non-existential constraints into DSm models. In such cases, we speak about *hybrid DSm models*.

An *exclusivity constraint* $\theta_1 \cap \theta_2 \overset{\mathcal{M}_1}{\equiv} \emptyset$ says that elements θ_1 and θ_2 are mutually exclusive in model $\mathcal{M}_1$, whereas both of them can overlap with θ_3. If we assume exclusivity constraints $\theta_1 \cap \theta_2 \overset{\mathcal{M}_2}{\equiv} \emptyset$, $\theta_1 \cap \theta_3 \overset{\mathcal{M}_2}{\equiv} \emptyset$, $\theta_2 \cap \theta_3 \overset{\mathcal{M}_2}{\equiv} \emptyset$, another exclusivity constraint directly follows: $\theta_1 \cap \theta_2 \cap \theta_3 \overset{\mathcal{M}_2}{\equiv} \emptyset$, as $\theta_1 \cap \theta_2 \cap \theta_3 \subseteq \theta_1 \cap \theta_2$, etc. In this case all the elements of the 3-element frame of discernment $\Theta = \{\theta_1, \theta_2, \theta_3\}$ are mutually exclusive as in the classic Dempster-Shafer theory, and we call such hybrid DSm model a *Shafer's model* $\mathcal{M}^0(\Theta)$.

A *non-existential constraint* $\theta_3 \overset{\mathcal{M}_3}{\equiv} \emptyset$ brings additional information about a frame of discernment saying that θ_3 is impossible; it forces all the gbbms of $X \subseteq \theta_3$ to be equal to zero for any gbba in model $\mathcal{M}_3$. It represents a sure meta-information with respect to generalized belief combination which is used in a dynamic fusion.

As a nice example of a hybrid DSm model we can present Black-R-G-B example: Let us suppose, now, a picture printed using Black, Red, Green, and Blue toners, where any pixel is either black or coloured, in such a way that its colour is composed from three basic colours R, G, B as it is above in the three colour example. Thus we have $\Theta_{Bk} = \{\theta_0, \theta_1, \theta_2, \theta_3\}$, where $\theta_0 = Black$ and $\theta_1, \theta_2, \theta_3$ as it is above. This time we have constraints that $\theta_0 \cap \theta_i \equiv \emptyset$ for $i = 1, 2, 3$ (or simply that $\theta_0 \cap (\theta_1 \cup \theta_2 \cup \theta_3) \equiv \emptyset$), and subsequently also $\theta_0 \cap \theta_1 \cap \theta_2 \equiv \emptyset$, $\theta_0 \cap \theta_1 \cap \theta_3 \equiv \emptyset$, $\theta_0 \cap \theta_2 \cap \theta_3 \equiv \emptyset$, and $\theta_0 \cap \theta_1 \cap \theta_2 \cap \theta_3 \equiv \emptyset$.

In a degenerated case of the *degenerated DSm model* $\mathcal{M}_\emptyset$ (*vacuous DSm model* in [13]) we always have $m(\emptyset) = 1$, $m(X) = 0$ for $X \neq \emptyset$. It is the only case where $m(\emptyset) > 0$ is allowed in DSmT.

The total ignorance on Θ is the union $I_t = \theta_1 \cup \theta_2 \cup ... \cup \theta_n$. $\boldsymbol{\emptyset} = \{\boldsymbol{\emptyset}_\mathcal{M}, \emptyset\}$, where $\boldsymbol{\emptyset}_\mathcal{M}$ is the set of all elements of D^Θ which are forced to be empty through the constraints of the model $\mathcal{M}$ and $\emptyset$ is the classical empty set[5].

For a given DSm model we can define (in addition to [13]) $\Theta_\mathcal{M} = \{\theta_i \,|\, \theta_i \in \Theta, \theta_i \notin \boldsymbol{\emptyset}_\mathcal{M}\}$, $\Theta_\mathcal{M} \overset{\mathcal{M}}{\equiv} \Theta$, and $I_\mathcal{M} = \bigcup_{\theta_i \in \Theta_\mathcal{M}} \theta_i$, i.e. $I_\mathcal{M} \overset{\mathcal{M}}{\equiv} I_t$, $I_\mathcal{M} = I_t \cap \Theta_\mathcal{M}$, $I_{\mathcal{M}_\emptyset} = \emptyset$. $D^{\Theta_\mathcal{M}}$ is a hyper-power set on the DSm frame of discernment $\Theta_\mathcal{M}$, i.e., on Θ without elements which are excluded by the constraints of model $\mathcal{M}$. $\Theta_\mathcal{M} = \Theta$, $D^{\Theta_\mathcal{M}} = D^\Theta$ and $I_\mathcal{M} = I_t$ holds true for any DSm model without non-existential constraint. Whereas a *reduced (or constrained) hyper-power set* $D^\Theta_\mathcal{M}$ (or $D^\Theta(\mathcal{M})$ from Chapter 4 in [13] arises from D^Θ by identifying of all $\mathcal{M}$-equivalent elements. $D^\Theta_{\mathcal{M}^0}$ corresponds to classic power set 2^Θ.

3.3 The DSm Rules of Combination

The *classic DSm rule DSmC* is defined on the free DSm model $\mathcal{M}^f(\Theta)$ as it follows[6]:

$$m_{\mathcal{M}^f(\Theta)}(A) = (m_1 \circledplus m_2)(A) = \sum_{X,Y \in D^\Theta,\, X \cap Y = A} m_1(X) m_2(Y).$$

Since D^Θ is closed under operators $\cap$ and $\cup$ and all the $\cap$s are non-empty on the free DSm model $\mathcal{M}^f(\Theta)$, the definition of the classic DSm rule guarantees, that is defined for any couple of gbbm's m_1, m_2 and $(m_1 \circledplus m_2)$ is a proper generalized basic belief assignment. The rule is commutative and associative. For n-ary version of the rule see [13]. In fact, we can observe, that DSmC rule is conjunctive rule of combination. It is really equivalent to generalized conjunctive rule of combination $\circledcirc$, moreover the following theorem holds true (see [5]).

Theorem 1 *Both classic and hybrid DSm rules and the following generalized rules: Dempster's rule, the non-normalized conjunctive rule, Yager's rule, and Dubois-Prade's rule are all mutually equivalent in the free DSm model* $\mathcal{M}^f(\Theta)$.

Hence it holds true that

$$(m_1 \circledplus m_2)(A) = (m_1 \oplus m_2)(A) = (m_1 \circledcirc m_2)(A) = (m_1 \circledY m_2)(A) \ = \ (m_1 \circledD m_2)(A)$$

for any m_1, m_2 and $A \in D^\Theta$ on $\mathcal{M}^f(\Theta)$.

When the free DSm model $\mathcal{M}^f(\Theta)$ does not hold true due to the nature of the problem under consideration, which requires to take into account some known integrity constraints, one has to work with a proper hybrid DSm model $\mathcal{M}(\Theta) \neq \mathcal{M}^f(\Theta)$. For such a case, the *hybrid DSm rule of combination DSmH* was defined in [7] (see also DSm book Vol. 1 [13]). In [5] it was shown that DSmH rule is a generalization of Dubois-Prade rule of combination, which is slightly extended to be applicable also for degenerated cases of a dynamic fusion. Nevertheless this rule has no relation to presented conditioning rules, thus we can skip it, for detail see both volumes of DSm book [13, 14].

Let us assume all elements X from D^Θ to be in CNF in the rest of this contribution, unless another form of X is explicitly specified. With $X = Y$ we mean that the formulas X and Y have the same CNF. With $X \equiv Y$ ($X \overset{\mathcal{M}}{\equiv} Y$) we mean that the formulas X and Y are equivalent in DSm model $\mathcal{M}$, i.e. their DNFs are the same up to unions with some constrained conjunctions of elements of Θ.

[5] $\boldsymbol{\emptyset}$ should be $\boldsymbol{\emptyset}_\mathcal{M}$ extended with the classical empty set $\emptyset$, thus more correct should be the expression $\boldsymbol{\emptyset} = \boldsymbol{\emptyset}_\mathcal{M} \cup \{\emptyset\}$.

[6] To distinguish the DSm rule from Dempster's rule, we use $\circledplus$ instead of $\oplus$ for the DSm rule in this text.

The *generalized Dempster's rule of combination* $\oplus$ is given as

$$(m_1 \oplus m_2)(A) = \sum_{X,Y \in D^\Theta,\, X \cap Y \equiv A} K m_1(X) m_2(Y)$$

for $\emptyset \neq A \in D^\Theta_{\mathcal{M}}$, where

$$K = \frac{1}{1-\kappa}, \;\; \kappa = \sum_{X,Y \in D^\Theta,\, X \cap Y \in \boldsymbol{\emptyset}} m_1(X) m_2(Y),$$

and

$$(m_1 \oplus m_2)(A) = 0$$

otherwise, i.e., for $A = \emptyset$ and for $A \notin D^\Theta_{\mathcal{M}}$.

Similarly to the classic case, the generalized Demspter's rule is not defined in fully contradictive cases[7] in hybrid DSm models, i.e. whenever $\kappa = 1$. Specially, the generalized Dempster's rule is not defined (and it cannot be defined) on the degenerated DSm model $\mathcal{M}_\emptyset$.

To complete our brief overview of DSm combination, we should mention also the generalized Yager's rule and the generalized Dubois-Prade rule, for both of them see [5], generalization of minC combination rule, see [3, 6], and a family of DSm proportional conflict redistribution (PCR) rules [14].

3.4 The DSm Belief Conditioning Rules

There is a long series of 31 Belief Conditioning Rules (BCR) defined in Chap. 9 of DSm book Vol. 2 [14, 15]. Their formulas are often rather complicated for a brief presentation of all of them. Thus we will focus only on 3 of them here. We will start with the simplest one BCR1 and beside it we will briefly recall also BCR12 and CBR17, which are recommended by its authors — Smarandache and Dezert — as the best to use.

BCR12 is moreover regarded by its authors as a generalization of SCR (i.e., of Shafer's conditioning rule, which is Dempsters' rule of conditioning in fact) from the power set to the hyper-power set in the free DSm model, where all intersections are non-empty, see [15] pg 260.

Let us suppose a sure evidence, that the truth is a given element B of $D^\Theta \setminus \{\emptyset\}$. Let us denote $s(B) = \{\theta_{i_1}, \theta_{i_2}, ..., \theta_{i_p}\}$, $1 \leq p \leq n$ the set of all elements of Θ, which compose B; for example $s(\theta_1 \cup (\theta_3 \cap \theta_4)) = \{\theta_1, \theta_3, \theta_4\}$. Let us define an auxiliary splitting of hyper-power set $D^\Theta \setminus \{\emptyset\}$ into three disjoint parts D_1, D_2, and D_3 as it follows:

$$D_1 = \mathcal{P}_\mathcal{D}(B) = \{X | \; \emptyset \neq X \in D^\Theta, \; X \subseteq B\},$$

$$D_2 = ((\Theta \setminus s(A)), \cap, \cup),$$

i.e. the subhyper-power set of D^Θ without emptyset, which is generated by $\Theta \setminus s(A)$,

$$D_3 = D^\Theta \setminus (D_1 \cup D_2 \cup \emptyset).$$

For example, let $\Omega = \{a, b, c\}$ and $B = b$. Then $D_1 = \{b, a \cap b, b \cap c, a \cap b \cap c\}$, $s(B) = \{b\}$, $D_2 = D^{\{a,c\}} \setminus \{\emptyset\} = (\{a, c\}, \cap, \cup) = \{a, c, a \cap c, a \cup c\}$, $D_3 = D^\Theta \setminus (D_1 \cup D_2 \cup \emptyset) = \{a \cup b, a \cup c, a \cup b \cup c, a \cup (b \cap c), b \cup (a \cap c), c \cup (a \cap b), (a \cap b) \cup (a \cap c), (b \cap c) \cup (a \cap b), (a \cap c) \cup (b \cap c), (a \cap b) \cup (a \cap c) \cup (b \cap c)\}$.

Belief Conditioning Rule no. 1 (*BCR1*) is defined for $X \in D_1$ by the formula[8]

$$m_{BCR1}(X|B) = \frac{m(X)}{\sum_{Y \in D_1} m(Y)}.$$

$m_{BCR1}(X|B) = 0$ for $X \in D^\Theta \setminus D_1$, the case, where $\sum_{Y \in D_1} m(Y) = 0$, is not referred to in [15].

Before defining other rules, we have to introduce another auxiliary notion from [15]: let $W \in D_3$, we say that $X \in D_1$ is the *k-largest*, $k \geq 1$, element from D_1 that is included in W, if $(\nexists Y \in D_1 \setminus \{X\})(X \subset Y, Y \subset W)$; depending on the model, there are $k \geq 1$ such elements (hence k is an integer number), (see [15],

[7] Note that in a static combination it means a full conflict/contradiction between input BFs. Whereas in the case of a dynamic combination it could be also a full conflict between mutually non-conflicting or partially conflicting input BFs and constraints of a used hybrid DSm model. E.g. $m_1(\theta_1 \cup \theta_2) = 1$, $m_2(\theta_2 \cup \theta_3) = 1$, where θ_2 is constrained in a used hybrid model.

[8] We have to put stress on the fact, that it is necessary to keep in mind, that definition of sets D_1, D_2, D_3, i.e. splitting of D^Θ, depends on the conditioning set B, which is included in the formula through the set D_1.

corrigenda of pg 240). The same is used also for $W \in D_2$, such that $W \cap D_1 \not\equiv \emptyset$, i.e., k-largest element from D_1 that is included in $W \in D_2$.

Belief Conditioning Rule no. 12 (*BCR12*) is defined for $X \in D_1$ by the formula

$$m_{BCR12}(X|B) = m(X) \; + \; [m(X) \; \cdot \sum_{\substack{Z \in D_2 \\ (\nexists Y \in D_1)(Y \subseteq Z)}} m(Z)] \; / \sum_{Y \in D_1} m(Y) \; + \sum_{\substack{W \in D_2 \cup D_3 \\ X \subseteq W, X \text{is } k\text{-largest}}} m(W)/k.$$

Belief Conditioning Rule no. 17 (*BCR17*) is defined for $X \in D_1$ by the formula

$$m_{BCR17}(X|B) = m(X) \; + \; [m(X) \; \cdot \sum_{\substack{Z \in D_2 \\ (\nexists Y \in D_1)(Y \subseteq Z)}} m(Z)] \; / \sum_{Y \in D_1} m(Y)$$

$$+ m(X) \; \cdot \sum_{\substack{W \in D_2 \cup D_3 \\ X \subseteq W, S(W) \neq 0}} \frac{m(W)}{S(W)} + \sum_{\substack{W \in D_2 \cup D_3 \\ X \subseteq W, X \text{is } k\text{-largest}, S(W)=0}} m(W)/k \; ,$$

where

$$S(W) = \sum_{Y \in D_1, \; Y \subset W} m(Y).$$

Analogously to BCR1, $m_{BCR12}(X|B) = m_{BCR17}(X|B) = 0$ for $X \in D^\Theta \setminus D_1$, and the case, where $\sum_{Y \in D_1} m(Y) = 0$, is again not referred to in [15].

Unfortunately, no DSm models are mentioned at definitions of BCR rules, no conditions for definition domains are presented in [15]. Hence it looks like all 31 BCR rules are defined just on D^Θ, which need not any additional conditions, i.e., only on the free DSm model. In contradiction to this, Shafer's model (a special case of hybrid DSm model) is mentioned in several examples in [15], unfortunately again without any other conditions and explanation.

Now let us introduce a generalization of the classic rules of conditioning; Dempster's rule of conditioning in the following section, and the focusing rule of conditioning in the subsequent one.

4 A Generalization of Dempster's Conditioning Rule

The *generalized Dempster's rule of conditioning* is given as

$$m(A|B) = K \sum_{X \in D^\Theta, \; B \cap X \equiv A} m(X)$$

for $\emptyset \neq A \subseteq B$, $A, B \in D^\Theta_{\mathcal{M}}$, where

$$K = \frac{1}{1 - \kappa}, \;\; \kappa = \sum_{X \in D^\Theta, \; B \cap X \in \emptyset} m(X),$$

and

$$m(A|B) = 0$$

otherwise, i.e., for $A = \emptyset$ and for $A \notin D^\Theta_{\mathcal{M}}$.

The rule is defined (applicable) whenever $\kappa < 1$, i.e., whenever there exists $X \in D^\Theta_{\mathcal{M}}$, $X \cap B \not\equiv \emptyset$, such that $m(X) > 0$. Specially, the rule is applicable for any couple of m and B on the free DSm model as $\kappa = 0$ there. It holds true that $\kappa = \sum_{X \in D^\Theta, \, B \cap X \in \emptyset} m(X) = 0$ for $\mathcal{M}^f(\Theta)$.

Similarly to the classic Dempster's conditioning, we have to underline, that the generalized Dempster's rule of conditioning is defined without usage of any combination rule, hence its definition is not based either on generalized Dempster's rule (of combination) $\oplus$ or on generalized conjunctive rule of combination $\circledcirc$. On the other hand, similarly to the classic case again, there is a relationship between generalized Dempster's conditioning and the combination rules, as it is formalized in the following statements.

Lemma 1 *For the generalized Dempster's conditioning rule and generalized conjunctive rule of combination the following holds true on the free DSm model* $\mathcal{M}^f(\Theta)$

$$m(A|B) = (m \circledcirc m_B)(A),$$

where $m_B(B) = 1$ *and* $m_B(X) = 0$ *otherwise, for* $A, B, X \in D^\Theta_{\mathcal{M}^f} = D^\Theta$.

Thus it immediately follows that

$$m(A|B) = (m \circledcirc m_B)(A) = (m \oplus m_B)(A) = (m \circledast m_B)(A)$$

on $\mathcal{M}^f(\Theta)$.

Theorem 2 *For the generalized Dempster's conditioning rule and generalized Dempster's rule of combination the following holds true on any hybrid DSm model* $\mathcal{M}(\Theta)$

$$m(A|B) = (m \oplus m_B)(A),$$

whenever the expression is defined, and where $m_B(B) = 1$ *and* $m_B(X) = 0$ *otherwise, for* $A, B, X \in D^\Theta_{\mathcal{M}}$.

Proof. A proof is a simple verification of the statements.

As in the classic case, we can work with conditioned generalized basic belief assignment as it is usual in the evidence theory, thus we can define *conditioned belief function* and *conditioned plausibility* as is follows: $Bel(\_|B), Pl(\_|B) : D^\Theta_{\mathcal{M}} \longrightarrow [0,1]$, such that

$$Bel(A|B) = \sum_{X \subseteq A, X \in D^\Theta_{\mathcal{M}}} m(X|B),$$

$$Pl(A|B) = \sum_{X \cap A \neq \emptyset, X \in D^\Theta_{\mathcal{M}}} m(X|B),$$

where $\emptyset \neq A \subseteq B$, $A, B \in D^\Theta_{\mathcal{M}}$. Similarly to $Pl(A)$, also for $Pl(A|B)$ it holds true, that $Pl(A|B) \equiv 1$ on on the free DSm model D^Θ, but this does not hold true on general hybrid models, see e.g. Shafer's model — the special case of DSm hybrid models.

Analogously to the classic case, we can express also the generalized Dempster's conditioning rule in several equivalent forms, namely in a generalization of the original plausibility form:

$$Pl(A|B) = \frac{Pl(A \cap B)}{Pl(B)} = \frac{Pl(A)}{Pl(B)}.$$

Let us verify it. $Pl(A|B) = \sum_{X \cap A \neq, X \in D^\Theta_{\mathcal{M}}} m(X|B) = \sum_{X \cap A \neq \emptyset, X \in D^\Theta_{\mathcal{M}}} (\frac{1}{1-\kappa} \sum_{Y \in D^\Theta, Y \cap B \equiv X} m(Y))$, where $\kappa = \sum_{Z \in D^\Theta, Z \cap B \in \emptyset} m(Z) = 1 - \sum_{Z \in D^\Theta, Z \cap B \notin \emptyset} m(Z) = 1 - Pl(B)$. Using it, we can rewrite $Pl(A|B)$ as it follows. $Pl(A|B) = \frac{1}{Pl(B)} \sum_{X \cap A \neq \emptyset, X \in D^\Theta_{\mathcal{M}}} \sum_{Y \in D^\Theta, Y \cap B \equiv X} m(Y) = \frac{\sum_{Y \in D^\Theta, Y \cap B \cap A \neq \emptyset} m(Y)}{Pl(B)} = \frac{\sum_{Y \in D^\Theta, Y \cap A \neq \emptyset} m(Y)}{Pl(B)} = \frac{Pl(A)}{Pl(B)} = \frac{Pl(A \cap B)}{Pl(B)}$, where the last and the last but two equations are based on the fact that, $A = A \cap B$, due to $A \subseteq B$.

As there is no complement in hyper-power sets, and there is no notion of complement in DSmT in general, we cannot simply generalize the belief form of Demspter's conditioning rule to the standard DSmT.

5 A Generalization of Belief Focusing

The *generalized belief focusing rule of conditioning* is given as

$$m(A||B) = \frac{m(A)}{Bel(B)} = \frac{m(A)}{\sum_{X \subseteq B, X \in D^\Theta_{\mathcal{M}}} m(X)}$$

for $\emptyset \neq A \in D^\Theta_{\mathcal{M}}$, where $A \subseteq B$,

$$m(A||B) = 0,$$

otherwise, i.e. for $A \not\subseteq B$ and for $A \in (D^\Theta \setminus D^\Theta_{\mathcal{M}}) \cup \{\emptyset\}$.

This rule is applicable whenever $Bel(B) > 0$, whenever there exist some $\emptyset \neq X \in D^\Theta_{\mathcal{M}}$, $X \subseteq B$, $m(X) > 0$.

Similarly to the Dempster's conditioning rule, we can generalize alternative forms also for belief focusing conditioning, namely its belief form. We have

$$Bel(A||B) = \frac{Bel(A \cap B)}{Bel(B)} = \frac{Bel(A)}{Bel(B)},$$

where $A \subseteq B \in D^\Theta_{\mathcal{M}}$ and $Bel(B) > 0$ on any DSm hybrid model $\mathcal{M}$.

A verification: $Bel(A||B) = \sum_{X\subseteq A, X\in D^\Theta_{\mathcal{M}}} m(X||B) = \sum_{X\subseteq A, X\in D^\Theta_{\mathcal{M}}} \frac{m(X)}{\sum_{Y\subseteq X,\, Y\in D^\Theta_{\mathcal{M}}} m(Y)} = \frac{\sum_{X\subseteq A, X\in D^\Theta_{\mathcal{M}}} m(X)}{\sum_{Y\subseteq X,\, Y\in D^\Theta_{\mathcal{M}}} m(Y)} = \frac{Bel(A)}{Bel(B)} = \frac{Bel(A\cap B)}{Bel(B)}$. We use $A \cap B = A$ for $A \subseteq B$ again.

Due to lack of complement, we cannot generalize the plausibility form of the rule again.

Fact: For Bayesian belief functions on Shafer's hybrid model the generalized focusing rule coincides with the generalized Dempster's rule of conditioning.

To generalize this obvious fact, we have to specify definition of splitting of D^Θ into 3 sets D_i also for hybrid DSm models:

$$D_1 \;\ldots\; {}_{1|B}D^\Theta_{\mathcal{M}} = \{X|\ \emptyset \neq X \in D^\Theta_{\mathcal{M}}, X \subseteq B\},$$

$$D_2 \;\ldots\; {}_{2|B}D^\Theta_{\mathcal{M}} = ((\Theta \setminus s(B)), \cap, \cup) \cap D^\Theta_{\mathcal{M}},$$

$$D_3 \;\ldots\; {}_{3|B}D^\Theta_{\mathcal{M}} = D^\Theta_{\mathcal{M}} \setminus ({}_{1|B}D^\Theta_{\mathcal{M}} \cup {}_{2|B}D^\Theta_{\mathcal{M}} \cup \emptyset).$$

Lemma 2 *(i) Let $B \in D^\Theta_{\mathcal{M}}$, and Bel be a belief function defined by gbba m on $D^\Theta_{\mathcal{M}}$. If $Bel(\bigcup({}_{3|B}D^\Theta_{\mathcal{M}}) \cap B) = 0$, i.e. if $m(X) = 0$ for all $X \subseteq \bigcup({}_{3|B}D^\Theta_{\mathcal{M}}) \cap B$, then*

$$m(A||B) = (m \oplus m_B)(A) = m(A|B).$$

When assuming dynamic combination and conditioning, i.e. situations, where input belief functions are not necessarily defined on used DSm model $D^\Theta_{\mathcal{M}}$, also the following stronger version of the lemma holds true (both the versions mutually coincide when $m(X) = 0$ for all $X \in \boldsymbol{\emptyset}_{\mathcal{M}}$):

6 A Brief Comparison of the Conditioning Rules

6.1 Generalized Conditioning Rules

In the previous section, we have presented the conditions, under which both the generalized rules of conditioning mutually coincide. In general, we have to note, that similarly to the classic case, the generalized Dempster's rule of conditioning has a larger definition domain than it has the generalized focusing, as $\emptyset \not\equiv X \subseteq B \;\&\; m(X) > 0$ implies $\emptyset \not\equiv X \cap B \;\&\; m(X) > 0$.

Further we have to note, that similarly to the classic case again, the focusing rule of conditioning is more sensitive to the values $m(X)$ for $X \subseteq B$, because only these values are considered for computing of conditioned belief masses, whereas the other values (non-conditioned belief masses), i.e. other $m(X)$ where $X \cap B \not\equiv \emptyset$, are completely ignored by this rule.

6.2 Generalized Conditioning Rules Versus BCR Rules

We know that $D_1 = \{X|\ \emptyset \neq X \in D^\Theta, X \subseteq B\}$ on the free DSm model, thus we can rewrite the belief conditioning rule no.1 as

$$\begin{aligned} m_{BCR1}(X|B) &= \frac{m(X)}{\sum_{Y\in D_1} m(Y)} \\ &= \frac{m(X)}{\sum_{Y\in\{X|\ \emptyset\neq X\in D^\Theta, X\subseteq B\}} m(Y)} \\ &= \frac{m(X)}{\sum_{Y\in D^\Theta, Y\subseteq B} m(Y)} \\ &= m(X||B), \end{aligned}$$

as we assume $m(\emptyset) = 0$.

Hence, we can see that the rules BCR1 and generalized focusing rule of conditioning mutually coincide on the free DSm model. Similarly, the same holds true also for a general DSm hybrid model, if we use full definition of D_1, from the previous section, and extend with it the original definition of BCR1 from [15].

In [15], the authors refer to BCR12 as to a generalization of Dempster's rule of conditioning from the power set to the hyper-power set, i.e. to the free DSm model. Unfortunately, this is not true in general. We can easily verify, that a part of belief masses is proportionalized with the respect to the sets X from D_1, and the rest of belief masses is divided into k equally sized parts by BCR12. Whereas using Dempster's rule of conditioning all belief masses are normalized, i.e., all belief masses are proportionalized with the respect to the sets X from$D_1 \cup D_3 \cup \{Z \in D_2 \,|\, (\exists Y \in D_1)(Y \subseteq Z)\}$.

Let us look at the definition of BCR12 now. What does k-largest element from D_1 that is included W mean? It means that $X \in D_1$ (i.e. $X \in D^\Theta$ and $X \subseteq B$) and $X \subseteq W$ such that $(\nexists Y \in D_1 \setminus \{X\})(X \subset Y, Y \subseteq W)$, where k is number of such sets. Hence X the largest set such that $X \subseteq B \cap W$, i.e. $X = B \cap W$ and $k = 1$ because intersection in unique. Thus we can simplify the expression of BCR12 to the following equivalent form:

$$m_{BCR12}(X|B) = m(X) \;+\; [m(X) \;\cdot \sum_{\substack{Z \in D_2 \\ (\nexists Y \in D_1)(Y \subseteq Z)}} m(Z)] \;/ \sum_{Y \in D_1} m(Y) \;+ \sum_{\substack{W \in D_2 \cup D_3 \\ W \cap B = X}} m(W).$$

Analogically $\nexists Y \in D_1$ is equivalent to $(\nexists Y)(\emptyset \neq Y \subseteq B)$ and $(\nexists Y \in D_1)(Y \subseteq Z)$ is equivalent to $(\nexists Y)(\emptyset \neq Y \subseteq B \;\&\; Y \subseteq Z)$, i.e. to $Z \cap B = \emptyset$. Thus, we have a further equivalent simplification of the expression of BCR12:

$$m_{BCR12}(X|B) = m(X) \;+\; [m(X) \cdot \sum_{\substack{Z \in D_2 \\ Z \cap B = \emptyset}} m(Z)] \;/ \sum_{Y \in D_1} m(Y) \;+ \sum_{\substack{W \in D_2 \cup D_3 \\ W \cap B = X}} m(W).$$

How BCR12 works on Shafer's model? On Shafer's model we have

$$D_1 = \{X \,|\, \emptyset \neq X \subseteq B\},$$

$$D_2 = \mathcal{P}(s(B)) = \{X \,|\, \emptyset \neq X \subseteq \Theta \setminus B\},$$

$$D_3 = \mathcal{P}(\Theta) \setminus (D_1 \cup D_2 \cup \emptyset) = \{X \,|\, X \cap B \neq \emptyset,\; X \nsubseteq B\}.$$

Thus

$$\begin{aligned}\{Z \,|\, Z \in D_2 \,\&\, (\nexists Y \in D_1)(Y \subseteq Z)\} &= \{Z \,|\, \emptyset \neq Z \subseteq \Theta \setminus B \,\&\, (\nexists Y \subseteq B)(Y \subseteq Z)\} \\ &= D_2 = \{Z \,|\, \emptyset \neq Z \subseteq \Theta \setminus B\},\end{aligned}$$

and

$$D_2 \cup D_3 = \{Z \,|\, \emptyset \neq Z \subseteq \Theta \;\&\; Z \nsubseteq B\}.$$

Hence we can rewrite BCR12 on Shafer's model as it follows:

$$m_{BCR12}(X|B) = m(X) \;+\; [m(X) \cdot \sum_{Z \subseteq \Theta \setminus B} m(Z)] \;/ \sum_{Y \subseteq B} m(Y) + \sum_{\substack{W \subseteq \Theta \,\&\, W \nsubseteq B \\ W \cap B = X}} m(W).$$

We can easily see, that this special case of BCR12 does not coincide with Dempster's conditioning in general, and subsequently that BCR12 does not generalize Dempster's rule of conditioning in full generality.

When does BCR12 coincide with Dempster's rule of conditioning? It coincides whenever belief masses of all Zs from definition of BCR12 are zero or whenever belief masses of all Ws are zero, i.e. whenever $m(X) = 0$ for all $X \subseteq \Theta \setminus B$ or $m(X) = 0$ for all X such that $X \cap B \neq \emptyset \;\&\; X \nsubseteq B$. And analogically in the generalized case. Hence we have proven the following lemma:

Lemma 3 *(o) BRC12 is not a generalization of Dempster's rule of conditioning in general.*

(i) BCR12 coincides with Dempster's rule of conditioning on Shafer's model for belief functions, such that $m(X) = 0$ *for all* $X \subseteq \Theta \setminus B$, *or*
$m(X) = 0$ *for all* X *such that* $X \cap B \neq \emptyset \,\&\, X \nsubseteq B$.

(ii) BCR12 is a generalization of Dempster's rule of conditioning only for belief functions, such that $m(X) = 0$ for all $X \in D_2$ such that $X \cap B \equiv \emptyset$, or $m(X) = 0$ for all $X \in D_2 \cup D_3$ such that $X \cap B \not\equiv \emptyset$.

The above results are not surprising, when we simply compare formulas of BCR12 and of the real complete generalization of Dempster's rule of conditioning, which is presented in this paper.

To conclude this section, we have to note, that there is still large open area for evaluation of BCR rules, for full comparison of BCR rules with the classic conditioning rules and their generalization presented in this contribution, and also for further development of belief conditioning in the context of DSmT.

7 Related Works

When speaking about belief representation and processing on a lattice based structure, we have to mention also other related approaches, namely works by Besnard [2] and his former PhD students Jaouen and Perin, who have proposed to replace the classical Boolean algebras with a distributive lattice, hoping, it might solve Smets' bomb issue. Their distributive lattice generated on a frame of discernment is the free DSm model, in fact. These authors use a conflicting relation for a construction of their evidential structure. There is no concept of negation, similarly to DSm approach. A comparison of their conflicting relation with DSm constraints, and comparison of their evidential structures with hybrid DSm models is still an open problem for a future research, to formulate a relationship between the two independently developed approaches to belief combination on distributive lattices.

Both the approaches are also related to minC combination [3], which manage classic belief functions on power set of a frame of discernment, but on internal working level it uses also a lattice based structure for representation of different types of conflicts.

Reviewing and comparative study of all these approaches was started, but unfortunately unfinished, by Philippe Smets in 2004/2005.

In the end we should mention also, that a comparison of DSm approach with Dempster-Shafer theory applied to frames of discernment with overlapping elements is just under development.

8 Conclusion

In this paper, we have briefly introduced Dempster-Shafer and DSm theories with the focus on belief conditioning. Classic belief conditioning rules were generalized to DSm hyper-power set and to any DSm hybrid model used in DSmT. These generalizations perform a solid theoretical background for a serious objective comparison of the DSm belief conditioning rules (BCR) with the classical ones. A relationship of the generalized conditioning rules to some of the DSm belief conditioning rules has been outlined.

The presented results enable a deeper understanding of belief conditioning in DSmT and a better placement of DSmT among other approaches to belief functions.

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