ISSN 1750-9823 (print)
International Journal of Sports Science and Engineering
Vol. 05 (2011) No. 01, pp. 027-042

Approximate Analytical Investigation of Projectile Motion in a Medium with Quadratic Drag Force

Peter S. Chudinov

Perm State Agricultural Academy, Perm, 614990, Russian Federation

(*Received October 1, 2010, accepted December 9, 2010*)

Abstract. The classic problem of the motion of a point mass (projectile) thrown at an angle to the horizon is reviewed. The air drag force is taken into account in the form of a quadratic function of velocity with the coefficient of resistance assumed to be constant. Analytical methods for the investigation are mainly used. With the help of simple approximate analytical formulas a full investigation of the problem was carried out. This study includes the determining of eight basic parameters of projectile motion (flight range, time of flight, maximum ascent height and others). The study also includes the construction of the basic functional dependences of the motion, the determination of the optimum angle of throwing, providing the greatest range; constructing of the envelope of a family of trajectories of the projectile and finding the vertical asymptote of projectile motion. The motion of a baseball is presented as examples.

Keywords: projectile motion, quadratic drag force, analytical formulas.

1. Introduction

The problem of the motion of a point mass (projectile) thrown at an angle to the horizon has a long history. The number of works devoted to this task is immense. It is a constituent of many introductory courses of physics. This task arouses interest of authors as before [1 – 3]. With zero air drag force, the analytic solution is well known. The trajectory of the point mass is a parabola. In situations of practical interest, such as throwing a ball, taking into account the impact of the medium the quadratic resistance law is usually used. In that case the problem probably does not have an exact analytic solution and therefore in most scientific publications it is solved numerically [4 – 9]. Analytic approaches to the solution of the problem are not sufficiently advanced. Meanwhile, analytical solutions are very convenient for a straightforward adaptation to applied problems and are especially useful for a qualitative analysis. Comparativly simple approximate analytical formulas to study the motion of the point mass in a medium with a quadratic drag force have been obtained using such an approach [10 – 15]. These formulas make it possible to carry out a complete qualitative and quantitative analysis without using numerical integration of differential equations of point mass motion. This article brings together these works [10 -15] within a unified approach and gives a full investigation of the problem. The proposed analytical solution differs from other solutions by easy formulas, ease of use and high accuracy. In this article the following stages of research are consistently described:

- equations of motion and the construction of the trajectory;
- analytical formulas for determining the basic parameters of projectile motion (flight range, time of flight, maximum ascent height and others);
- analytical formulas for the basic functional dependences of the problem;
- the determination of the optimum angle of throwing, providing the greatest range;
- constructing the envelope of a family of trajectories of the projectile;
- finding the vertical asymptote of projectile motion.

All these characteristics are determined directly from the initial conditions of projectile motion - the initial velocity and angle of throwing. The proposed formulas make it possible to carry out a complete analytical investigation of the motion of a point mass in a medium with the resistance in the way it is done for the case of no drag. In this article the term "point mass" means the centre of mass of a smooth spherical object of finite radius r and cross-sectional area S = πr2. The conditions of applicability of the quadratic resistance law are deemed to be fulfilled, i.e. Reynolds number Re lies within $1\times 103 < Re < 2\times 105$ [2].

These values corresponds to the velocity of motion of a point, lying in the range between 0.25 m/s and 53 m/s.

2. Equations of motion and the construction of the trajectory

Suppose that the force of gravity affects the point mass together with the force of air resistance **R** (Figure 1), which is proportional to the square of the velocity of the point and directed opposite the velocity vector. For the convenience of further calculations, the drag force will be written as $R = mgkV^2$. Here m is the mass of the projectile, g is the acceleration due to gravity, k is the proportionality factor. Vector equation of the motion of the point mass has the form

$$m\mathbf{w} = m\mathbf{g} + \mathbf{R},$$

where **w** – acceleration vector of the point mass. Differential equations of the motion, a commonly used in ballistics, are as follows [16]

$$\frac{dV}{dt} = -g\sin\theta - gkV^2 , \quad \frac{d\theta}{dt} = -\frac{g\cos\theta}{V} , \quad \frac{dx}{dt} = V\cos\theta , \quad \frac{dy}{dt} = V\sin\theta \tag{1}$$

Here V is the velocity of the point mass, θ is the angle between the tangent to the trajectory of the point mass and the horizontal, x, y are the Cartesian coordinates of the point mass, $k = \dfrac{\rho_a c_d S}{2mg} = const$ is the proportionality factor, ρ_a is the air density, c_d is the drag factor for a sphere, and S is the cross-section area of the object (Figure 1). The first two equations of the system (1) represent the projections of the vector equation of motion for the tangent and principal normal to the trajectory, the other two are kinematic relations connecting the projections of the velocity vector point mass on the axis x, y with derivatives of the coordinates.

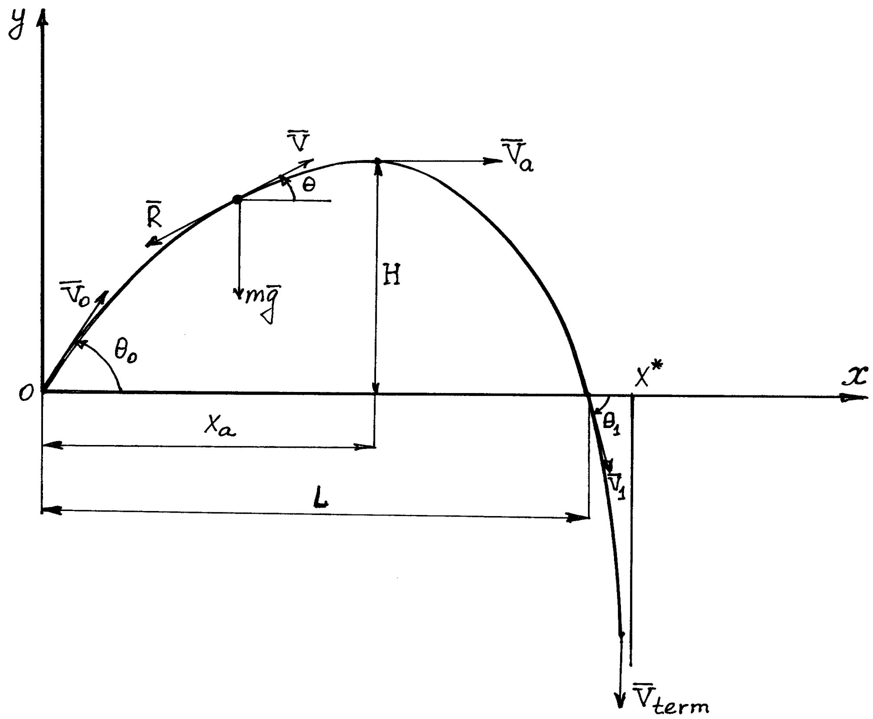

Fig. 1 : Basic motion parameters.

The well-known solution of Equations (1) consists of an explicit analytical dependence of the velocity on the slope angle of the trajectory and three quadratures

$$V(\theta) = \frac{V_0\cos\theta_0}{\cos\theta\sqrt{1 + kV_0^2\cos^2\theta_0\,(f(\theta_0) - f(\theta))}} , \quad f(\theta) = \frac{\sin\theta}{\cos^2\theta} + \ln\tan\left(\frac{\theta}{2} + \frac{\pi}{4}\right) \tag{2}$$

$$t=t_0-\frac{1}{g}\int_{\theta_0}^{\theta}\frac{V}{\cos\theta}d\theta\ ,\quad x=x_0-\frac{1}{g}\int_{\theta_0}^{\theta}V^2 d\theta\ ,\quad y=y_0-\frac{1}{g}\int_{\theta_0}^{\theta}V^2\tan\theta\, d\theta \tag{3}$$

Here V_0 and θ_0 are the initial values of the velocity and the slope of the trajectory respectively, t_0 is the initial value of the time, $x_{0,}y_0$ are the initial values of the coordinates of the point mass (usually accepted $t_0=x_0=y_0=0$). The derivation of the formulas (2) is shown in the well-known monograph [17].

The integrals on the right-hand sides of (3) cannot be expressed in terms of elementary functions. Hence, to determine the variables *t, x* and *y* we must either integrate (1) numerically or evaluate the definite integrals (3).

It turns out [10] that, using a special form of organised integration of quadratures (3) by parts in a fairly small interval $[\theta_0, \theta]$, the variables t, x and y can be written in the form

$$t=t_0+\frac{2(V_0\sin\theta_0-V\sin\theta)}{g(2+\varepsilon)}\ ,\quad x=x_0+\frac{V_0^2\sin 2\theta_0-V^2\sin 2\theta}{2g(1+\varepsilon)}$$

$$y=y_0+\frac{V_0^2\sin^2\theta_0-V^2\sin^2\theta}{g(2+\varepsilon)}\ ,\quad \varepsilon=k(V_0^2\sin\theta_0+V^2\sin\theta) \tag{4}$$

We will obtain the first of the formulae (4). The method of calculating the quadratures is based on the use of the relation between an auxiliary variable $u=V\cos\theta$ and the independent variable θ. This relation has the following differential form [16]

$$\frac{du}{u^3}=k\frac{d\theta}{\cos^3\theta} \tag{5}$$

We will consider the first of the quadratures (3) and we write it, using the relation $u=V\cos\theta$, in the form

$$t=t_0-\frac{1}{g}\int_{\theta_0}^{\theta}\frac{V}{\cos\theta}d\theta=t_0-\frac{1}{g}\int_{\theta_0}^{\theta}\frac{u}{\cos^2\theta}d\theta \tag{6}$$

We take the integral (6) by parts

$$t=t_0-\frac{u\tan\theta}{g}\Big|_{\theta_0}^{\theta}+\frac{1}{g}\int_{\theta_0}^{\theta}\tan\theta\, du=t_0-\frac{V\sin\theta}{g}\Big|_{\theta_0}^{\theta}+\frac{1}{g}\int_{\theta_0}^{\theta}\tan\theta\, du$$

Using relation (5) we convert the last term

$$\frac{1}{g}\int_{\theta_0}^{\theta}\tan\theta\, du=\frac{k}{g}\int_{\theta_0}^{\theta}V^3\tan\theta\, d\theta=-k\int_{\theta_0}^{\theta}V^2\sin\theta\, dt$$

Hence

$$t=t_0-\frac{V\sin\theta}{g}\Big|_{\theta_0}^{\theta}-ktV^2\sin\theta\Big|_{\theta_0}^{\theta}+k\int_{\theta_0}^{\theta}t\, d(V^2\sin\theta) \tag{7}$$

Suppose the range of integration $\theta-\theta_0=\Delta\theta$ is fairly small. Then the integral in (7) can be calaulated as the area of a trapezium with bases t_0, t and height $h=V^2\sin\theta-V_0^2\sin\theta_0$. We have

$$k\int_{\theta_0}^{\theta}t\, d(V^2\sin\theta)\approx\frac{k(t_0+t)}{2}\int_{\theta_0}^{\theta}d(V^2\sin\theta)=\frac{1}{2}k(t_0+t)(V^2\sin\theta-V_0^2\sin\theta_0)$$

As a result, formula (7) takes the form

$$t(1+\frac{\varepsilon}{2})=t_0(1+\frac{\varepsilon}{2})+\frac{V_0\sin\theta_0-V\sin\theta}{g}$$

(the variable ε is defined by last of relations (4)). Finally

$$t = t_0 + \frac{2(V_0 \sin\theta_0 - V\sin\theta)}{g(2+\varepsilon)}$$

We can similarly derive the other two formulas (4).

Hence, in a small interval $[\theta_0, \theta]$ the trajectory of the point mass can be approximated by Eqs (4). These formulas have a local nature. We can calculate the whole trajectory very accurately in steps by calculating $V(\theta)$, $t(\theta)$, $x(\theta)$, $y(\theta)$ using Eqs (2), (4) at the right-hand end of the interval $[\theta_0, \theta]$ and taking them as the initial values for the following step

$$V_0 = V(\theta)\ ,\quad t_0 = t(\theta)\ ,\quad x_0 = x(\theta)\ ,\quad y_0 = y(\theta)$$

This cyclical procedure replaces both numerical integration of system (1) and the evaluation of the integrals (3). The smaller the value of k the greater the range $[\theta_0, \theta]$ of applicability of the formulas obtained. When $k = 0$, i.e. when there is no drag, formulas (4) transform to the well-known accurate formulas of the theory of the parabolic motion pf a point mass and become valid for any values θ_0 and θ. Moreover, formulas (4) are accurate in those finite intervals of $[\theta_0, \theta]$ where the variables t, x and y depend linearly on the auxiliary variable $z = V^2\sin\theta$.

As calculations show, the trajectory obtained by integrating system of equations (1) and the trajectory constructed using formulas (2) and (4), are identical. Here, to construct the trajectory it is sufficient to use a step $\Delta\theta = \theta - \theta_0$ of the order of 0.1°.

3. Analytical formulas for determining the main parameters of motion of the point mass

Equations (4) enable us to obtain simple analytical formulas for the main parameters of motion of the point mass. In Figure 2 we have drawn a graph of the coordinate y (measured in meters) against the auxiliary dimensionless variable $R_y = -kV^2\sin\theta$, where R_y is the projection of the normalized drag of the medium on the y axis. We used the following values V_0, θ_0 and the coefficient k, the corresponding to the movement of the baseball [6]

$V_0 = 44.7$ m/s , $\theta_0 = 60°$, $k = 0.000548$ s^2/m^2 , g = 9.81 m/s^2.

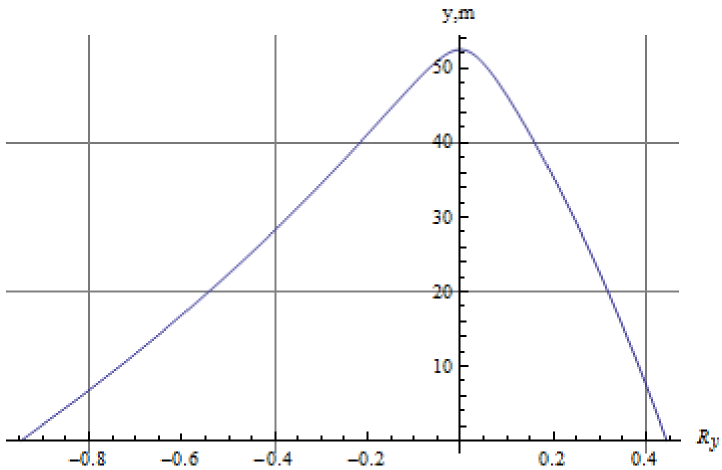

Fig. 2: The graph of the function $y = y(R_y)$.

The variable R_y is similar to the above-mentioned variable z. It can be seen that, both at the ascending stage (the left part of the graph) and the descending stage (the right part) this graph is close to linear. Hence it follows that the maximum height of ascent of the poin mass H can be obtained approximately using formula (4) for y in the finite interval $[\theta_0, 0]$, i.e. by taking $\theta = 0$ in this formula.

From the relation for the maximum height of ascent H we can derive comparatively simple approximate analytical formulas for the other parameters of motion of the poin mass. The four parameters correspond to the top of the trajectory, four – point of drop. We will give a complete summary of the formulas for the maximum height of ascent of the point mass H, motion time T, the velocity at the trajectory apex V_a, flight range L, the time of ascent t_a, the abscissa of the trajectory apex x_a, impact angle with respect to the horizontal θ_1 and the final velocity V_1:

$$H=\frac{V_0^2\sin^2\theta_0}{g(2+kV_0^2\sin\theta_0)}\,,\quad T=2\sqrt{\frac{2H}{g}}$$

$$V_a=\frac{V_0\cos\theta_0}{\sqrt{1+kV_0^2\left(\sin\theta_0+\cos^2\theta_0\ln\tan\left(\frac{\theta_0}{2}+\frac{\pi}{4}\right)\right)}}$$

$$L=V_aT\,,\quad t_a=\frac{T-kHV_a}{2}\,,\quad x_a=\sqrt{LH\cot\theta_0}$$

$$\theta_1=-\arctan\left[\frac{LH}{(L-x_a)^2}\right],\quad V_1=V(\theta_1) \tag{8}$$

In formulas (8) V_0 and θ_0 are the initial values of the velocity and the slope of the trajectory of the point mass, respectively. Formulas (8) enable us to calculate the basic parameters of motion of a point mass directly from the initial data V_0, θ_0, as in the theory of parabolic motion. With zero drag ($k = 0$), formulas (8) go over into the respective formulas of point mass parabolic motion theory.

As an example of the use of formulas (8) we calculated the motion of a baseball with the folowing initial conditions

$$V_0 = 45\ \text{m/s}\quad;\quad \theta_0 = 40°\quad;\quad k = 0.000548\ \text{s}^2/\text{m}^2,\ g = 9.81\ \text{m/s}^2.$$

Table 1.

Parameter	Numerical value	Analytical value	Error (%)
H, m	30.97	31.43	+1.5
T, sec	5.00	5.06	+1.2
V_a, m/s	23.19	23.19	0.0
L, m	117.8	117.4	-0.3
t_a, sec	2.35	2.33	-0.9
x_a, m	65.36	66.32	+1.5
θ_1, deg	-53.04	-54.73	+3.2
V_1, m/s	27.45	27.99	+2.0

The results of calculations are recorded in Table 1. The second column shows the values of parameters obtained by numerical integration of the motion equations (1) by the fourth-order Runge-Kutta method. The third column contains the values calculated by formulas (8). The deviations from the exact values of parameters are shown in the fourth column of the table.

Figure 3 is an interesting geometric picture for Table 1. If we use motion parameters L, H, x_a to construct the ABC triangle with the height BD = LH, segments AD = x_a^2 and CD = $(L-x_a)^2$, then in this triangle $\angle$A $\cong$ θ_0 . . . . C $\triangleq$ θ_1 ˜ . . hus . . . for the values $L = 117.8$, $H = 30.97$, x_a = 65.36 we have: $\angle$A = 40.5°, $\angle$C = 53°.

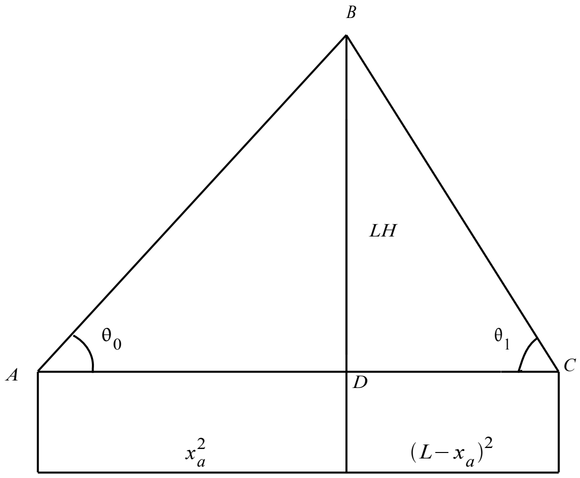

Fig.3: Motion parameters.

4. Analitical formulas for basic functional relationships of the problem

Formulas (8) make it possible to obtain simple approximate analitical expressions for the basic functional relationships of the problem *y(x), y(t), y(θ), x(t), x(θ), t(θ).*

We construct the first of these dependencies. In the absence of a drag force, the trajectory of a point mass is a parabola, whose equation using parameters H, L, x_a can be written as

$$y(x)=\frac{H}{x_a^2}\,x\,(L-x) \tag{9}$$

When the point mass is under a drag force, the trajectory becomes asymmetrycal. The top of the trajectory is shifted toward the point of incidence. In addition, a vertical asymptote appears near the trajectory. Taking these circumstances into account, we shall construct the function *y(x)* as

$$y(x)=\frac{Hx(L-x)}{x_a^2+ax},$$

where a – is a negative coefficient to be determined. We define it by the condition $y(x_a)=H$. We get $a=L-2x_a$. Then the sought dependence *y(x)* has the form

$$y(x)=\frac{Hx(L-x)}{x_a^2+(L-2x_a)x} \tag{10}$$

Constructed dependence *y(x)* provides shift of the apex of the trajectory to the right and has a vertical asymptote, since the coefficient $a<0$. In the case of no drag $L=2x_a$, relationship (10) goes over into (9).

Exactly the same way, we construct the function *y(t)* described as

$$y(t)=\frac{Ht(T-t)}{t_a^2+(T-2t_a)t} \tag{11}$$

Since $T-2t_a>0$, the maximum of the function *y(t)* drifts to the left, to the launching side.

From the equations of motion (1) there follows the equation *dy/dx* = tan *θ*. From this equation, upon differentiation of Eq. (10) and putting the results in it, we get the expression for *x(θ)*

$$x(\theta)=a_3\left(1+\frac{1-a_1}{\sqrt{1+a_2\tan\theta}}\right) \tag{12}$$

Here $a_1=L/x_a$, $a_2=(L-2x_a)/H$, $a_3=x_a(2-a_1)^{-1}$.

Putting (12) to (10), we get the function *y(θ)*

$$y(\theta)=b_1\left(b_2-\frac{2+a_2\tan\theta}{\sqrt{1+a_2\tan\theta}}\right), \tag{13}$$

where $b_1=H(a_1-1)(2-a_1)^{-2}$, $b_2=2+H/b_1$.

Using (10) and (11), we get the *x(t)* function

$$x(t)=\frac{L(w_1^2+w_2+w_1\sqrt{w_1^2+cw_2})}{2w_1^2+a_1w_2}, \tag{14}$$

where $w_1(t)=t-t_a, w_2(t)=2t(T-t)/a_1$, $c=2(a_1-1)/a_1$.

Using (11) and (13), we construct the function $t = t(\theta)$ described as

$$t(\theta)=\frac{T}{2}+d_1y(\theta)\mp\sqrt{(H-y(\theta))(d_2-d_1^2y(\theta))}, \tag{15}$$

where $d_1=\frac{1}{H}(t_a-\frac{T}{2})$, $d_2=\frac{T^2}{4H}$. The minus sign in front of the radical in (15) is taken on the interval $0\le\theta\le\theta_0$ and the plus sign is taken on the interval $\theta_1\le\theta\le 0$.

Another form of the function *t(θ)* can be obtained from the equation of the motion $dy/dt=V\sin\theta$ likewise formula (12)

$$t(\theta)=l_2\left(1+\frac{1-l_1}{\sqrt{1+l_3V\sin\theta}}\right), \tag{16}$$

where $l_1=T/t_a$, $l_2=t_a(2-l_1)^{-1}$, $l_3=(T-2t_a)/H$. The function *V(θ)* in (16) is defined by relation (2).

Thus, with the known motion parameters H, L, T, x_a , t_a formulas (10) - (16) make it possible to consruct functions *y(x), y(t), y(θ), x(t), x(θ), t(θ).*

Here is an example of using these formulas. The following values of the parameters are taken to calculate the motion of a baseball

$$\rho_a = 1.2 \text{ kg/m}^3,\ c_d = 0.25,\ r = 0.0366 \text{ m},\ m = 0.145 \text{ kg},$$
$$V_0 = 50 \text{ m/s} \ ;\ \theta_0 = 40° \quad g = 9.81 \text{ m/s}^2,\quad k = 0.00044 \text{ s}^2/\text{m}^2.$$

The results of calculations are shown in Figures 4 – 9. In all Figures thin solid lines are obtained by numeric integration of motion equations (1), broken lines in the Figures show the same functions constructed from formulas (8), (10) — (16). Numerical integration of system (1) was realized with the aid of the 4-th order Runge-Kutta method. Analisis of the curves in Fig. 4 through 9 shows that analytical dependencies *y(x), y(t), y(θ), x(t), x(θ), t(θ)* approximate numerically obtained functions rather well.

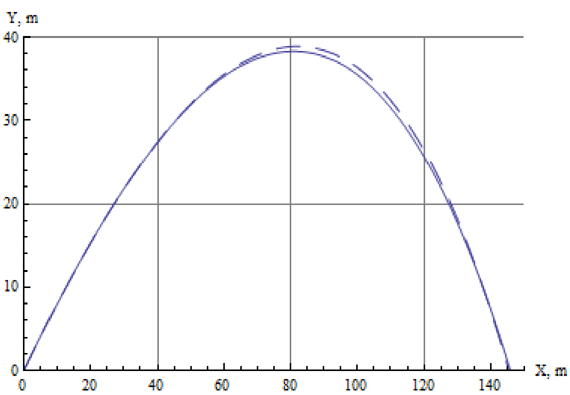

Fig. 4: The graph of the function *y= y(x).*

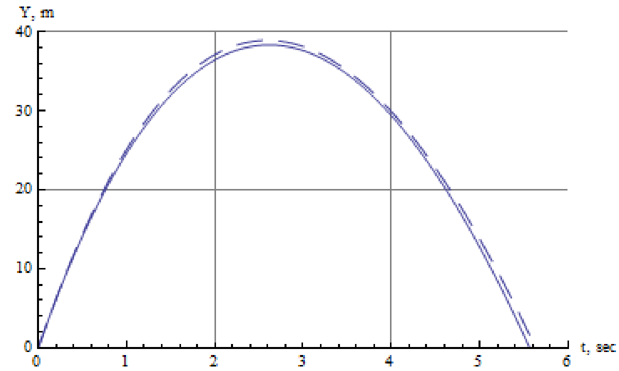

Fig.5: The graph of the function *y=y(t)*.

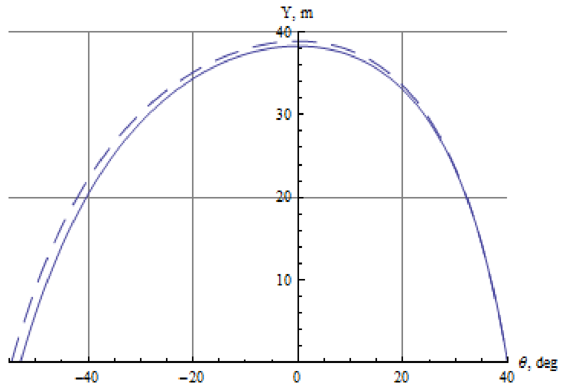

Fig. 6: The graph of the function *y=y(θ)*.

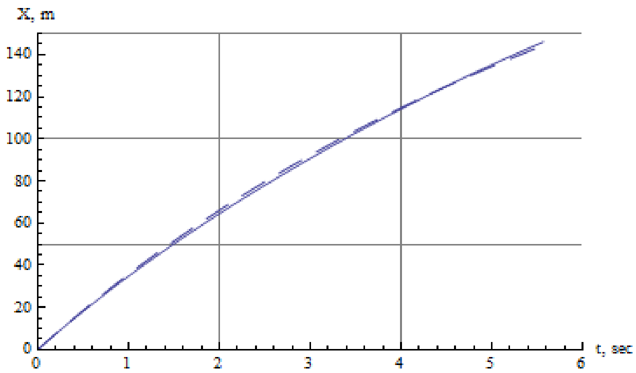

Fig. 7: The graph of the function *x= x(t)*.

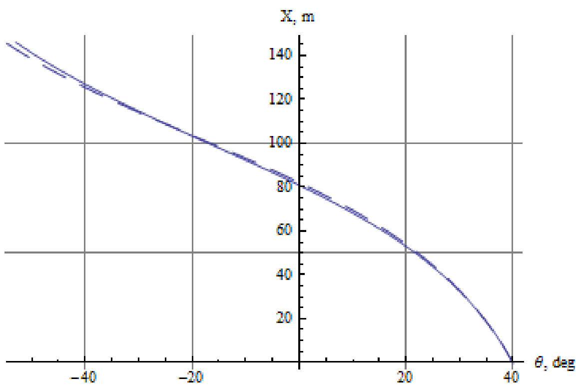

Fig. 8: The graph of the function *x= x(θ)*.

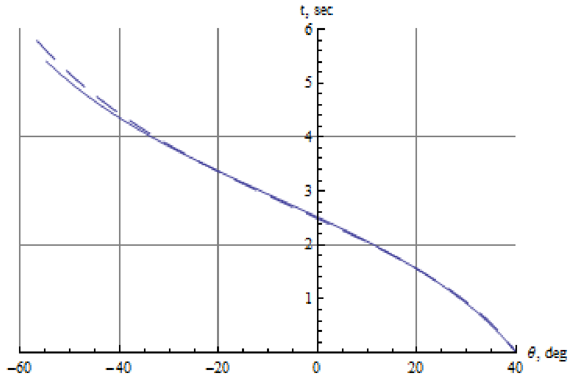

Fig. 9: The graph of the function *t= t(θ)*.

5. The determination of the optimum angle of throwing, providing the maximum range

The formula for the range of throw of the point mass is written as $L(\theta_0)=V_a(\theta_0)T(\theta_0)$ and is defined by relations (8). The optimal angle of throwing α, which provides the maximum distance of flight, is a root of equation

$$\frac{dL(\theta_0)}{d\theta_0}=0$$

Differentiating the $L(\theta_0)$ function with respect to θ_0, after certain transformations, we obtain the equation for finding the angle α when the points of throwing and downs are on the same horizontal

$$\tan^2\alpha+\frac{p\sin\alpha}{4+4p\sin\alpha}=\frac{1+p\lambda}{1+p(\sin\alpha+\lambda\cos^2\alpha)} \tag{17}$$

Here $p=kV_0^2$, $\lambda(\alpha)=\ln\tan(\frac{\alpha}{2}+\frac{\pi}{4})$. When $k=0$, equation (17) gives the known solution $\alpha = 45°$. When $k \neq 0$, equation (17) is easily solved graphically or numerically. The value of the optimum angle α depends on the value of the parameter p. This parameter represents the force of air resistance at the start of motion, referred to the weihgt of the object.

With a condition $p=kV_0^2=const$ it is possible to change values k and V_0 simultaneously. The optimal angle of throwing α will be alike. But main parameters of motion H, L, T will change as it follows from the formulas (8).

Let the values of motion parameters H_1, L_1, T_1 correspond to drag coefficient k_1, and values H_2, L_2, T_2 to drag coefficient $k_2=qk_1$. Then with the condition

$$p=k_1V_{(01)}^2=k_2V_{(02)}^2=const \tag{18}$$

we get the correlations

$H_2=\frac{H_1}{q}$, $L_2=\frac{L_1}{q}$, $T_2=\frac{T_1}{\sqrt{q}}$. The trajectories of the point mass will be similar when the condition (18) are fulfilled.

The graph of function $\alpha = \alpha(p)$ is submitted in Figure 10. The solid line in the Figure is based on the results of solving equation (17). The dots denote the values of the optimum angle of the throwing obtained by numerical integration (1). The Figure shows that at a sufficiently large interval of the parameter p the solution (17) well approximates the numerical solution.

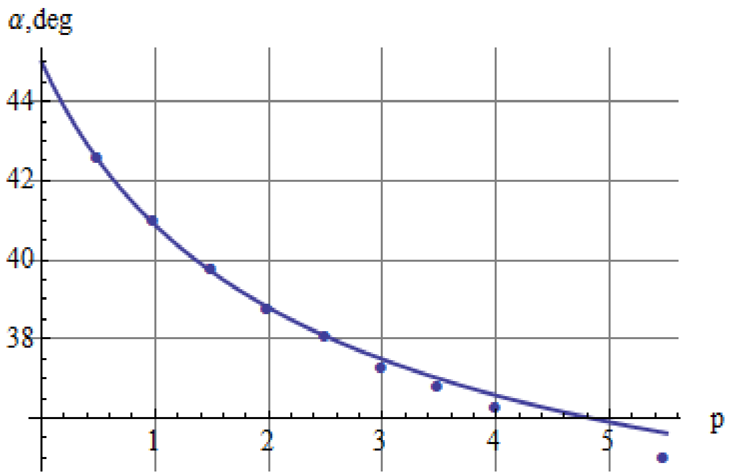

Fig.10: The graph of the function $\alpha = \alpha(p)$.

6. Constructing the envelope of a family of trajectories of the projectile

In the case of no air drag the trajectory of a point mass is a parabola. For the different angles of throwing under one and same initial velocity projectile trajectories form a family of parabolas. Maximum range and maximum height for limiting parabolas are given by formulas

$$L_{max} = \frac{V_0^2}{g}, \quad H_{max} = \frac{V_0^2}{2g} \tag{19}$$

The envelope of this family is also a parabola, equation of which is usually written as

$$y(x) = \frac{V_0^2}{2g} - \frac{g}{2V_0^2}x^2 \tag{20}$$

Using (19), we will convert the equation (20) as

$$y(x) = \frac{H_{max}(L_{max}^2 - x^2)}{L_{max}^2} \tag{21}$$

We will set up an analytical formula similar to (21) for the envelope of the point mass trajectories taking into account the air drag force. Taking into account the formula (21), we will construct an equation of the envelope as

$$y(x) = \frac{H_{max}(L_{max}^2 - x^2)}{L_{max}^2 - ax^2} \tag{22}$$

Such structure of equation (22) takes into account the fact that the envelope has a maximum under $x = 0$. Besides, function (22) under $a > 0$ has a vertical asymptote, as well as any point mass trajectory accounting resistance of air. In formula (22) H_{max} is the maximum height, reached by the point mass when throwing with initial conditions V_0 , $\theta_0 = 90^\circ$; L_{max} - the maximum range, reached when throwing a point mass with the initial velocity V_0 under some optimum angle $\theta_0 = \alpha$. In the parabolic theory an angle $\alpha = 45°$ under any initial velocity V_0 . Taking into account the resistance of air, an optimum angle of throwing α is less than 45° and depends on the value of parameter $p = kV_0^2$. Parameter H_{max} with the preceding notation is defined by formula [16]

$$H_{max} = \frac{1}{2gk}\ln(1 + kV_0^2) \tag{23}$$

A choice of a positive factor a in the formula (22) is sufficiently free. However it must satisfy the condition $a = 0$ in the absence of resistance ($k = 0$). We shall find this coefficient under the following considerations.

It was shown above, that while taking into account air resistance, the trajectory of a point mass is well approximated by the function

$$y(x)=\frac{Hx(L-x)}{x_a^2+(L-2x_a)x}, \tag{24}$$

here x, y are the Cartesian coordinates of the point mass; parameters $H,\ L,\ x_a$ are shown in Figure 1. Thus, for the generation of the equation of the maximum range trajectory three parameters are required : $H,\ L_{max},\ x_a$. We will calculate these parameters as follows. Under a given value of quantity $p=kV_0^2$ we will find the root α of equation (17). An angle α ensures the maximum range of the flight. By integrating numerically system (1) with the initial conditions V_0 , α , we obtain the values $H(\alpha)$, $L_{max} = L(\alpha)$, $x_a(\alpha)$ for the maximum range trajectory. The parameter a in the formula (22) we find as follows. We set the equal slopes of the tangents to envelope (22) and to the maximum range trajectory

$$y(x)=\frac{H(\alpha)\,x(L_{max}-x)}{x_a^2(\alpha)+(L_{max}-2x_a(\alpha))\,x}$$

in the spot of incidence $x=L_{max}$. It follows that parameter a is defined by formulas

$$a=1-\frac{2H_{max}}{H(\alpha)}\left(1-\frac{x_a(\alpha)}{L_{max}}\right)^2 \tag{25}$$

In the absence of air resistance parameter $a = 0$.

The equation of the envelope can be used for the determination of the maximum range if the spot of falling lies above or below the spot of throwing. Let the spot of falling be on a horizontal straight line defined by the equation $y=y_1=const$. We will substitute a value y_1 in the equation (22) and solve it for *x*. We obtain the formula

$$x_{max}=L_{max}\sqrt{\frac{H_{max}-y_1}{H_{max}-ay_1}} \tag{26}$$

The correlation (26) allows us to find a maximum range under the given height of the spot of falling.

As an example we will consider the moving of a baseball with the resistance factor $k = 0.000548$ s^2/m^2 [6]. Other parameters of motion are given by values

$$g = 9.81\ \text{m/s}^2\quad V_0 = 50\ \text{m/s},\quad y_1 = \pm 20, \pm 40, \pm 60\ \text{m}.$$

Substituting values k and V_0 in the formula (23), we get H_{max} = 80.26 m. Hereinafter we solve an equation (17) at the value of non-dimensional parameter $p=kV_0^2$ = 1.37. The root of this equation gives the value of an optimum angle of throwing. This angle ensures the maximum range: $\alpha = 40°$. By integrating the system of equations (1) with the initial conditions

$$V_0 = 50,\ \theta_0 = 40°, x_0 = 0, y_0 = 0\ ,$$

we find meanings

$$H(\alpha) = 36.2\ \text{m},\quad L_{max} = L(\alpha) = 133.6\ \text{m},\quad x_a(\alpha) = 75.1\ \text{m}$$

According to the formula (25) the factor is $a = 0.149$. The graph of the envelope (22) is plotted in Figure 11 together with the family of trajectories. We note that family of trajectories is received by means of numerical integrating of the equations of motion of a point mass (1). A standard fourth-order Runge-Kutta method was used.

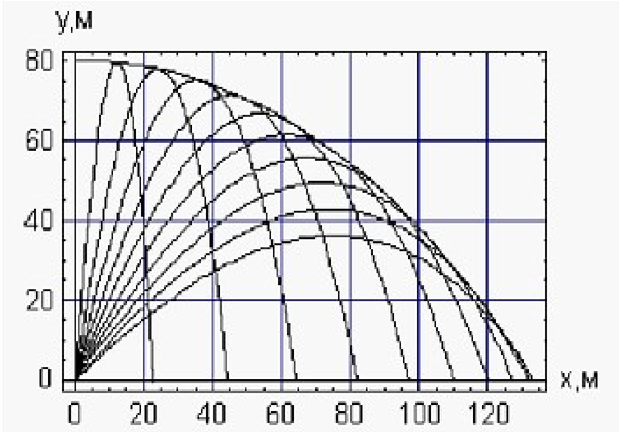

Fig.11: The family of projectile trajectories and the envelope of this family.

The results of calculations using the formula (26) are presented in Table 2. The second column of the table contains range values calculated analytically by the formula (26). The third column of the table contains range values from integration of the equations of motion (1). The fourth column presents an error of the calculation of the range in the percentage. Error does not exceed 0.4 %. Formula (26) gives almost the exact value of the maximum range in a wide range of height point drop (120 m). The tabulated data show that formulas (8), (17), (22), (26) ensure sufficiently pinpoint accuracy of the calculation of parameters of motion.

Note that values of $H(\alpha)$, $L_{max} = L(\alpha)$, $x_a(\alpha)$ can be obtained by using the formulas (8), without numerical integration of the system (1). Substituting V_0 and α in the formulas (8), we have

$$H(\alpha) = 36.5 \text{ m},\ L_{max} = L(\alpha) = 132.4 \text{ m},\ x_a(\alpha) = 76.0 \text{ m}$$

For these values $a = 0.2$. The graph of the envelope does not nearly change. The right end of the graph shift along the x axis is less than 1%.

Table 2. Maximum range under different heights of the spot of the falling

y_1 , m	Analytical value x_{max} , m	Numerical value x_{max} , m	Error (%)
60	71.2	71.1	0.1
40	98.3	98.3	0.
20	118.0	118.0	0.
0	133.6	133.6	0.
-20	146.6	146.5	0.1
-40	157.8	157.5	0.2
-60	167.5	166.9	0.4

7. Finding the vertical asymptote of projectile motion

It is well known that the projectile trajectory has a vertical asymptote in the resistant medium. We will deduce an appropximate analytical formula for the value of the asymptote $x_{as} = x^*$ (Figure 1). Note that different assumtion formulas can be taken to yield the formula for x^*. Accordingly, the final formula for x^* will be also different. This matter is well worth another look. In this paper we are using the first of formulas (1) to solve the task :

$$\frac{dV}{dt} = -g \sin\theta - gkV^2$$

Multiplying the two parts of the formula by the expression $\frac{dt}{V}\cos\theta$, we get the equation

$$\cos\theta\frac{dV}{V}=-\frac{g\sin\theta\cos\theta\, dt}{V}-gkV\cos\theta\, dt \tag{27}$$

From the second and third equations of (1) it follows that

$$dt=-\frac{V\,d\theta}{g\cos\theta},\quad dx=V\cos\theta\, dt \tag{28}$$

Using the ratio (28), we transform the equation (27) to the form

$$\cos\theta\frac{dV}{V}=\sin\theta\, d\theta-gk\, dx \tag{29}$$

In turn, the ratio (29) can be written as

$$gk\, dx=-d(\cos\theta)-\cos\theta\, d(\ln V) \tag{30}$$

Flight range L is determined by the appropriate formula (8). There is no need to integrate the equation (30) along the whole trajectory. We will integrate the equation (30) only in the interval from $x = L$ to $x = x_{as} = x^*$ (Figure 1). We take into account that the value $x = L$ corresponds to the value of the trajectory angle $\theta = \theta_1$. The value θ_1 is also calcalated with the aid of the formulas (8). We have

$$gk\int_L^{x_{as}} dx=-\int_{\theta_1}^{-\frac{\pi}{2}} d(\cos\theta)-\int_{\theta_1}^{-\frac{\pi}{2}}\cos\theta\, d(\ln V) \tag{31}$$

After the substitution of limits in the first two integrals we get

$$gk(x_{as}-L)=\cos\theta_1-\int_{\theta_1}^{-\frac{\pi}{2}}\cos\theta\, d(\ln V) \tag{32}$$

To calculate the integral in equation (32) we apply a trapezoidal rule. We split the interval of integration $[\theta_1, -\frac{\pi}{2}]$ in two equal segments with a point $\theta_2=\frac{1}{2}(\theta_1-\frac{\pi}{2})$. At each of the segments $[\theta_1, \theta_2]$ and $[\theta_2, \frac{-\pi}{2}]$ we calculate the integral in the equation (32) as an area of appropriate trapezoid. We have

$$gk(x_{as}-L)=\cos\theta_1-\frac{1}{2}(\cos\theta_1+\cos\theta_2)(\ln V_2-\ln V_1)-\frac{1}{2}(\cos\theta_2+\cos(\frac{-\pi}{2}))(\ln(\frac{1}{\sqrt{k}})-\ln V_2)$$

Here symbols $V_1=V(\theta_1)$, $V_2=V(\theta_2)$ are introduced . The values V_1 , V_2 are calculated by formula (2). We take into account that when $\theta = -\frac{\pi}{2}$ the point mass velocity accepts the terminal value V_{term}: $V(\frac{-\pi}{2})=V_{term}=\frac{1}{\sqrt{k}}$. After some transformations we get the final formula

$$x^* = x_{as}=L+\frac{1}{2gk}\ln[(e^2\frac{V_1}{V_2})^{(\cos\theta_1)}(\sqrt{k}V_1)^{(\cos\theta_2)}] \tag{33}$$

Here e = 2. 71828. Note that using formulas (2) , (8), (33) the value of x^* is directly determined by the initial conditions of throwing V_0 , θ_0 .

As an example, we consider the motion of the baseball for the following parameters [6]:

$$k = 0.000548\ \mathrm{s^2/m^2},\quad g = 9.81\ \mathrm{m/s^2}. \tag{34}$$

The calculations are carried out in the following ranges of changes of launch angle and initial velocity: $0 \le \theta_0 \le 90°$, $20 \le V_0 \le 50$ m/s. The results of calculations are recorded in Table 3 and shown in Fig.12 and 13.

Table 3. Numerical and analytical values of asymptote x^* for different values V_0 , θ_0

	θ_0 ,degree	0°	15°	30°	45°	60°	75°
V_0 , m/s	20	120.4 122.8	121.5 124.6	114.5 117.3	98.8 100.9	73.8 75.3	40.2 40.9
	30	166.4 165.8	169.1 170.8	160.9 162.7	140.4 141.7	106.7 107.5	59.1 59.7
	40	205.0 200.8	208.8 209.3	200.0 199.9	175.4 175.0	134.5 134.3	75.1 75.5
	50	237.8 230.3	242.7 241.7	232.4 230.3	204.8 202.0	157.6 155.8	89.0 88.4

Upper value of each table cell represents the value of x^* (in metres) obtained numerically. Lower value of each cell is calculated by formula (33). The numerical values are found by integrating the system of equations (1) with the fourth-order Runge-Kutta method. The table data show that the formulas (2), (8), (33) ensure an accuracy high enough for calcucating asymptote. In Fig. 12 the graphs of dependencies $x^* = x^*(\theta_0)$ are plotted for three different values of initial velocity V_0 with parameters (34).

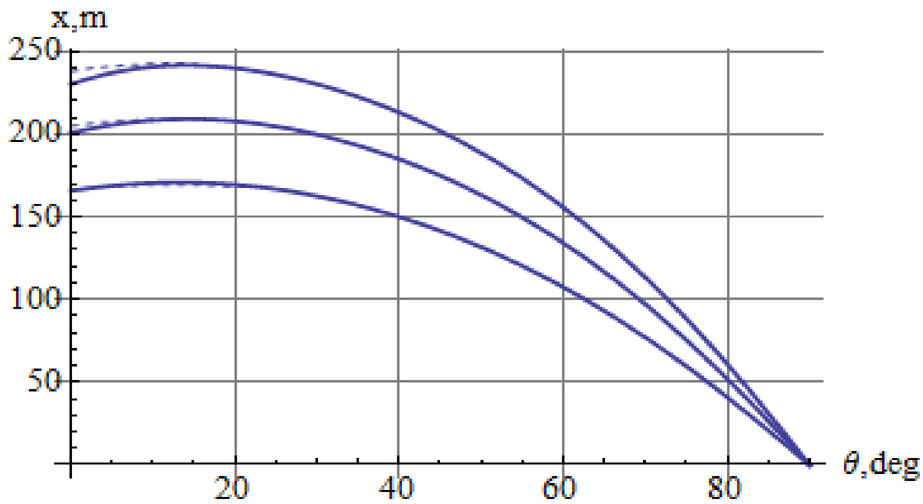

Fig. 12: The graph of the function $x^* = x^*(\theta_0)$.

The upper curve is for the value V_0 = 50 m/s ; the average one– for the V_0 = 40 m/s; the lower one – for the value V_0 = 30 m/s. The solid line is calculated using the formula (33), the dotted line is plotted on the results of numerical integration of the system (1). It is seen from the graphs that in the considered range of parameters V_0 , θ_0 formula (33) approximates the function $x^*(\theta_0)$ rather well not only qualitativly but also quantitatively. The maximum value of asymptote is achieved at the significance $\theta_0 \approx 15°$. The analytical formula (33) makes it easy to plot not only the curves $x^*(\theta_0)$, $x^*(V_0)$, but also the surface $x^*(\theta_0, V_0)$.

In Fig. 13 the surface $x^* = x^*(\theta_0, V_0)$ is constructed in the range of arguments mentioned above.

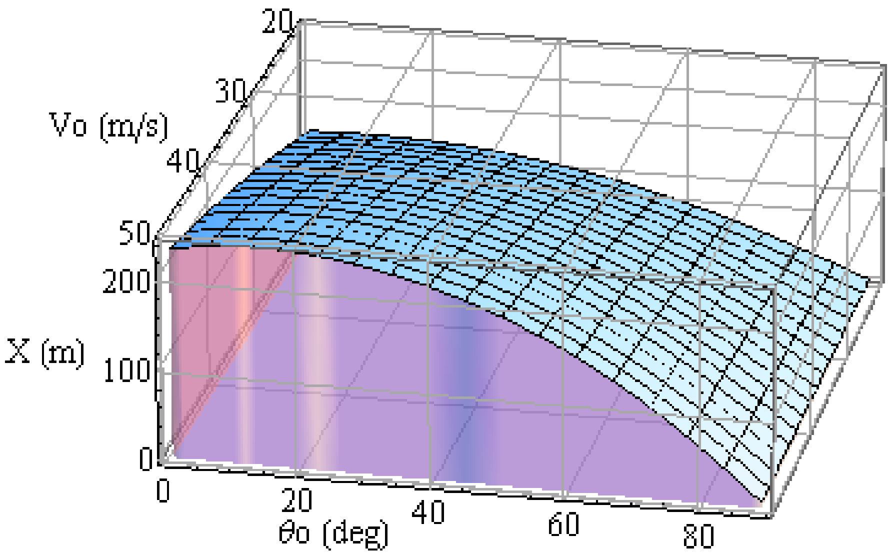

Fig. 13: The $x^* = x^*(\theta_0, V_0)$ surface.

8. Conclusion

The proposed approach based on the use of analytical formulas make it possible to simplify significantly a qualitative analysis of the motion of a point mass with air drag taken into account. All basic parameters are described by simple analytic formulas. Moreover, numerical values of the sought variables are determined with acceptable accuracy. Thus, proposed formulas make it possible to carry out a complete analytical investigation of the motion of a point mass in a medium with drag in the way it is done for the case of no drag.

9. References

[1] D. A. Morales. Exact expressions fot the range and the optimal angle of a projectile with linear drag. *Canadian Journal of Physics.* 2005, **83**: 67-83.

[2] A. Vial. Horisontal distance travelled by a mobile experiencing a quadratic drag force: normalised distance and parametrization. *European Journal of Physics*. 2007, **28**: 657-663.

[3] K. Yabushita, M. Yamashita and K.Tsuboi. An analytic solution of projectile motion with the quadratic resistance law using the homotopy analysis method. *Journal of Physics A: Mathematical and Theoretical*. 2007, **40**: 8403-8416.

[4] G. W. Parker. Projectile motion with air resistance quadratic in the speed. *American Journal of Physics*. 1997, **45**(7): 606- 610.

[5] H. Erlichson. Maximum projectile range with drag and lift, with particular application to golf. *American Journal of Physics*. 1983, **51**: 357-362.

[6] A. Tan, C.H. Frick and O.Castillo. The fly ball trajectory: an older approach revisited. *American Journal of Physics.* 1987, **55**(1): 37-40.

[7] C.W. Groetsch. On the optimal angle of projection in general media. *American Journal of Physics.* 1997, **65**: 797-799.

[8] J. Lindemuth. The effect of air resistance on falling balls. *American Journal of Physics.* 1971, **39**: 757-759.

[9] R.H. Price and J.D. Romano. Aim high and go far – optimal projectile launch angles greater than 45°. *American Journal of Physics*. 1998, **66**: 109-113.

[10] P.S. Chudinov. The motion of a point mass in a medium with a square law of drag.*Journal of Applied Mathematics and Mechanics.* 2001, **65**(3): 421-426.

[11] P.S. Chudinov. The motion of a heavy particle in a medium with quadratic drag force. *International Journal of Nonlinear Sciences and Numerical Simulation*. 2002,**3**: 121-129.

[12] P.S. Chudinov. An optimal angle of launching a point mass in a medium with quadratic drag force. *Indian Journal of Physics*. 2003, B 77: 465-468.

[13] P.S. Chudinov. Analytical investigation of point mass motion in midair. *European Journal of Physics*. 2004, **25**: 73 -79.

[14] P.S. Chudinov. Approximate formula for the vertical asymptote of projectile motion in midair. *International Journal of Mathematical Education in Science and Technology*. 2010, **41**: 92-98.

[15] P.S. Chudinov. 2009 Numerical-analytical algorithm for constructing the envelope of the projectile trajectories in midair. *http://arxiv.org/abs/0902.0520v1*.

[16] B.N. Okunev. *Ballistics*. Voyenizdat, Moscow, 1943.

[17] S. Timoshenko and D.H. Young. *Advanced Dynamics*. New York: McGrow-Hill Book Company, 1948.