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A New Stock Model for Credibilistic Option Pricing

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Abstract

Thirty years ago, Black and Scholes assumed that stock price follows geometric Brownian motion, and stochastic financial mathematics was then founded based on this assumption. Recently, professor Baoding Liu proposed the notion of fuzzy process as well as a spectrum of fuzzy calculus. After that, he gave a conjecture that the stock price may also follow geometric Liu process and proposed a fuzzy counterpart of the Black and Scholes model. This paper presents the fuzzy counterpart of the Black-Karasinski model. After a discussion of the new fuzzy stock model, we give an application to option pricing.

Keywords: fuzzy process, Liu process, fuzzy differential equation, fuzzy stock model, option pricing formula

1 Introduction

Randomness is well-known as one type of uncertainty, and probability theory is a branch of mathematics for dealing with random phenomena. Stochastic process is concerned with modeling and analysis of random phenomena over time or space. In 1944, Itô proposed the stochastic calculus and established the celebrated Itô formula. In 1969, Merton introduced stochastic calculus into the study of finance. Then, Merton [15] and Black and Scholes [2], independently, used the geometric Brownian motion to construct a theory for determining the stock options price. Their work won the 1997 Nobel Prize in Economics, and stochastic financial mathematics was then founded on this work.

Fuzziness mainly concerns with phenomena with vague and subjective information, especially those lack of or even without historical records. So fuzziness is not an opposition but a complementary of randomness. The concept of fuzzy set was initially proposed by Zadeh [21] via membership function in 1965. The credibility theory was founded [10] and refined [12] by Liu. Different from other fuzzy theories (e.g., possibility theory [21][4]), the credibility theory is based on an axiomatic system, which makes it a mathematical branch for dealing with fuzzy phenomena.

In order to find out the evolution of fuzzy phenomena with time, Liu introduced the concept of fuzzy process. As two special fuzzy processes, Liu process and Geometric Liu process [13] play the role of Brownian motion and geometric Brownian motion, respectively. Meanwhile, a spectrum of fuzzy calculus is developed [5][13][14]. By assuming that stock price follows geometric Liu process, Liu [13] presented a basic reference stock model–Liu stock model, which is a fuzzy counterpart of Black-Scholes stock model [2]. These work, seemingly, will open a new field of fuzzy financial engineering. In this paper, we go further by presenting and investigating a new fuzzy stock model, a fuzzy counterpart of the Black-Karasinski model.

The rest of this paper is organized as follows. In the following section, some basic concepts and properties about fuzzy process are recalled. Then in Section 3, we propose the new fuzzy stock model as well as its solution. In Section 4, we present the formula for valuing European call and put option based on our model. And in Section 5, we assume that the short rate follows a standard Liu process, and discuss the pricing of a zero-coupon. Finally, some conclusions are listed.

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2 Preliminaries

Definition 1 (Liu [13]) *Let T be an index set and let $(\Theta, \mathcal{P}, \mathrm{Cr})$ be a credibility space. A fuzzy process is a function from $T \times (\Theta, \mathcal{P}, \mathrm{Cr})$ to the set of real numbers.*

That is, a fuzzy process $X(t;\theta)$ is a function of two variables such that the function $X(t^*;\theta)$ is a fuzzy variable for each $t^* \in T$. For each fixed θ^*, the function $X(t;\theta^*)$ is called a sample path of the fuzzy process. A fuzzy process $X(t;\theta)$ is said to be sample-continuous if the sample path is continuous for almost all θ. Instead of longer notation $X(t;\theta)$, sometimes we use the symbol X_t.

Definition 2 (Liu [13]) *A fuzzy process X_t is said to have independent increments if*

$$X_{t_1} - X_{t_0}, X_{t_2} - X_{t_1}, \cdots, X_{t_k} - X_{t_{k-1}} \tag{1}$$

are independent fuzzy variables for any times $t_0 < t_1 < \cdots < t_k$. A fuzzy process X_t is said to have stationary increments if for any given $t > 0$, the $X_{s+t} - X_s$ are identically distributed fuzzy variables for all $s > 0$.

Definition 3 (Liu [13]) *A fuzzy process C_t is said to be a Liu process if*
(i) $C_0 = 0$,
(ii) C_t has stationary and independent increments,
(iii) every increment $C_{s+t} - C_s$ is a normally distributed fuzzy variable with expected value et and variance $\sigma^2 t^2$, whose membership function is

$$\mu(x) = 2\left(1 + \exp\left(\frac{\pi \mid x - et \mid}{\sqrt{6}\sigma t}\right)\right)^{-1}, \quad \boldsymbol{x} \in \Re.$$

The parameters e and σ are called the drift and diffusion coeffients, respectively. The Liu process is said to be standard if $e = 0$ and $\sigma = 1$. Any Liu process may be represented by $et + \sigma C_t$, where C_t is a standard Liu process.

Definition 4 (Liu [13]) *Let C_t be a standard Liu process. Then $et + \sigma C_t$ is a Liu process, and the fuzzy process*

$$X_t = exp(et + \sigma C_t) \tag{2}$$

is called a geometric Liu process.

Theorem 1 (Liu [13]) *Let C_t be a standard Liu process, and h(t,c) a continuously differentiable function. Denote $X_t = h(t, C_t)$. Then we have the following chain rule*

$$dX_t = \frac{\partial h}{\partial t}(t, C_t)dt + \frac{\partial h}{\partial c}(t, C_t)dC_t. \tag{3}$$

Traditionally, it was assumed that stock price follows geometric Brownian motion, and stochastic financial mathematics was founded based on this assumption. With the introduction of the fuzzy process Liu [13] as well as a spectrum of fuzzy calculus [5][13][14], Liu presented an alternative assumption that stock price follows geometric Liu process. Based on this assumption, it is expected to reconsider option pricing, optimal stopping, portfolio selection and so on, thus producing a totally new fuzzy financial mathematics. Liu [13] presented a basic stock model for fuzzy financial market in which the bond price X_t and the stock price Y_t follow

$$X_t = X_0 \exp(rt)$$
$$Y_t = Y_0 \exp(et + \sigma C_t)$$

or equivalently

$$\begin{aligned} \mathrm{d}X_t &= rX_t\mathrm{d}t \\ \mathrm{d}Y_t &= eY_t\mathrm{d}t + \sigma Y_t\mathrm{d}C_t, \end{aligned} \tag{4}$$

where r is the riskless interest rate, e is the stock drift, and σ is the stock diffusion. It is just a fuzzy counterpart of Black-Scholes stock model [2].

3 A New Fuzzy Stock Model

The Black-Karasinski model [1] is a basic model in stochastic financial engineering. In this section, we will give a fuzzy counterpart of the Black-Karasinski model by assuming that the stock price follows geometric Liu process.

Let X_t be the bond price and Y_t the stock price.

$$\begin{aligned} dX_t &= rX_t dt \\ dY_t &= a(b - Y_t)dt + \sigma dC_t. \end{aligned} \tag{5}$$

This model incorporates a general economic behavior: mean reversion. Mean reversion means that the stock prices appear to be pulled back to some long-run average level over time. That is, when the stock price Y_t is high, mean reversion tends to cause it to have a negative drift; when the stock price Y_t is low, mean reversion tends to cause it to have a positive drift. In our model, the stock price is pulled to a level b at rate a. Superimposed upon this pull is a normally distributed fuzzy term σdC_t.

Now, we derive the analytical solution for our model (5). Rewrite $dY_t = a(b - Y_t)dt + \sigma dC_t$ as

$$\exp(at)dY_t + a\exp(at)Y_t dt = ab\exp(at)dt + \sigma\exp(at)dC_t. \tag{6}$$

Here it is tempting to relate the left hand side to $d(\exp(at)Y_t)$. Using Liu's formula, we obtain

$$d(\exp(at)Y_t) = a\exp(at)Y_t dt + \exp(at)dY_t. \tag{7}$$

Substituted this in equation (6) gives

$$d(exp(at)Yt) = ab\exp(at)dt + \sigma\exp(at)dC_t. \tag{8}$$

Then we have

$$\exp(at)Y_t - Y_0 = ab\int_0^t \exp(as)ds + \sigma\int_0^t \exp(as)dC_s \tag{9}$$

or

$$Y_t = \exp(-at)\left(Y_0 + \sigma\exp(at)C_t + a\int_0^t \exp(as)(b - \sigma C_s)ds\right). \tag{10}$$

by integration by parts.

4 Valuing European Option

A European call option gives the holder the right, but not the obligation, to buy a stock at a specified time for a specified price. Considering the stock model (5), we assume that a European call option has strike price K and expiration time T. Then the payoff from buying a European call option is $(Y_T - K)^+$. Considering the time value of money, the present value of this payoff is $\exp(-rT)(Y_T - K)^+$.

Definition 5 *European call option price f for the stock model (5) is defined as*

$$f(Y_0, K, a, b, \sigma, r) = \exp(-rT)\mathrm{E}\left[\left(\exp(-rT)(Y_0 + \sigma\exp(aT)C_T + a\int_0^T \exp(at)(b - \sigma C_t)dt) - K\right)^+\right],$$

where K is the strike price at time T.

A European put option gives the holder the right, but not the obligation, to sell a stock at a specified time for a specified price. Similarly, we can get the European put option price formula as follows.

Definition 6 *European call option price f for the stock model (5) is defined as*

$$f(Y_0, K, a, b, \sigma, r) = \exp(-rT)\mathrm{E}\left[\left(K - \exp(-rT)(Y_0 + \sigma\exp(aT)C_T + a\int_0^T \exp(at)(b - \sigma C_t)dt)\right)^+\right],$$

where K is the strike price at time T.

5 Valuing Zero-Coupon

In this section, we consider the zero-coupon when the short rate follows our stock model (5) and discuss the problem of pricing zero-coupon.

Equilibrium models usually start with assumptions about economic variables and derive a process for the short rate r. Then they explore what the process for r implies about the bond and option price. The short rate, r, at time t is the rate that applies to an infinitesimally short period of time at time t. Suppose that r is described by an standard Liu process of the form

$$dr_t = a(b - r_t)dt + \sigma dC_t, \tag{11}$$

where a, b and σ are constants. By this model, the short rate is pulled to a level b at rate a. Superimposed upon this "pull" is a normally distributed fuzzy term σdC_t.

A zero-coupon bond is one type of bond that pays off 1 at time T. Let $P = P(r, t, T)$ as the price at time t of a zero-coupon. Then by the fuzzy chain rule in Theorem 1, we have

$$dP(r_t, t, T) = \frac{\partial P}{\partial r_t} dr_t + \frac{\partial P}{\partial t} dt = \left(a(b - r_t) \frac{\partial P}{\partial r_t} + \frac{\partial P}{\partial t} \right) dt + \frac{\partial P}{\partial r_t} \sigma dC_t. \tag{12}$$

Start from this partial differential equation, we may use analytical or numerical approaches to value the zero-coupon.

6 Conclusion

In this paper, we proposed a new fuzzy stock model, and derived its analytical solution. Then we presented the European call and put option price formula for this model. As the fuzzy counterpart of the Black and Karasinski model, this model can be used to value bonds and options as directed in Sections 4 and 5.

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