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A General Stock Model for Fuzzy Markets

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Abstract

This paper presents a general stock model for fuzzy markets based on a class of fuzzy process known as Liu process. Firstly, a brief history of stock models and some methodologies used in stochastic stock models are reviewed. Next, some useful concepts and properties about fuzzy process are presented. Then, a general stock model for fuzzy markets is formulated by the way of fuzzy differential equation. This basic single-factor model leads quite naturally to multi-factor extensions. Some option pricing formulas on the proposed fuzzy stock model are investigated. Finally, some remarks are made in the concluding section.

Keywords: fuzzy stock model, fuzzy process, Liu process, fuzzy differential equation, option pricing formula

1 Introduction

Since the introduction of the well-known Black-Scholes model in 1973 [2], stochastic processes have played an increasingly important role in mathematical finance. While the celebrated Black-Scholes model, based on geometric Brownian motion, has been widely used in the analysis of derivatives pricing and portfolio management (see [1, 6, 7, 8, 9]). In many cases mathematical finance models are expressed in terms of stochastic differential equations [19, 20]. Usually, the analytic solutions of stochastic differential equations are rarely known. In general, these equations must be solved by using numerical approximation schemes [21].

Randomness is a basic type of objective uncertainty, and probability theory is a branch of mathematics for studying the behavior of random phenomena. The traditional stock models in a stochastic environment are described by real stochastic process. However, we are often faced with the case that the value of variables are partially observed by dimness of perception or measurement imprecision. Therefore, fuzzy stock models in an uncertain world are needed.

To reflect the uncertainty in a fuzzy stock market, some researchers model it as a fuzzy system [22, 23]. It has been recognized that there are needs for suitable models to better capture the price movements of the underlying securities in fuzzy environments. Fuzzy calculus has attracted a growing interest during the last few years and various models have been proposed [5, 16, 18].

The concept of fuzzy set was initiated by Zadeh [24] via membership function in 1965. In order to measure a fuzzy event, Liu and Liu [12] introduced the concept of credibility measure in 2002. Li and Liu [10] gave a sufficient and necessary condition for credibility measure in 2006 (cf. Liu [14]). Credibility theory was founded by Liu [13] in 2004 and refined by Liu [15] in 2007 as a branch of mathematics for studying the behavior of fuzzy phenomena. Credibility theory is deduced from the normality, monotonicity, self-duality, and maximality axioms. Liu [16] recently introduced the concepts of fuzzy process, Liu process and the geometric Liu process which will be commonly used model in finance for the value of an asset in a fuzzy environment. The two types of fundamental and important fuzzy processes, Liu process and the geometric Liu process, are the counterparts of Brownian motion and the geometric Brownian motion, respectively.

The remainder of this paper is structured as follows. The next section is intended to introduce some useful concepts of fuzzy process as they are needed. Section 3 reviews two types of stock models. A general stock model for fuzzy markets is formulated in Section 4. Some option pricing formulas on the proposed fuzzy stock model are investigated in Section 5. Finally, some remarks are made in Section 6.

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2 Preliminaries

In this section, we will introduce some useful definitions and properties about fuzzy process.

2.1 Fuzzy Process

Definition 1 *(Liu [16]) Given an index set T and a credibility space $(\Theta, \mathcal{P}, \mathrm{Cr})$, a fuzzy process is a function from $T \times (\Theta, \mathcal{P}, \mathrm{Cr})$ to the set of real numbers.*

In other words, a fuzzy process $X(t,\theta)$ is a function of two variables such that the function $X(t^*,\theta)$ is a fuzzy variable for each t^*. For simplicity, sometimes we use the symbol X_t instead of longer notation $X(t,\theta)$.

Definition 2 *(Liu[16]) A fuzzy process X_t is said to have independent increments if*

$$X_{t_1} - X_{t_0},\, X_{t_2} - X_{t_1},\, \cdots,\, X_{t_k} - X_{t_{k-1}} \tag{1}$$

are independent fuzzy variables for any times $t_0 < t_1 < \cdots < t_k$. A fuzzy process X_t is said to have stationary increments if, for any given $s > 0$, $X_{t+s} - X_t$ are identically distributed fuzzy variables for all t.

2.2 Liu Process

Definition 3 *(Liu[16]) A fuzzy process C_t is said to be a Liu process if (i) $C_0 = 0$, (ii) C_t has stationary and independent increments, and (iii) every increment $C_{s+t} - C_s$ is a normally distributed fuzzy variable with expected value et and variance $\sigma^2 t^2$, whose membership function is*

$$\mu(x) = 2\left(1 + \exp\left(\frac{\pi|x - et|}{\sqrt{6}\sigma t}\right)\right)^{-1}, \quad x \in \Re.$$

The parameters e and σ ($\sigma > 0$) are called the drift and diffusion coefficients, respectively. The Liu process is said to be standard if $e = 0$ and $\sigma = 1$.

The Liu process plays the role of the counterpart of Brownian motion.

Definition 4 *(Liu[16]) Let C_t be a standard Liu process. Then the fuzzy process*

$$G_t = \exp(et + \sigma C_t)$$

is called a geometric Liu process, or sometimes exponential Liu process.

The geometric Liu process is expected to model stock prices in a fuzzy environment. Li and Qin [11] has deduced that G_t is of a lognormal membership function

$$\mu(x) = 2\left(1 + \exp\left(\frac{\pi|\ln x - et|}{\sqrt{6}\sigma t}\right)\right)^{-1}, \quad x \geq 0.$$

2.3 Fuzzy Differential Equation

Suppose C_t is the standard Liu process, and f and g are some given functions. It is interesting to find an unknown fuzzy process X_t such that

$$\mathrm{d}X_t = f(t, X_t)\mathrm{d}t + g(t, X_t)\mathrm{d}C_t \tag{2}$$

which is called a fuzzy differential equation ([16]). A fuzzy process is called the solution of (2) if it satisfies (2) identically in t.

It is easy to see that the fuzzy differential equation

$$\mathrm{d}X_t = a\mathrm{d}t + b\mathrm{d}C_t$$

has a solution $X_t = at + bC_t$ which is just the Liu process with drift coefficient a and diffusion coefficient b.

It is not difficult to verify that the fuzzy differential equation

$$\frac{\mathrm{d}X_t}{X_t} = a\mathrm{d}t + b\mathrm{d}C_t$$

has a solution $X_t = \exp(at + bC_t)$ which is just a geometric Liu process.

3 Some Fuzzy Stock Models

In this section, we take a closer look at two types of fuzzy stock models for fuzzy financial markets. These models are given by the fuzzy differential equation and frequently used to simulate future bond prices, interest rates, or financial derivatives.

3.1 Liu's Stock Model

Let X_t be the bond price and Y_t the stock price. Assume that stock price follows a geometric Liu process. Then Liu [16] characterizes the price dynamics as follows,

$$\begin{cases} \mathrm{d}X_t = rX_t\mathrm{d}t \\ \mathrm{d}Y_t = eY_t\mathrm{d}t + \sigma Y_t\mathrm{d}C_t \end{cases} \tag{3}$$

where r is the riskless interest rate, e is the stock drift, and σ is the stock diffusion.

3.2 Gao's Stock Model

Let X_t be the bond price and Y_t the stock price. Let X_t be the bond price and Y_t the stock price. Then Gao [5] represents the price dynamics as follows:

$$\begin{cases} \mathrm{d}X_t = rX_t\mathrm{d}t \\ \mathrm{d}Y_t = (m - \alpha Y_t)\mathrm{d}t + \sigma\mathrm{d}C_t \end{cases} \tag{4}$$

where m, α, σ are constants.

4 A General Stock Model

Here our goal is to show a general stock model. Let X_t be the bond price, and Y_t the stock price. Then we express the price dynamics as follows:

$$\begin{cases} \mathrm{d}X_t = rX_t\mathrm{d}t \\ \mathrm{d}Y_t = (m - \alpha Y_t)\mathrm{d}t + \sigma Y_t^{\beta}\mathrm{d}C_t \end{cases} \tag{5}$$

where r is the constant interest rate, m, α, σ, β are constants, σY_t^{β} is the variable stock volatility and $(m - \alpha Y_t)$ is the variable stock drift.

Specifically, it has different special cases. When $\beta = 0.5$, it becomes a counterpart of Cox-Ingersoll-Ross model [3, 4] which follows a square-root process. When $\beta = 0$, it is just Gao's stock model. When $m = 0, \beta = 1$, it is just Liu's stock model.

There is a natural generalization for the above stock model. The fuzzy differential equations for the generalized model are

$$\begin{cases} \mathrm{d}X_t = r_tX_t\mathrm{d}t \\ \mathrm{d}Y_t = (m_t - \alpha_t Y_t)\mathrm{d}t + \sigma_t Y_t^{\beta_t}\mathrm{d}C_t \end{cases} \tag{6}$$

where $r_t, m_t, \alpha_t, \beta_t, \sigma_t$ are deterministic functions of time t.

In order to adapt the above model to be more consistent with the general cases, we introduce an extension of the model with time-dependent parameters. The most general model satisfies the following fuzzy differential equations

$$\begin{cases} \mathrm{d}X_t = r_tX_t\mathrm{d}t \\ \mathrm{d}Y_t = a(t, Y_t)\mathrm{d}t + b(t, Y_t)\mathrm{d}C_t \end{cases} \tag{7}$$

where $r_t, a(t, \cdot), b(t, \cdot)$ are deterministic functions of time t.

Different from the stock price dynamics driven by a single factor, multiple stocks can be driven by multiple Liu processes. Instead of just one Liu process, we will have, in the n-factor case, n independent Liu processes $C_{1t}, C_{2t}, \cdots, C_{nt}$. That means that each $C_{jt} (j = 1, 2, \cdots, n)$ behaves as a Liu process, and the behaviour of any one of them is completely uninfluenced by the movements of the others. Letting X_t be the bond price, and

Y_{it} the stock price, $i = 1, 2, \cdots, m$. Now this basic single-factor model leads quite naturally to multi-factor extensions. Then the multi-factor price dynamics can be modelled as

$$\begin{cases} \mathrm{d}X_t = rX_t\mathrm{d}t \\ \mathrm{d}Y_{it} = (m_i - \alpha_i Y_{it})\mathrm{d}t + \sum_{j=1}^{n} \sigma_{ij} Y_{jt}^{\beta_j} \mathrm{d}C_{jt}, \;\; i = 1, 2, \cdots, m \end{cases} \tag{8}$$

where $r, m_i, \alpha_i, \beta_j, \sigma_{ij}$ are deterministic real numbers for $i = 1, 2, \cdots, m$ and $j = 1, 2, \cdots, n$, respectively. Thus, this multi-factor model has n independent Liu processes and $m + 1$ equations.

5 Some Option Pricing Formulas

5.1 American Options

An option is a financial instrument which gives the holder a right without being under obligation to trade the underlying asset at or by expiry date for a certain prescribed price known as exercise or strike price (see [17]).

Let X_t be the bond price and Y_t the stock price. Let us denote by Y_T the price of the underlying asset at the expiry time T and K the strike or exercise price.

American options are financial instruments that give their holders the right to trade an underlying asset at any time $t \leq T$ for prescribed price K without being obliged to do so.

American call option is an option that provides right to the holder to buy an underlying asset at any time $t \leq T$ for the strike price K without being under obligation to do so. American put option gives the right to the holder to sell the underlying asset at time $t \leq T$ as long as the seller wishes to do so. In other words, the holder of the American call option will decide to buy the asset at any time before the option expires while the holder of American put option will decide to sell the asset at any time before expiration of the option.

The payoff of American call option is given by

$$G(Y, K) = (Y_t - K)^+ = \begin{cases} Y_t - K, & \text{if } Y_t > K, \; t \leq T \\ 0, & \text{otherwise.} \end{cases} \tag{9}$$

This equation means that the American call option can only be exercised when the price Y_t of the underlying asset is greater than the strike price K. Likewise the payoff of the American put option is given as

$$G(Y, K) = (K - Y_t)^+ = \begin{cases} K - Y_t, & \text{if } Y_t < K, \; t \leq T \\ 0, & \text{otherwise.} \end{cases} \tag{10}$$

There are American options with no expiry dates. Such options are called perpetual American options or American options with infinite time horizon. American options whose expiry dates are known in advance are said to be of finite time horizon. The difference between perpetual American options from other American options is that they cannot be traded but they are dealt as mathematical problems.

5.2 Option Pricing Formulas

In this section we give a derivation of the price in our general stock model. In finance, the options pricing formulas may provide a numerical method for the valuation of options.

Definition 5 *Let X_t be the bond price and Y_t the stock price. American call option price f for the general stock model (5) is defined as*

$$f(Y_0, K, m, \alpha, \beta, \sigma, r) = \max_{0 \leq t \leq T} E[\exp(-rt)(Y_t - K)^+] \tag{11}$$

where K is the strike price at exercise time t.

Theorem 1 *(American Call Option Pricing Formula) Let X_t be the bond price, and Y_t the stock price. Suppose that X_t and Y_t satisfying the price dynamics described by the general stock model (5). Then the American call option pricing formula is given by*

$$\begin{aligned} &f(Y_0, K, m, \alpha, \beta, \sigma, r) \\ =& \max_{0\le t\le T} \exp(-rt) \int_K^{\infty} \mathrm{Cr}\left\{\frac{m}{\alpha} + \exp(-\alpha t)(Y_0 - \frac{m}{\alpha}) + \sigma \exp(-\alpha t)\int_0^t \exp(\alpha u) Y_u^{\beta} \mathrm{d}C_u \ge s\right\} \mathrm{d}s. \end{aligned} \tag{12}$$

Proof. Firstly, it follows from Liu formula that

$$\begin{aligned} \mathrm{d}(\exp(\alpha t)Y_t) &= \alpha \exp(\alpha t) Y_t \mathrm{d}t + \exp(\alpha t)\mathrm{d}Y_t \\ &= \alpha \exp(\alpha t) Y_t \mathrm{d}t + \exp(\alpha t)(m - \alpha Y_t)\mathrm{d}t + \exp(at)\sigma Y_t^{\beta}\mathrm{d}C_t \\ &= m\exp(\alpha t)\mathrm{d}t + \sigma \exp(\alpha t) Y_t^{\beta} \mathrm{d}C_t. \end{aligned}$$

Integration of both sides of above equation yields

$$\exp(\alpha t)Y_t - Y_0 = m\int_0^t \exp(\alpha s)\mathrm{d}s + \sigma\int_0^t \exp(\alpha s) Y_s^{\beta}\mathrm{d}C_s.$$

This means

$$Y_t = \frac{m}{\alpha} + \exp(-\alpha t)(Y_0 - \frac{m}{\alpha}) + \sigma\exp(-\alpha t)\int_0^t \exp(\alpha s) Y_s^{\beta}\mathrm{d}C_s.$$

Secondly, according to the definition of expected value of fuzzy variable, we have

$$\begin{aligned} &f(Y_0, K, m, \alpha, \beta, \sigma, r) \\ =& \max_{0\le t\le T} \exp(-rt) E[(Y_t - K)^+] \\ =& \max_{0\le t\le T} \exp(-rt) \int_0^{\infty} \mathrm{Cr}\{(Y_t - K)^+ \ge s\}\mathrm{d}s \\ =& \max_{0\le t\le T} \exp(-rt) \int_0^{\infty} \mathrm{Cr}\{Y_t - K \ge s\}\mathrm{d}s \\ =& \max_{0\le t\le T} \exp(-rt) \int_0^{\infty} \mathrm{Cr}\{Y_t \ge K + s\}\mathrm{d}s \\ =& \max_{0\le t\le T} \exp(-rt) \int_K^{\infty} \mathrm{Cr}\{Y_t \ge s\}\mathrm{d}s \\ =& \max_{0\le t\le T} \exp(-rt) \int_K^{\infty} \mathrm{Cr}\left\{\frac{m}{\alpha} + \exp(-\alpha t)(Y_0 - \frac{m}{\alpha}) + \sigma \exp(-\alpha t)\int_0^t \exp(\alpha u) Y_u^{\beta} \mathrm{d}C_u \ge s\right\} \mathrm{d}s. \end{aligned}$$

This yields the desired result and completes the proof.

Similarly, we can deduce the following result.

Definition 6 *Let X_t be the bond price and Y_t the stock price. American put option price f for the general stock model (5) is defined as*

$$f(Y_0, K, m, \alpha, \beta, \sigma, r) = \max_{0\le t\le T} E[\exp(-rt)(K - Y_t)^+] \tag{13}$$

where K is the strike price at exercise time t.

Theorem 2 *(American Put Option Pricing Formula) Let X_t be the bond price and Y_t the stock price. Suppose that X_t and Y_t satisfying the price dynamics described by the general stock model (5). Then the American put option pricing formula is given by*

$$\begin{aligned} &f(Y_0, K, m, \alpha, \beta, \sigma, r) \\ =& \max_{0\le t\le T} \exp(-rt) \int_K^{\infty} \mathrm{Cr}\left\{\frac{m}{\alpha} + \exp(-\alpha t)(Y_0 - \frac{m}{\alpha}) + \sigma \exp(-\alpha t)\int_0^t \exp(\alpha u) Y_u^{\beta} \mathrm{d}C_u \le s\right\} \mathrm{d}s. \end{aligned} \tag{14}$$

Proof. According to the definition of expected value of fuzzy variable, we have

$$
\begin{aligned}
& f(Y_0, K, m, \alpha, \beta, \sigma, r) \\
= & \max_{0\le t\le T} \exp(-rt) E[(K - Y_t)^+] \\
= & \max_{0\le t\le T} \exp(-rt) \int_0^{\infty} \mathrm{Cr}\{(K - Y_t)^+ \ge s\} \mathrm{d}s \\
= & \max_{0\le t\le T} \exp(-rt) \int_0^{\infty} \mathrm{Cr}\{K - Y_t \ge s\} \mathrm{d}s \\
= & \max_{0\le t\le T} \exp(-rt) \int_0^{\infty} \mathrm{Cr}\{Y_t \le K + s\} \mathrm{d}s \\
= & \max_{0\le t\le T} \exp(-rt) \int_K^{\infty} \mathrm{Cr}\{Y_t \le s\} \mathrm{d}s \\
= & \max_{0\le t\le T} \exp(-rt) \int_K^{\infty} \mathrm{Cr}\left\{ \frac{m}{\alpha} + \exp(-\alpha t)(Y_0 - \frac{m}{\alpha}) + \sigma \exp(-\alpha t) \int_0^t \exp(\alpha u) Y_u^{\beta} \mathrm{d}C_u \le s \right\} \mathrm{d}s.
\end{aligned}
$$

This yields the desired result and completes the proof.

6 Conclusion

The main contribution of the present paper is to suggest a general stock model for fuzzy markets by means of Liu process. Some option pricing formulas on the proposed fuzzy stock model are investigated.

The proposed model can be further extended in many straightforward ways. Based on the methodology or results, some potential applications of fuzzy stock models will be an interesting topic of further research.

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