

Measures of Departure from Bradley-Terry Model for Square Tables: Application to Baseball

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Abstract. For square contingency tables with nominal categories, Tahata, Miyamoto and Tomizawa (2004) considered power-divergence type measure to represent the degree of departure from the quasi-symmetry (QS) model which is essentially equivalent to the Bradley-Terry (BT) model. This paper proposes other measures to represent the degree of departure from the QS (BT) model. The proposed measures are defined by using the Matusita distance (1954, 1955), which is the true distance measure. The proposed measures would be useful for comparing the degree of departure from the QS (BT) model in several tables. As example, the measure is applied to the win-loss standings of professional baseball league in Japan.

Keywords: Bradley-Terry model, Contingency table, Matusita distance, Measure, Quasi-symmetry.

1. Introduction

Consider the athletic competitions with the outcome for the play of any two teams of R teams. Then, let π_{ij} for $i < j$ denote the probability that team i defeats team j when team i plays team j , and let π_{ji} for $i < j$ denote the probability that team j defeats team i when team i plays team j . Note that $\pi_{ji} = 1 - \pi_{ij}$, for $i < j$.

The Bradley-Terry (BT) model is defined by

$$\pi_{ij} = \frac{\gamma_i}{\gamma_i + \gamma_j} \quad (i \neq j);$$

see Bradley and Terry (1952). The BT model can also be expressed as

$$Q_{ijk} = Q_{kji} \quad (i < j < k),$$

where

$$Q_{ijk} = \pi_{ij}\pi_{jk}\pi_{ki}, \quad Q_{kji} = \pi_{kj}\pi_{ji}\pi_{ik}.$$

This model indicates that for the plays of any two teams of teams i , j and k , the probability that team i defeats team j , team j defeats team k , and team k defeats team i , is equal to the probability that k defeats j , j defeats i , and i defeats k . The BT model is essentially equivalent to the quasi-symmetry (QS) model (Causinus, 1965) applied to the data of the square contingency table with same nominal row and column categories (see the details in Section 5).

When the BT (QS) model does not hold for the given data, we are interested in measuring the degree of departure from the structure of BT (QS). Tahata, Miyamoto and Tomizawa (2004) considered the power-divergence type measure $\Phi^{(\lambda)}$ which represents the degree of departure from the BT (QS) model (see Appendix 1).

In general, a distance d is defined on a set W if for any two elements $x, y \in W$, a real number $d(x, y)$ is assigned that satisfies the following postulates:

- (a) $d(x, y) \geq 0$ with equality if and only if $x = y$,
- (b) $d(y, x) = d(x, y)$,

(c) $d(x, z) \leq d(x, y) + d(y, z)$ for $x, y, z \in W$ (the triangle inequality),

(see also Read and Cressie, 1988, p.111). Then, the power-divergence $I^{(\lambda)}$ does not satisfy the postulate (c). However, the square root of $I^{(-\frac{1}{2})}$ satisfies all three postulates. The definition of power-divergence is given in Appendix 1. It is a true distance measure known as the Matusita distance,

$$M = \left[\sum_{i=1}^R \sum_{j=1}^R (\sqrt{p_{ij}} - \sqrt{q_{ij}})^2 \right]^{\frac{1}{2}},$$

for the probability distributions $\{p_{ij}\}$ and $\{q_{ij}\}$ (Matusita, 1954, 1955; Read and Cressie, 1988, p.112). Therefore the measure proposed by Tahata et al. (2004) does not satisfy the postulate (c). So, we are interested in considering the measure which satisfies all three postulates.

The purpose of this paper is to propose some measures which represent the degree of departure from the structure of BT (QS) by using Matusita distance. The measure proposed in this paper would be useful for *comparing* the degree of departure from the BT (QS) model in several tables. Examples are given.

2. Measures

Let

$$\Delta = \sum_{i < j < k} (Q_{ijk} + Q_{kji}),$$

and, for $i < j < k$,

$$Q_{ijk}^* = \frac{Q_{ijk}}{\Delta}, \quad Q_{kji}^* = \frac{Q_{kji}}{\Delta}, \quad C_{ijk}^* = C_{kji}^* = \frac{1}{2}(Q_{ijk}^* + Q_{kji}^*).$$

Then, assuming that $Q_{ijk} + Q_{kji} \neq 0$ for $i < j < k$, we shall consider a measure to represent the degree of departure from the BT model as follows:

$$\Phi^* = \left[\frac{2 + \sqrt{2}}{2} \sum_{i < j < k} \left\{ \left(\sqrt{Q_{ijk}^*} - \sqrt{C_{ijk}^*} \right)^2 + \left(\sqrt{Q_{kji}^*} - \sqrt{C_{kji}^*} \right)^2 \right\} \right]^{\frac{1}{2}}.$$

Since the measure Φ^* is the root of $\Phi^{(-\frac{1}{2})}$ (see Appendix 1 for $\Phi^{(\lambda)}$), the measure Φ^* satisfies all three postulates of distance. Namely, Φ^* represents essentially the true distance between $\{Q_{ijk}^*, Q_{kji}^*\}$ and $\{C_{ijk}^*, C_{kji}^*\}$ with the structure of BT.

We note that the measure Φ^* lies between 0 and 1. Also (i) $\Phi^* = 0$ if and only if the BT model holds in the $R \times R$ table, and (ii) $\Phi^* = 1$ if and only if the degree of departure from the BT model is a maximum, in the sense that $Q_{ijk} = 0$ (then $Q_{kji} > 0$) or $Q_{kji} = 0$ (then $Q_{ijk} > 0$) for $i < j < k$.

We shall now say that the stochastic three-way deadlock arises when the probability that i defeats j , j defeats k , and k defeats i is larger or smaller than the probability for the reverse order. Then, e.g., for the case of athletic competitions, (i) $\Phi^* = 0$ indicates that for any three teams of R teams, the stochastic three-way deadlock does not arise; because then $Q_{ijk} = Q_{kji}$ for $i < j < k$, and, (ii) $\Phi^* = 1$ indicates that for any three teams of R teams, the *strongest* stochastic three-way deadlock arises; namely, the probability that i defeats j , j defeats k , and k defeats i is positive (then the probability for reverse order is zero) or zero (then the probability for reverse order is positive).

Let

$$Q_{ijk}^c = \frac{Q_{ijk}}{Q_{ijk} + Q_{kji}}, \quad Q_{kji}^c = \frac{Q_{kji}}{Q_{ijk} + Q_{kji}} \quad (i < j < k).$$

Using $\{Q_{ijk}^c\}$ and $\{Q_{kji}^c\}$, the measure $\Phi^{(\lambda)}$ can also be expressed as

$$\Phi^{(\lambda)} = \frac{\lambda(\lambda+1)}{2^\lambda - 1} \sum_{i < j < k} (Q_{ijk}^* + Q_{kji}^*) I_{ijk}^{(\lambda)},$$

where

$$I_{ijk}^{(\lambda)} = \frac{1}{\lambda(\lambda+1)} \left[Q_{ijk}^c \left\{ \left(\frac{Q_{ijk}^c}{1/2} \right)^\lambda - 1 \right\} + Q_{kji}^c \left\{ \left(\frac{Q_{kji}^c}{1/2} \right)^\lambda - 1 \right\} \right].$$

Note that $I_{ijk}^{(\lambda)}$ is the power-divergence between the two distributions $\{Q_{ijk}^c, Q_{kji}^c\}$ and $\{1/2, 1/2\}$. Therefore, the measure $\Phi^{(\lambda)}$ would essentially represent the weighted sum of the power-divergence $I_{ijk}^{(\lambda)}$.

Now, we consider another measure defined by

$$\Phi^{**} = \sum_{i < j < k} (Q_{ijk}^* + Q_{kji}^*) M_{ijk},$$

where

$$M_{ijk} = \left[\frac{2+\sqrt{2}}{2} \left\{ \left(\sqrt{Q_{ijk}^c} - \frac{\sqrt{2}}{2} \right)^2 + \left(\sqrt{Q_{kji}^c} - \frac{\sqrt{2}}{2} \right)^2 \right\} \right]^{\frac{1}{2}}.$$

This measure is the weighted sum of the Matusita distance for the two distributions $\{Q_{ijk}^c, Q_{kji}^c\}$ and $\{1/2, 1/2\}$. We point out that Φ^{**} is not equivalent to Φ^* although $\Phi^{(\lambda)}$ can be expressed by the weighted sum of the power-divergence as described above. Also, the inequality $\Phi^{**} \leq \Phi^*$ holds from Jensen's inequality.

When the BT model does not hold, the reader may be interested in seeing which triad of i , j and k contribute most to the degree of departure from the BT model. The measure M_{ijk} may be useful in such situation. We point out that (i) $0 \leq M_{ijk} \leq 1$, (ii) $M_{ijk} = 0$ if and only if $Q_{ijk}^c = Q_{kji}^c$ (i.e., $Q_{ijk} = Q_{kji}$), and (iii) $M_{ijk} = 1$ if and only if $Q_{ijk}^c = 1$ (then $Q_{kji}^c = 0$) or $Q_{ijk}^c = 0$ (then $Q_{kji}^c = 1$). Thus, the measure Φ^{**} lies between 0 and 1. Also (i) $\Phi^{**} = 0$ if and only if the BT model holds, and (ii) $\Phi^{**} = 1$ if and only if the degree of departure from the BT model is a maximum, in the sense that $Q_{ijk}^c = 1$ (then $Q_{kji}^c = 0$) or $Q_{ijk}^c = 0$ (then $Q_{kji}^c = 1$) for $i < j < k$.

3. Approximate confidence intervals for measures

Consider a set of data from $R(R-1)/2$ paired comparison experiments for R treatments. Let r_{ij} be the number of comparisons for the treatment pair (i, j) , and n_{ij} be the number that the treatment i exceeds the treatment j in the r_{ij} comparisons. Assuming that there is no tie we have $r_{ij} = r_{ji} = n_{ij} + n_{ji}$. Let π_{ij} be the probability that the treatment i exceeds the treatment j in a single comparison of the pair. We have $\pi_{ij} + \pi_{ji} = 1$ excluding the possibility of tie. The probability for $\{n_{ij}\}$, $i \neq j$, is then the product of $R(R-1)/2$ binomials,

$$\prod_{1 \leq i < j \leq R} \frac{r_{ij}!}{n_{ij}! n_{ji}!} \pi_{ij}^{n_{ij}} \pi_{ji}^{n_{ji}}.$$

We shall consider an approximate standard error and large-sample confidence interval for the measure Φ^* , using the delta method, as described by Bishop, Fienberg and Holland (1975, Section 14.6) and Agresti (1990, Section 12.1). The estimated measures $\hat{\Phi}^*$, $\hat{\Phi}^{**}$ and \hat{M}_{ijk} are given by Φ^* , Φ^{**} and M_{ijk} with $\{\pi_{ij}\}$ replaced by $\{\hat{\pi}_{ij}\}$, where $\hat{\pi}_{ij} = n_{ij} / r_{ij}$, respectively. Using the delta method, $\hat{\Phi}^*$ has asymptotically a

normal distribution with mean Φ^* and variance $\sigma^2[\Phi^*]$ (see Appendix 2).

Let $\hat{\sigma}^2[\Phi^*]$ denote $\sigma^2[\Phi^*]$ with $\{\pi_{ij}\}$ replaced by $\{\hat{\pi}_{ij}\}$. Then, $\hat{\sigma}[\Phi^*]$ is an estimated standard error for $\hat{\Phi}^*$, and $\hat{\Phi}^* \pm z_{p/2} \hat{\sigma}[\Phi^*]$ is an approximate $100(1-p)\%$ confidence interval for Φ^* , where $z_{p/2}$ is the percentage point from the standard normal distribution that corresponds to a two-tail probability equal to p .

The approximate $100(1-p)\%$ confidence intervals for Φ^{**} and M_{ijk} can be obtained by similar manner. Also, the asymptotic variances for $\hat{\Phi}^{**}$ and \hat{M}_{ijk} are given in Appendixes 3 and 4, respectively.

4. Examples

4.1. Win-loss standings in 2008

Consider the data in Tables 1a and 1b which are obtained from the official website of Nippon Professional Baseball (<http://www.npb.or.jp/>). These data are the results of the professional baseball league in Japan in 2008. For instance, in the data in Table 1a, from Giants' perspective, the (Giants, Tigers) results in 2008 correspond to 14 successes and 10 failures in 24 trials.

Firstly, we shall apply the measure Φ^* for these data. Since the confidence intervals for the measure Φ^* applied to the data in Tables 1a and 1b do not contain zero (see Table 3a), this would indicate that there is not a structure of the BT model between the teams in Central League and Pacific League in 2008.

When the degrees of departure from the BT model in Tables 1a and 1b are compared using the estimated measure $\hat{\Phi}^*$, the value of $\hat{\Phi}^*$ is greater for Table 1a than for Table 1b. Namely, the data in 2008 Central League rather than in 2008 Pacific League are estimated to be close to a situation with the *strongest* stochastic three-way deadlock, which indicates that for any three teams, i , j and k , the probability that team i defeats team j , team j defeats team k , and team k defeats team i is positive (or zero) and the probability for reverse order is zero (or positive).

Secondly, we consider the comparison between Central and Pacific Leagues by using the measure Φ^{**} . From Table 3a, the value of $\hat{\Phi}^{**}$ is greater for Table 1a than for Table 1b. Therefore, we get the same result as described above.

Lastly, we see which triad of i , j and k contribute most to the degree of departure from the BT model. For the data in Table 1a, $\hat{M}_{123} = 0.50$, $\hat{M}_{135} = 0.52$ and $\hat{M}_{236} = 0.49$. Also, the approximate 95% confidence intervals for M_{123} , M_{135} and M_{236} are (0.16, 0.84), (0.19, 0.85) and (0.13, 0.85), respectively, and the confidence intervals for other three teams in Table 1a contain zero although the detail is omitted. Therefore, we could infer that triads of (Giants, Tigers, Dragons), (Giants, Dragons, Swallows) and (Tigers, Dragons, Bay Stars) rather than other three teams in the Central League in 2008 are close to a situation with strongest stochastic three-way deadlock. On the other hand, for the Pacific League in 2008, the confidence intervals for all three teams contain zero. So, we could not see clearly which triads of three teams contribute most.

4.2. Win-loss standings in 2009

Consider the data in Tables 2a and 2b which are obtained from the official website of Nippon Professional Baseball (<http://www.npb.or.jp/>). These data are the results of the professional baseball league in Japan in 2009.

Firstly, we shall apply the measure Φ^* . Since the confidence intervals for the measure Φ^* applied to the data in Tables 2a and 2b do not contain zero (see Table 3b), this would indicate that there is not a structure of the BT model between the teams in Central League and Pacific League in 2009.

When, the degrees of departure from the BT model in Tables 2a and 2b are compared using the value of $\hat{\Phi}^*$, it is greater for Table 2b than for Table 2a. Namely, the data in 2009 Pacific League rather than in 2009 Central League are estimated to be close to a situation with the *strongest* stochastic three-way deadlock.

Secondly, we consider the comparison between Central and Pacific Leagues by using the measure Φ^{**} .

From Table 3b, the value of $\hat{\Phi}^{**}$ is greater for Table 2b than for Table 2a. Therefore, we get the same result as described above.

Lastly, we see which triad of i , j and k contribute most to the degree of departure from the BT model. For the data in Table 2a, $\hat{M}_{125} = 0.52$ and $\hat{M}_{356} = 0.43$. Also, an approximate 95% confidence intervals for M_{125} and M_{356} are (0.17, 0.86) and (0.06, 0.80), respectively, and the confidence intervals for other three teams in Table 2a contain zero although the detail is omitted. Therefore, we could infer that triads of (Giants, Tigers, Swallows) and (Dragons, Swallows, Bay Stars) rather than other three teams in the Central League in 2009 are close to a situation with strongest stochastic three-way deadlock. On the other hand, for the Pacific League in 2009, $\hat{M}_{124} = 0.39$, $\hat{M}_{134} = 0.43$, $\hat{M}_{245} = 0.53$, $\hat{M}_{256} = 0.46$ and $\hat{M}_{346} = 0.43$. Although the detail is omitted, an approximate 95% confidence intervals for M_{124} , M_{134} , M_{245} , M_{256} and M_{346} do not contain zero, and the confidence intervals for other three teams in Table 2b contain zero. Therefore, we could infer that triads of (Lions, Buffaloes, Marines), (Lions, Fighters, Marines), (Buffaloes, Marines, Eagles), (Buffaloes, Eagles, Hawks) and (Fighters, Marines, Hawks) rather than other three teams in the Pacific League in 2009 are close to a situation with strongest stochastic three-way deadlock.

5. Discussion

Consider an $R \times R$ square contingency table with the same nominal row and column classifications, let p_{ij} denote the probability that an observation will fall in the i th row and j th column of the table ($i = 1, \dots, R; j = 1, \dots, R$). The symmetry (S) model is defined by

$$p_{ij} = \psi_{ij} \quad (i = 1, \dots, R; j = 1, \dots, R),$$

where $\psi_{ij} = \psi_{ji}$ (Bowker, 1948). When the S model does not hold for the given data, Tomizawa (1994) and Tomizawa, Seo and Yamamoto (1998) proposed the power-divergence type measure which represents the degree of departure from the S model. Moreover, Yamamoto, Miyamoto, Tsuboi and Tomizawa (2008) proposed the true distance type measure.

As the extension of the S model, the QS model is defined by

$$p_{ij} = \mu \alpha_i \beta_j \psi_{ij} \quad (i = 1, \dots, R; j = 1, \dots, R),$$

where $\psi_{ij} = \psi_{ji}$ (Caussinus, 1965). This model can also be expressed as

$$p_{ij} p_{jk} p_{ki} = p_{kj} p_{ji} p_{ik} \quad (i < j < k).$$

Let

$$p_{ij}^c = \frac{p_{ij}}{p_{ij} + p_{ji}} \quad (i \neq j).$$

Then the QS model can be expressed as

$$G_{ijk} = G_{kji} \quad (i < j < k),$$

where

$$G_{ijk} = p_{ij}^c p_{jk}^c p_{ki}^c, \quad G_{kji} = p_{kj}^c p_{ji}^c p_{ik}^c.$$

So the QS model is essentially equivalent to the BT model. Therefore, we shall define the measure Φ_{QS}^* , which represents the degree of departure from the QS model, by Φ^* with $\{\pi_{ij}\}$ replaced by $\{p_{ij}^c\}$.

Next, we shall consider an approximate standard error and large-sample confidence interval for the measure Φ_{QS}^* . Let n_{ij} denote the observed frequency in the i th row and j th column of the table ($i = 1, \dots, R; j = 1, \dots, R$). We assume that $\{n_{ij}\}$ have a multinomial distribution. The sample version of Φ_{QS}^* , i.e., $\hat{\Phi}_{QS}^*$ is given by Φ_{QS}^* with $\{p_{ij}\}$ replaced by $\{\hat{p}_{ij}\}$, where $\hat{p}_{ij} = n_{ij} / n$, where $n = \sum \sum n_{ij}$.

Using the delta method, $\hat{\Phi}_{QS}^*$ has asymptotically a normal distribution with mean Φ_{QS}^* and variance $\sigma^2[\Phi_{QS}^*]$. The measure $\hat{\Phi}_{QS}^*$ is applied to a multinomial sampling, and $\hat{\Phi}^*$ is applied to the independent binomial sampling. So, $\sigma^2[\Phi_{QS}^*]$ with $\{p_{ij}^c\}$ replaced by $\{\pi_{ij}\}$, $i \neq j$, is not always identical to $\sigma^2[\Phi^*]$ except when $\{p_{ij} + p_{ji}\}$ are equal to $\{(n_{ij} + n_{ji})/n\}$ in $\sigma^2[\Phi_{QS}^*]$ (see for the detail, Tahata et al., 2004). Let $\hat{\sigma}^2[\Phi_{QS}^*]$ denote $\sigma^2[\Phi_{QS}^*]$ with $\{p_{ij}\}$ replaced by $\{\hat{p}_{ij}\}$. Then we note that $\{\hat{p}_{ij} + \hat{p}_{ji} = (n_{ij} + n_{ji})/n\}$ in $\hat{\sigma}^2[\Phi_{QS}^*]$. Therefore we point out that the estimated variance $\hat{\sigma}^2[\Phi_{QS}^*]$ is theoretically identical to $\hat{\sigma}^2[\Phi^*]$.

The proposed measure Φ^* is defined by putting $\lambda = -\frac{1}{2}$ and applying square root for $\Phi^{(\lambda)}$. Let $\sigma^2[\Phi^{(-\frac{1}{2})}]$ be the asymptotic variance for the measure $\hat{\Phi}^{(-\frac{1}{2})}$ (see Tahata et al., 2004). From the delta method, the asymptotic variance for Φ^* is $(4\Phi^{(-\frac{1}{2})})^{-1}$ times $\sigma^2[\Phi^{(-\frac{1}{2})}]$. Therefore, (i) when the value of $\Phi^{(-\frac{1}{2})}$ is 0.25, the variance $\sigma^2[\Phi^*] = \sigma^2[\Phi^{(-\frac{1}{2})}]$ since the coefficient value is 1, (ii) when the value of $\Phi^{(-\frac{1}{2})}$ is greater than 0.25, the variance $\sigma^2[\Phi^*]$ is less than the variance $\sigma^2[\Phi^{(-\frac{1}{2})}]$ since the coefficient value is less than 1, and (iii) when the value of $\Phi^{(-\frac{1}{2})}$ is less than 0.25, the variance $\sigma^2[\Phi^*]$ is greater than the variance $\sigma^2[\Phi^{(-\frac{1}{2})}]$ since the coefficient value is greater than 1.

Let G^2 denote the likelihood ratio chi-squared statistic for testing goodness-of-fit of the BT model with $(R-1)(R-2)/2$ degrees of freedom. It may seem to many readers that G^2 is a reasonable measure for representing the degree of departure from the BT model. However, G^2 is not a reasonable measure. Indeed, the measure Φ^* is useful when we want to measure what degree the departure from the BT model is toward the *strongest* three-way deadlock by the true distance measure; although we cannot measure it by the test statistic G^2 .

Moreover, consider the artificial data in Table 4. The values of G^2 are 25.755 for Table 4a and 30.491 for Table 4b, respectively. Thus, the value of G^2 is less for Table 4a than for Table 4b. On the other hand, the value of estimated measure Φ^* is 0.605 for Table 4a and 0.518 for Table 4b. Thus, the value of $\hat{\Phi}^*$ is greater for Table 4a than for Table 4b. Note that Q_{kji}/Q_{ijk} is 1 for $i < j < k$ when the BT model holds. In terms of $\hat{Q}_{kji}/\hat{Q}_{ijk}$, $i < j < k$, (see Table 4c) where \hat{Q}_{ijk} denote Q_{ijk} with $\{\pi_{ij}\}$ replaced by $\{\hat{\pi}_{ij}\}$, it seems natural to conclude that the degree of departure from the BT model is greater for Table 4a than for Table 4b. Therefore $\hat{\Phi}^*$ would also be preferable to the test statistic G^2 for *comparing* the degree of departure from the BT model in several tables.

Assume that the order of categories is not interchanged. Let for $i < j < k$ and $l < m < n$ (where $(l, m, n) \neq (i, j, k)$),

$$M(\{Q_{ijk}^c, Q_{kji}^c\}; \{Q_{lmn}^c, Q_{nml}^c\}) = \left[\frac{2 + \sqrt{2}}{2} \left\{ \left(\sqrt{Q_{ijk}^c} - \sqrt{Q_{lmn}^c} \right)^2 + \left(\sqrt{Q_{kji}^c} - \sqrt{Q_{nml}^c} \right)^2 \right\} \right]^{\frac{1}{2}}.$$

This indicates the true distance between conditional distribution for teams i, j and k and that for teams l, m and n . Note that M_{ijk} is $M(\{Q_{ijk}^c, Q_{kji}^c\}; \{1/2, 1/2\})$. Since $M(\cdot; \cdot)$ satisfies all three postulates, we have

$$\begin{aligned} 0 &\leq M(\{Q_{ijk}^c, Q_{kji}^c\}; \{Q_{lmn}^c, Q_{nml}^c\}) \\ &\leq M(\{Q_{ijk}^c, Q_{kji}^c\}; \{1/2, 1/2\}) + M(\{1/2, 1/2\}; \{Q_{lmn}^c, Q_{nml}^c\}). \end{aligned}$$

This result is not obtained from the power-divergence type measure such as $\Phi^{(\lambda)}$. In deed,

$$0 \leq M(\{Q_{ijk}^c, Q_{kji}^c\}; \{Q_{lmn}^c, Q_{nml}^c\}) \leq \sqrt{2 + \sqrt{2}}.$$

We point out that (i) $M(\cdot; \cdot) = 0$ when $Q_{ijk}^c = Q_{lmn}^c$, and (ii) $M(\cdot; \cdot) = \sqrt{2 + \sqrt{2}}$ when $Q_{ijk}^c = 1$ and $Q_{lmn}^c = 0$ (or $Q_{ijk}^c = 0$ and $Q_{lmn}^c = 1$). Therefore the $M(\{Q_{ijk}^c, Q_{kji}^c\}; \{Q_{lmn}^c, Q_{nml}^c\})$ may be useful for comparing the two conditional distributions.

6. Concluding remarks

The measure Φ^* is the true distance measure which satisfies all three postulates of distance, although the measure $\Phi^{(\lambda)}$ is not the true distance measure. Also, the measure Φ^{**} is the weighted sum of the true distance measure and is different from Φ^* . The reader may be interested in considering which the measures Φ^* and Φ^{**} are preferable. We think that (i) if user want to measure by using the true distance measure, we recommend to use the measure Φ^* , and (ii) if user want to measure by using the weighted sum of the true distance and want to see the partial degree of departure, we recommend to use the measure Φ^{**} .

Since the proposed measures in present paper always range between 0 and 1 independent of the dimension R and $\{r_{ij}\}$ and sample size n , those may be useful for *comparing* the degree of departure from the BT and QS models in several tables.

The measures Φ^* and Φ^{**} would be useful when we want to see with a single summary measure, for example, for the athletic competitions, how strong the stochastic three-way deadlock for any three teams of R teams arises toward a situation with the *strongest* stochastic three-way deadlock, which indicates that the probability that i defeats j , j defeats k , and k defeats i is positive (then the probability for reverse order is zero) or zero (then the probability for reverse order is positive); thus, this indicates that at least one team, e.g., team i , among any three teams i , j and k of R teams, is always defeated by one of the other teams. Also, for the analysis of square contingency tables the measure Φ_{QS}^* would be useful when we want to see with a single summary measure what degree the departure from the QS model.

The proposed measures are invariant under the arbitrary same permutations of row and column categories, and therefore it is possible to apply these measures for analyzing the data on a nominal scale, and also possible for analyzing the data on an ordinal scale if one may not use the information about the order of listing the categories.

7. References

- [1] Agresti, A. *Categorical Data Analysis*. New York: John Wiley, 1990.
- [2] Bishop, Y. M. M., Fienberg, S. E. and Holland, P. W. *Discrete Multivariate Analysis: Theory and Practice*. Cambridge, Massachusetts: The MIT Press, 1975.
- [3] Bowker, A. H. A test for symmetry in contingency tables. *Journal of the American Statistical Association*. 1948, **43**: 572-574.
- [4] Bradley, R. A. and Terry, M. E. Rank analysis of incomplete block designs I. The method of paired comparisons. *Biometrika*. 1952, **39**: 324-345.
- [5] Caussinus, H. Contribution à l'analyse statistique des tableaux de corrélation. *Annales de la Faculté des Sciences de l'Université de Toulouse*. 1965, **29**: 77-182.
- [6] Matusita, K. On the estimation by the minimum distance method. *Annals of the Institute of Statistical Mathematics*. 1954, **5**: 59-65.
- [7] Matusita, K. Decision rules, based on the distance, for problems of fit, two samples, and estimation. *Annals of Mathematical Statistics*. 1955, **26**: 631-640.
- [8] Read, T. R. C. and Cressie, N. *Goodness-of-Fit Statistics for Discrete Multivariate Data*. New York, Springer, 1988.
- [9] Tahata, K., Miyamoto, N. and Tomizawa, S. Measure of departure from quasi-symmetry and Bradley-Terry models for square contingency tables with nominal categories. *Journal of the Korean Statistical Society*. 2004, **33**: 129-147.

- [10] Tomizawa, S. Two kinds of measures of departure from symmetry in square contingency tables having nominal categories. *Statistica Sinica*. 1994, **4**: 325-334.
- [11] Tomizawa, S., Seo, T. and Yamamoto, H. Power-divergence-type measure of departure from symmetry for square contingency tables that have nominal categories. *Journal of Applied Statistics*. 1998, **25**:387-398.
- [12] Yamamoto, K., Miyamoto, N., Tsuboi, H. and Tomizawa, S. Distance measure of departure from symmetry for square contingency tables with nominal categories. *International Journal of Pure and Applied Mathematics*. 2008, **48**: 483-489.

Appendix 1

Assuming that $Q_{ijk} + Q_{kji} \neq 0$ for $i < j < k$, Tahata et al. (2004) proposed the measure $\Phi^{(\lambda)}$, defined by

$$\Phi^{(\lambda)} = \frac{\lambda(\lambda+1)}{2^\lambda - 1} I^{(\lambda)} \quad (\lambda > -1),$$

where

$$I^{(\lambda)} = \frac{1}{\lambda(\lambda+1)} \sum_{i < j < k} \left[Q_{ijk}^* \left\{ \left(\frac{Q_{ijk}^*}{C_{ijk}^*} \right)^\lambda - 1 \right\} + Q_{kji}^* \left\{ \left(\frac{Q_{kji}^*}{C_{kji}^*} \right)^\lambda - 1 \right\} \right],$$

and $\{Q_{ijk}^*, Q_{kji}^*\}$ and $\{C_{ijk}^*, C_{kji}^*\}$ are defined in Section 2, and the value at $\lambda = 0$ is taken to be the limit as $\lambda \rightarrow 0$.

Appendix 2

Using the delta method, $\hat{\Phi}^*$ has asymptotically a normal distribution with mean Φ^* and variance

$$\sigma^2[\Phi^*] = \frac{1}{4(\Phi^*)^2 \Delta^2} \sum_{s=1}^{R-1} \sum_{t=s+1}^R \frac{1}{r_{st}} \left[\frac{1}{\pi_{st}} W_{st}^2 + \frac{1}{\pi_{ts}} V_{ts}^2 - (W_{st} + V_{ts})^2 \right],$$

where

$$\begin{aligned} W_{st} = & -\left(1 + \sqrt{2}\right) \left[\sum_{k=t+1}^R \left\{ \frac{Q_{stk}}{\sqrt{Q_{stk}^c}} - \frac{1}{2} Q_{stk} Q_{kts}^c \frac{\sqrt{Q_{kts}^c} - \sqrt{Q_{stk}^c}}{\sqrt{Q_{stk}^c} Q_{kts}^c} \right\} \right. \\ & + \sum_{i=1}^{s-1} \left\{ \frac{Q_{ist}}{\sqrt{Q_{ist}^c}} - \frac{1}{2} Q_{ist} Q_{tsi}^c \frac{\sqrt{Q_{tsi}^c} - \sqrt{Q_{ist}^c}}{\sqrt{Q_{ist}^c} Q_{tsi}^c} \right\} \\ & + \sum_{j=s+1}^{t-1} \left\{ \frac{Q_{tjs}}{\sqrt{Q_{tjs}^c}} - \frac{1}{2} Q_{tjs} Q_{sjt}^c \frac{\sqrt{Q_{sjt}^c} - \sqrt{Q_{tjs}^c}}{\sqrt{Q_{tjs}^c} Q_{sjt}^c} \right\} \\ & \left. - \left(\sum_{k=t+1}^R Q_{stk} + \sum_{i=1}^{s-1} Q_{ist} + \sum_{j=s+1}^{t-1} Q_{tjs} \right) \left((1 - \sqrt{2})(\Phi^*)^2 + \sqrt{2} \right) \right], \\ V_{ts} = & -\left(1 + \sqrt{2}\right) \left[\sum_{k=t+1}^R \left\{ \frac{Q_{kts}}{\sqrt{Q_{kts}^c}} - \frac{1}{2} Q_{kts} Q_{stk}^c \frac{\sqrt{Q_{stk}^c} - \sqrt{Q_{kts}^c}}{\sqrt{Q_{kts}^c} Q_{stk}^c} \right\} \right. \\ & + \sum_{i=1}^{s-1} \left\{ \frac{Q_{tsi}}{\sqrt{Q_{tsi}^c}} - \frac{1}{2} Q_{tsi} Q_{ist}^c \frac{\sqrt{Q_{ist}^c} - \sqrt{Q_{tsi}^c}}{\sqrt{Q_{tsi}^c} Q_{ist}^c} \right\} \\ & + \sum_{j=s+1}^{t-1} \left\{ \frac{Q_{sjt}}{\sqrt{Q_{sjt}^c}} - \frac{1}{2} Q_{sjt} Q_{tjs}^c \frac{\sqrt{Q_{tjs}^c} - \sqrt{Q_{sjt}^c}}{\sqrt{Q_{sjt}^c} Q_{tjs}^c} \right\} \\ & \left. - \left(\sum_{k=t+1}^R Q_{kts} + \sum_{i=1}^{s-1} Q_{tsi} + \sum_{j=s+1}^{t-1} Q_{sjt} \right) \left((1 - \sqrt{2})(\Phi^*)^2 + \sqrt{2} \right) \right]. \end{aligned}$$

Appendix 3

Using the delta method, $\hat{\Phi}^{**}$ has asymptotically a normal distribution with mean Φ^{**} and variance

$$\sigma^2[\Phi^{**}] = \frac{1}{\Delta^2} \sum_{s=1}^{R-1} \sum_{t=s+1}^R \frac{1}{r_{st}} \left[\frac{1}{\pi_{st}} W_{st}^2 + \frac{1}{\pi_{ts}} V_{ts}^2 - (W_{st} + V_{ts})^2 \right],$$

where

$$\begin{aligned} W_{st} = & \left(2 + \sqrt{2}\right)^{\frac{1}{2}} \left[\sum_{k=t+1}^R \left\{ \varrho_{stk} \sqrt{A_{stk}} + \frac{\sqrt{2}}{8} \left(\frac{\varrho_{stk} \varrho_{kts}^c}{A_{stk}} \right)^{\frac{1}{2}} (\sqrt{\varrho_{stk}} - \sqrt{\varrho_{kts}}) \right\} \right. \\ & + \sum_{i=1}^{s-1} \left\{ \varrho_{ist} \sqrt{A_{ist}} + \frac{\sqrt{2}}{8} \left(\frac{\varrho_{ist} \varrho_{tsi}^c}{A_{ist}} \right)^{\frac{1}{2}} (\sqrt{\varrho_{ist}} - \sqrt{\varrho_{tsi}}) \right\} \\ & + \sum_{j=s+1}^{t-1} \left\{ \varrho_{tjs} \sqrt{A_{tjs}} + \frac{\sqrt{2}}{8} \left(\frac{\varrho_{tjs} \varrho_{sjt}^c}{A_{tjs}} \right)^{\frac{1}{2}} (\sqrt{\varrho_{tjs}} - \sqrt{\varrho_{sjt}}) \right\} \\ & \left. - \Phi^{**} \left(2 + \sqrt{2}\right)^{-\frac{1}{2}} \left(\sum_{k=t+1}^R \varrho_{stk} + \sum_{i=1}^{s-1} \varrho_{ist} + \sum_{j=s+1}^{t-1} \varrho_{tjs} \right) \right], \\ V_{ts} = & \left(2 + \sqrt{2}\right)^{\frac{1}{2}} \left[\sum_{k=ts}^R \left\{ \varrho_{kts} \sqrt{A_{kts}} + \frac{\sqrt{2}}{8} \left(\frac{\varrho_{kts} \varrho_{stk}^c}{A_{kts}} \right)^{\frac{1}{2}} (\sqrt{\varrho_{kts}} - \sqrt{\varrho_{stk}}) \right\} \right. \\ & + \sum_{i=1}^{s-1} \left\{ \varrho_{tsi} \sqrt{A_{tsi}} + \frac{\sqrt{2}}{8} \left(\frac{\varrho_{tsi} \varrho_{ist}^c}{A_{tsi}} \right)^{\frac{1}{2}} (\sqrt{\varrho_{tsi}} - \sqrt{\varrho_{ist}}) \right\} \\ & + \sum_{j=s+1}^{t-1} \left\{ \varrho_{sjt} \sqrt{A_{sjt}} + \frac{\sqrt{2}}{8} \left(\frac{\varrho_{sjt} \varrho_{tjs}^c}{A_{sjt}} \right)^{\frac{1}{2}} (\sqrt{\varrho_{sjt}} - \sqrt{\varrho_{tjs}}) \right\} \\ & \left. - \Phi^{**} \left(2 + \sqrt{2}\right)^{-\frac{1}{2}} \left(\sum_{k=t+1}^R \varrho_{kts} + \sum_{i=1}^{s-1} \varrho_{tsi} + \sum_{j=s+1}^{t-1} \varrho_{sjt} \right) \right], \end{aligned}$$

with

$$A_{ijk} = A_{kji} = 1 - \frac{\sqrt{2}}{2} \left(\sqrt{\varrho_{ijk}^c} + \sqrt{\varrho_{kji}^c} \right) \quad (i < j < k).$$

Appendix 4

Using the delta method, \hat{M}_{ijk} ($i < j < k$) has asymptotically a normal distribution with mean M_{ijk} and variance

$$\sigma^2[M_{ijk}] = \left\{ \frac{1}{r_{ij}} \left(\frac{1}{\pi_{ij}} + \frac{1}{\pi_{ji}} \right) + \frac{1}{r_{jk}} \left(\frac{1}{\pi_{jk}} + \frac{1}{\pi_{kj}} \right) + \frac{1}{r_{ki}} \left(\frac{1}{\pi_{ki}} + \frac{1}{\pi_{ik}} \right) \right\} W_{ijk}^2,$$

where

$$W_{ijk} = \left(2 + \sqrt{2} \right)^{\frac{1}{2}} \frac{\sqrt{2}}{8} \left(\frac{Q_{ijk}^c Q_{kji}^c}{A_{ijk}} \right)^{\frac{1}{2}} \left(\sqrt{Q_{ijk}^c} - \sqrt{Q_{kji}^c} \right),$$

with

$$A_{ijk} = 1 - \frac{\sqrt{2}}{2} \left(\sqrt{Q_{ijk}^c} + \sqrt{Q_{kji}^c} \right).$$

Table 1

The results of the professional baseball league in Japan in 2008. For instance, from Giants' perspective, the (Giants, Tigers) results correspond to 14 successes and 10 failures in 24 trials.

(a) 2008 Central League

	Giants	Tigers	Dragons	Carp	Swallows	Bay Stars	Total
Giants	-	14	10	10	18	18	70
Tigers	10	-	17	14	13	13	67
Dragons	14	6	-	13	9	17	59
Carp	12	10	9	-	12	13	56
Swallows	6	10	13	11	-	15	55
Bay Stars	5	10	7	11	9	-	42
Total	47	50	56	59	61	76	349

(b) 2008 Pacific League

	Lions	Buffaloes	Fighters	Marines	Eagles	Hawks	Total
Lions	-	14	14	13	14	11	66
Buffaloes	10	-	13	14	13	14	64
Fighters	9	11	-	12	10	17	59
Marines	11	10	12	-	16	14	63
Eagles	10	10	13	7	-	12	52
Hawks	10	10	7	10	12	-	49
Total	50	55	59	56	65	68	353

Table 2

The results of the professional baseball league in Japan in 2009. For instance, from Giants' perspective, the (Giants, Tigers) results correspond to 11 successes and 11 failures in 22 trials.

(a) 2009 Central League

	Giants	Tigers	Dragons	Carp	Swallows	Bay Stars	Total
Giants	-	11	16	14	18	18	77
Tigers	11	-	10	13	9	15	58
Dragons	8	14	-	16	11	18	67
Carp	7	11	8	-	12	13	51
Swallows	5	15	13	12	-	11	56
Bay Stars	6	9	6	11	13	-	45
Total	37	60	53	66	63	75	354

(b) 2009 Pacific League

	Lions	Buffaloes	Fighters	Marines	Eagles	Hawks	Total
Lions	-	15	12	10	12	10	59
Buffaloes	9	-	8	14	4	13	48
Fighters	12	16	-	18	13	11	70
Marines	14	9	6	-	11	13	53
Eagles	12	19	11	13	-	13	68
Hawks	12	11	12	10	11	-	56
Total	59	70	49	65	51	60	354

Table 3

Estimates of Φ^* and Φ^{**} , estimated approximate standard errors for $\hat{\Phi}^*$ and $\hat{\Phi}^{**}$, approximate 95% confidence intervals for Φ^* and Φ^{**} , applied to Tables 1a, 1b, 2a and 2b.

(a) For Table 1

Measure	Applied data	Estimated measure	Standard error	Confidence interval
Φ^*	Table 1a	0.294	0.072	(0.154, 0.435)
	Table 1b	0.197	0.073	(0.053, 0.340)
Φ^{**}	Table 1a	0.254	0.070	(0.117, 0.391)
	Table 1b	0.165	0.067	(0.033, 0.296)

(b) For Table 2

Measure	Applied data	Estimated measure	Standard error	Confidence interval
Φ^*	Table 2a	0.247	0.074	(0.103, 0.392)
	Table 2b	0.284	0.071	(0.145, 0.423)
Φ^{**}	Table 2a	0.203	0.069	(0.067, 0.339)
	Table 2b	0.234	0.070	(0.096, 0.372)

Table 4

Tables 4a and 4b are the artificial data, and Table 4c is the values of $\hat{Q}_{kji} / \hat{Q}_{ijk}$ for $i < j < k$.

(a)			
-	6	12	23
16	-	7	6
11	15	-	4
9	16	18	-

(b)			
-	12	23	26
32	-	16	12
21	28	-	8
18	32	36	-

(c)		
	For Table 4a	For Table 4b
$\hat{Q}_{321} / \hat{Q}_{123}$	6.23	5.11
$\hat{Q}_{421} / \hat{Q}_{124}$	18.17	10.27
$\hat{Q}_{431} / \hat{Q}_{134}$	10.54	5.93
$\hat{Q}_{432} / \hat{Q}_{234}$	3.62	2.95