ISSN 1750-9823 (print)
International Journal of Sports Science and Engineering
Vol. 1 (2007) No. 2, pp. 75-86

A Triple Pendulum Robotic Model and a Set Of Simple Parametric Functions for the Analysis of the Golf Swing

Michele Aicardi [+]

Department of Communications, Computers and Systems Science (DIST), University of Genoa, Via Opera Pia 11/A, 16145, Genova, ITALY

(*Received May 15 2007, accepted June 16 2007*)

Abstract. The paper deals with the analysis the golf swing using a triple pendulum model and a simple class of feasible trajectories for the joints. The basic idea stems from the analysis of motions in sports. Many tasks in sport are not specified in term of equivalent joint trajectories but mainly with boundary conditions of joint position and velocity (e.g. weight lifting, punching, swinging in golf). In such tasks even if the 'optimal' trajectory is not a-priori known, some properties can be however inferred and therefore translated in the shape of the representing functions. For the golf swing, the influence of the joints' trajectory as well as that of the initial posture is analyzed with respect to the applied torques and impact velocity.

Keywords: triple pendulum, parametric functions, robotic model, golf swing

1. Introduction

The golf swing has been often matter of research in the last years (for an excellent review paper see [10]). Different models have been considered from a classical double-pendulum up to a full body representation of a player ([4],[8],[11]). The basic steps are however quite standard and are those used in robotics: set-up the model, define the goal, define the optimization parameters/functions, optimize via a numerical approach. The analysis and optimization of robotic motions, with reference to human tasks, can be performed with a variety of techniques among which, a well established approach is that of using some set of parametrized functions (polynomial, harmonic, neural networks, different kind of spline) able to represent the unknown control and or joint signals and then finding the optimal set of parameters by means of some procedure (direct optimization, learning) [1],[5],[3],[2],[6],[9]. What is generally left out from the direct approach is the use of some information about the trajectories to be found. There are a number of tasks for which, even if the optimal time-trajectory is not known, it is possible to safely assume some property (e.g. monotonic behavior). Consider the golf swing: if we consider the system from the shoulder to the wrist, it is very unlikely that the time trajectories of the joints located in the shoulder, elbow and wrist can have non monotonic behavior from the beginning of the downswing to the impact. So, if we would analyze such motion using any 'standard' set of functions, we should have to introduce a sufficient number of parameters and trust the results of the optimization, with little (if any) possibility of forcing the monotonic property in other way than introducing additional constraints within the optimization procedure itself. As an example, with nevertheless very remarkable results, see [6].

In this paper the basic approach is the classic one, i.e. parameterize the time-trajectories and then find the set of optimal parameters. The difference is first in a trivial but useful decomposition of the trajectory in term of component functions and second to characterize such functions with a structure where the parameters influence in a predictable and controllable way their shape. This aspect is most useful for reducing the nominal number of parameters to a subset of them depending on the characteristics of the task to be analyzed.

The paper is organized as follows: Section 2 introduces the triple-pendulum model relevant to the upper arms+torso/lower arms/hands+club. Section 3 is devoted to the definition of the proposed basis functions starting first with a very simple (albeit representative) structure and then making it more complex

[+] Corresponding author. *Tel.*: +39-10-3532746 , *Fax*.: +39-10-3532154
E-mail address: michele@dist.unige.it.

(basically 'nesting' the simple structure onto itself) in order to account for a broader class of boundary conditions. Section 4 will validate the proposed approach, even with a reduced set of parameters, with the analysis of the golf swing with respect to the joints' trajectory as well as to the posture at the beginning of the downswing.

2. The triple pendulum model

Here a triple-pendulum model of the upper/lower arms and golf club is introduced. The first link represents the upper arms plus some contribution of the torso, the second link is for the lower arms, whereas the third link represents the hands and the golf club (a driver). The lengths, masses and moments of inertia of the links reported in Table 1, have been determined to replicate the experiments as reported in [4] and [8] and therefore validate the model.

Table 1: Triple pendulum parameters

	Length	Mass	M.o.I.
Link 1	0.3 m	5.168 Kg	0.155 Kg m^2
Link 2	0.315 m	4.1 Kg	0.123 Kg m^2
Link 3	1.105 m	0.394 Kg	0.77 Kg m^2

Angles in Figure 1 identify the position of the shoulders, the elbows and the wrists respectively. In such a setting, the swing can be represented in terms of time evolution of the angles.

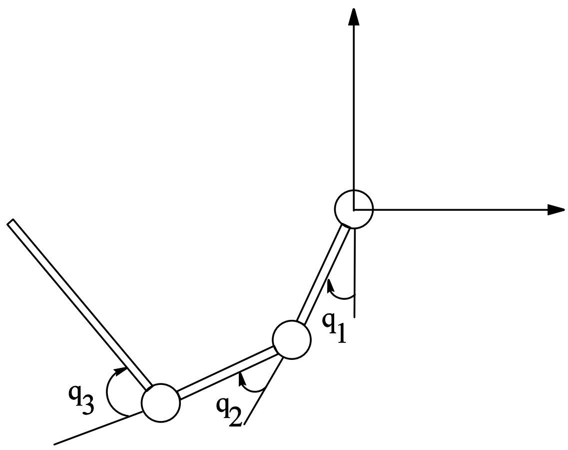

Figure 1. Triple link model

So, the next step will be the definition of a suitable structure for any in order to analyze, and, eventually, optimize the motion.

3. The set of functions used to characterize the motion

Consider a robotic structure with N degrees of freedom, in which each joint variable q_i has to start from $q_i(0)$ and reach $q_i(T)$ at time T. Any trajectory $q_i(t)$ can be written as

$$q_i(t) = q_i(0) + [q_i(T) - q_i(0)]\, f_{i,01}(t) + \varepsilon\, f_{i,00}(t) \quad t \in [0,T] \tag{1}$$

for some constant ε and with functions $f_{i,01}(t)$ and $f_{i,00}(t)$ such that

$$f_{i,01}(0) = 0\ ;\ f_{i,01}(T) = 1 \tag{2}$$

$$f_{i,00}(0) = 0\ \ ;\ f_{i,00}(T) = 0 \tag{3}$$

Where needed, we shall use (superscript T denotes the transpose)

$$Q(t) = [q_1(t), q_2(t), q_3(t)]^T \tag{4}$$

In general, to find the optimal motion with respect to some criterion, one has to represent in a parametric way either $f_{i,01}(t)$ and $f_{i,00}(t)$ and proceed to find the optimal parameters. An usual issue about this approach is the relationship between the number of parameters and the control on the shape of the solution.

3.1. A very simple structure

First, a very simplified structure of functions $f_{i,01}$ and $f_{i,00}$ is proposed just to illustrate the related properties. The very basic element of the proposed decomposition is the following (dropping the subscript i):

$$f_{01}(x,t,T) = \frac{1-e^{xt/T}}{1-e^{x}} \tag{5}$$

Such a function, for different values of x represents different shapes that are, even at this early stage, very usable since with a single parameter can span a family of shapes each of which increasing with time.

With the same approach, it is also possible to express $f_{00}(t)$ as

$$f_{00}(y,t,T) = \frac{-e^{y} + e^{y(1-t/T)} + e^{yt/T} - 1}{1-e^{y}} \tag{6}$$

As a consequence a three parameter expression of a possible trajectory is readily built:

$$q(x,y,t,T) = q(0) + [q(T) - q(0)]\frac{1-e^{xt/T}}{1-e^{x}} + \varepsilon\frac{-e^{y} + e^{y(1-t/T)} + e^{yt/T} - 1}{1-e^{y}} \tag{7}$$

A set of shapes corresponding to $q(0)=0$, $q(T)=1$, $\varepsilon=1$, and for some couple x,y is reported in Figure 2.

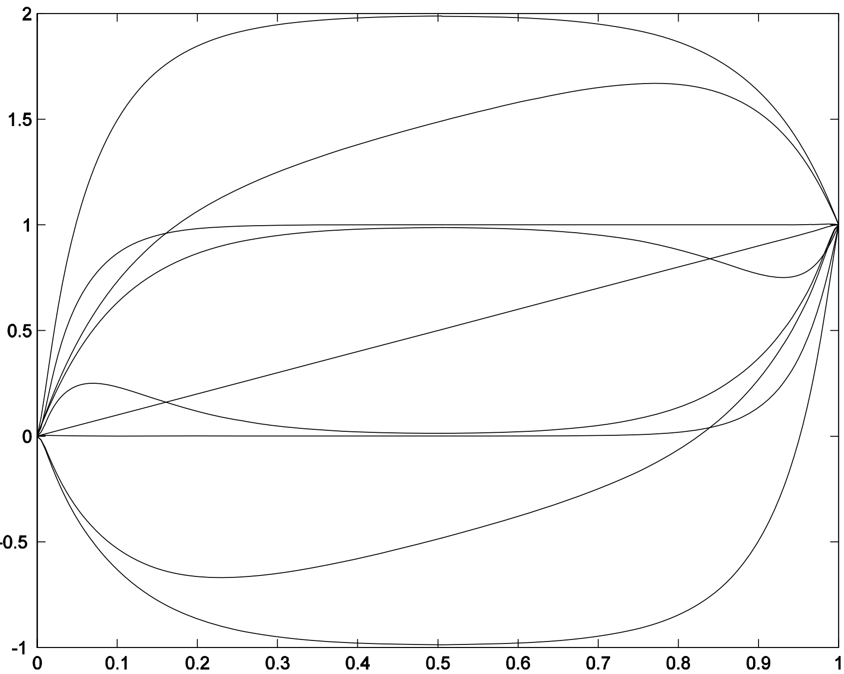

Figure 2. Set of $q(x,y,t)$ for $t \in [0,1]$ and different values of (x,y)

It is quite evident that for fixed $q(0)$ and $q(T)$ even such a simple structure can represent a quite useful family of functions. However, in order to represent time behaviors suitable to be used in the analysis of robotic/human motions, one has to assure that some additional set of boundary conditions can be fulfilled with the minimal loss of generality. To clarify this point, consider for example the following:

Problem 1: With $T=1$, $q(0)=0$ and $q(1)=1$, find a triple (x,y,ε) such that $\dot{q}(0)=0$ and $\dot{q}(1)=\dot{q}^*$

For such a problem, differentiating (7) with respect to time and computing the time derivatives in $t=0$ and $t=T$ respectively lead to the following set of conditions:

$$-\frac{x}{1-e^x}+\varepsilon y=0 \tag{8}$$

$$-\frac{xe^x}{1-e^x}-\varepsilon y=\dot{q}^* \tag{9}$$

which imply to find x such that

$$-\frac{x(1+e^x)}{1-e^x}=\dot{q}^* \tag{10}$$

Now, since $-\frac{x(1+e^x)}{1-e^x}\geq 2$ the above problem can be solved only if $\dot{q}^*\geq 2$

3.2. The proposed structure

Having put into evidence the limits of (5) consider now:

$$f_{01}(x(t),t)=\frac{1-e^{x(t)t/T}}{1-e^{x(T)}} \tag{11}$$

Obviously, (11) gives little indications since $x(t)$ may be almost any function. For this reason the following particular structure for $x(t)$ is proposed:

$$x(t)=\alpha_{in}+[\alpha_{fin}-\alpha_{in}]\frac{1-e^{\gamma t/T}}{1-e^{\gamma}} \tag{12}$$

so that, defining $\alpha=[\alpha_{in},\alpha_{fin}]$ the complete form of f_{01} becomes:

$$f_{01}(\alpha,\gamma,t,T)=\frac{1-e^{\left[\alpha_{in}+[\alpha_{fin}-\alpha_{in}]\frac{1-e^{\gamma t/T}}{1-e^{\gamma}}\right]t/T}}{1-e^{\alpha_{fin}}} \tag{13}$$

Basically, the proposed structure, which introduces a time-varying 'time constant', is a kind of nesting of a function onto itself. In this case, the welcome property of being monotonically increasing is not guaranteed any more, and depends on the new parameter introduced. However, in the special case of $\alpha_{in}=0$, differentiating (13) with respect to time, we get (with some trivial but cumbersome computation)

$$\dot{f}_{01}(\alpha_{fin},\gamma,t,T)>0 \quad \forall(t\in[0,T],\alpha_{fin},\gamma) \tag{14}$$

Exactly in the same way we can modify f_{00} introducing a similar time-varying behavior for $y(t)$:

$$y(t)=\beta_{in}+[\beta_{fin}-\beta_{in}]\frac{1-e^{\delta t/T}}{1-e^{\delta}} \tag{15}$$

which is to be used in the following form of $f_{00}(t)$

$$f_{00}(\beta,\delta,t,T)=\frac{-e^{\beta_{fin}}+e^{y(T-t)(1-t/T)}+e^{y(t)t/T}-1}{1-e^{\beta_{fin}}} \tag{16}$$

So, the complete trajectory can be expressed as (dropping all the arguments from q to simplify the notation

$$q(\cdot)=q(0)+[q(T)-q(0)]f_{01}(\alpha,\gamma,t,T)+\varepsilon f_{00}(\beta,\delta,t,T) \tag{17}$$

with $f_{01}(\alpha,\gamma,t,T)$, $f_{00}(\beta,\delta,t,T)$, and $y(t)$ given by (13), (16) and (15) respectively. Just for completeness, Figure 3 reports some plots of the function $q(\alpha,\beta,\gamma,\delta,\varepsilon,t,T)$ for $T=1$, $\alpha_{in}=\beta_{in}=0$, $\gamma=\delta$, $\varepsilon=1$, and for some values of ($\alpha_{fin},\beta_{fin},\gamma$).

Of course, it might be possible to go further in the complexity, defining a time-varying structure for γ,δ, but it would not have an effect comparable to the computational effort needed to deal with the new complexity. As a matter of fact it can be shown that each time a new time-varying parameter is introduced, it is possible to fix the value of an higher order derivative computed in some time instant (e.g. $t=0$ as one has in mind the McLaurin expansion). This means that the proposed procedure may allow to build set of functions 'dense' with respect to the continuous functions. As will be apparent in the section devoted to the analysis, the proposed set reported in (17) is sufficient to fix the most important aspects of the swing.

Looking at (13), (16) and (17) we have that the number of parameters are now 7. It is not a small number but there is a fact to consider: each of them has an apparent effect. In this respect one has a great control on the parameters themselves as regards different aspects to be taken into account. For instance both α_{fin} and β_{fin} control the shape of the functions f_{01} and f_{00} in the sense that they control the speed at which the final value (1 or 0 respectively) is reached. In the same way γ and δ just modulate the speed at which the relevant exponents, namely $x(t)$ and $y(t)$ approach their final values α_{fin} and β_{fin} respectively. With this in mind is not difficult to fix a-priori some of such values on the basis of a qualitative analysis of the problem to be solved as will be done in the following section.

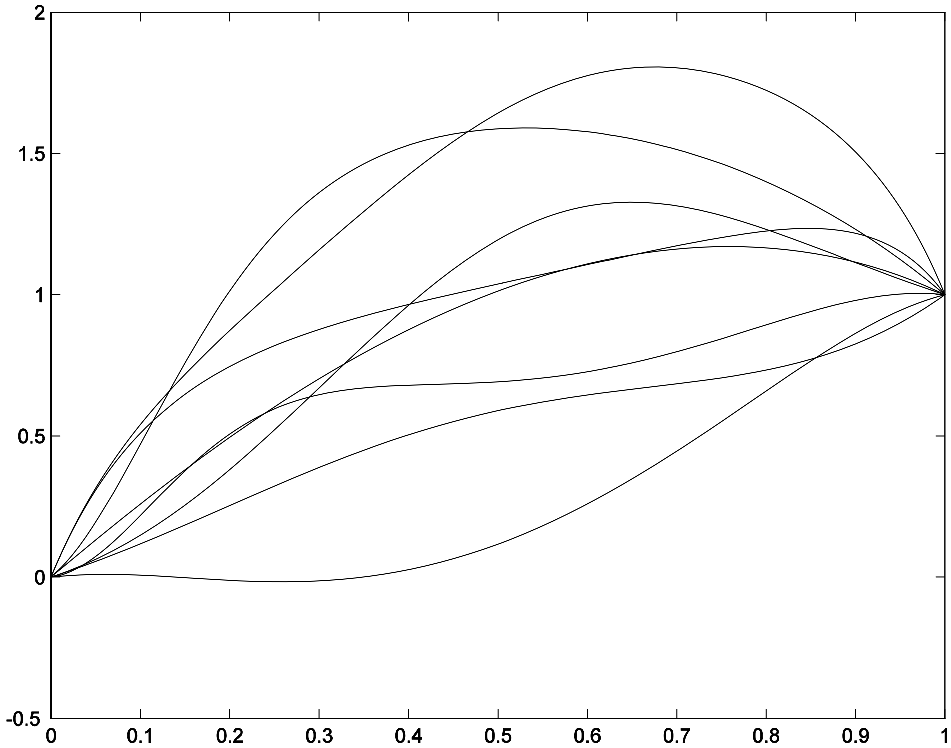

Figure 3. Set of $q(\cdot,t)$ for $t \in [0,1]$

4. Analysis of the swing

The motion to be analyzed has its own characteristics that can be taken into account. This will allow to eliminate most parameters to research for just on the basis of some physical considerations. First, let us consider only the downswing, and assume that the impact position corresponds to $Q(T)=[0,0,0]^T$.

Moreover fix the boundary conditions about initial joint velocities and the downswing time to:

$$\dot{q}_i(0)=0 \ \ \forall i \ ; \ T=0.34$$

since at the top of the back-swing we have no velocity (as to the specification on the downswing time it is taken from real measures as reported in [8]). Before going on, we assume that

1. All the joints have a monotonic time behavior from the top of the backswing to the impact position since it seems unlikely that during the downswing a joint has to invert its velocity;
2. The impact position should be reached with null torques in order to let the follow-through go by itself (as reported in [11]).

These specifications immediately cut the number of the parameters to be considered. Consider first assumption 1. In order to fulfill the monotonic requirements, with the aim of reducing the set of parameters to be optimized, we may set for any joint $\varepsilon=0$ therefore dropping the search for β and δ and leaving to functions f_{01} the tasks of generating monotonic behaviors subject to null initial velocities. Go now with assumption 2. The triple link is subject to the usual dynamic equations

$$A(Q)\ddot{Q}+B(Q,\dot{Q})\dot{Q}+C(Q)=m \tag{18}$$

where $A(Q)$ is the inertia matrix, $B(Q,\dot{Q})\dot{Q}$ takes into account centrifugal and Coriolis terms, $C(Q)$ represents the effects of gravity and m is the vector of applied torques. We consider the swing on a vertical plane dropping also any friction effects (these assumption are made just to compare the obtained results with the ones obtained in literature since it would not make the problem under analysis any more complex). In these conditions, the requirement about "zero torque at impact" together with $Q(T)=[0,0,0]^T$ translate immediately to

$$\ddot{Q}(T)=[0,0,0]^T$$

Summing up the boundary conditions to be satisfied are

$$Q(T)=[0,0,0]^T \tag{19}$$

$$\dot{Q}(0)=[0,0,0]^T \tag{20}$$

$$\ddot{Q}(T)=[0,0,0]^T \tag{21}$$

The first is actually obvious and can be directly substituted in (17). Remember now that we have fixed $\varepsilon=0$, therefore we have to work only with (13). More specifically we have for a generic joint:

$$\dot{q}(t)=-q(0)\,\dot{f}_{01}(\alpha,\gamma,t,T) \tag{22}$$

$$\ddot{q}(t)=-q(0)\,\ddot{f}_{01}(\alpha,\gamma,t,T) \tag{23}$$

which, computed in $t=0$ and $t=T$ respectively have to fulfill the relevant (20) and (21). Condition (20) leads to

$$\dot{f}_{01}(\alpha,\gamma,0,T)=0$$

To this end compute $\dot{f}_{01}(\alpha,\gamma,t,T)$. Recalling expression (11), and taking into account that $x(T)=\alpha_{fin}$, we have

$$\dot{f}_{01}=\frac{-e^{x(t)t/T}}{T(1-e^{\alpha_{fin}})}(\dot{x}(t)t+x(t)) \tag{24}$$

which, computed in $t=0$ leads to

$$\frac{-1}{T(1-e^{\alpha_{fin}})}(\alpha_{in})=0 \tag{25}$$

that in turn gives, for all joints $\alpha_{in}=0$. Such a result gives also the solution to the problem of fulfilling Assumption 1. In fact, as we have already seen in the previous section, if $\alpha_{fin}=0$ then $\dot{f}_{01}>0$ (see (14)).

So far, consider now the requirements about the second time-derivative computed in $t=T$ (Assumption 2). Considering $\alpha_{in}=0$ and with some computation we get the condition:

$$\frac{-\alpha_{fin}e^{\alpha_{fin}}}{T^2(1-e^{\alpha_{fin}})}\left[\alpha_{fin}\left(1-\frac{\gamma e^{\gamma}}{1-e^{\gamma}}\right)^2-(\gamma+2)\frac{\gamma e^{\gamma}}{1-e^{\gamma}}\right]=0 \tag{26}$$

Therefore, to satisfy (26), for each joint (reintroducing index i for the i-th joint), $\alpha_{i,fin}$ must be chosen as follow

$$\alpha_{i,fin}=\frac{(\gamma_i+2)\frac{\gamma_i e_i^{\gamma}}{1-e_i^{\gamma}}}{\left(1-\frac{\gamma_i e_i^{\gamma}}{1-e_i^{\gamma}}\right)^2} \tag{27}$$

Note that the range of $\alpha_{i,fin}$ for varying γ_i is $-1\le\alpha_{i,fin}\le 0.134$.

At this point we are left with a single parameter per joint, namely $\gamma_i, i=1,2,3$, and we have to deal with the ’real’ problem that is how to fix such parameters in order to make a ’good’ shot without using 'unfeasible' joint torques. First recall that the flight on the ball is a function of the club-head speed at impact [7]. Since the club-head speed is the speed of the end effector we can easily compute such quantity using the Jacobian matrix at the impact position. Due to the assumption on the impact position (19), the Jacobian matrix turns out to be singular with only one row being meaningful for the computation.

Namely, (reporting the only non-zero row) the Jacobian at impact results

$$J(Q(T)) = [-1.72 - 1.42 - 1.105]$$

and therefore the club-head speed v at impact can be computed as (the minus sign in the Jacobian's components is due to the way of measuring angles q_i, see Figure 1):

$$v(T) = J(Q(T))\dot{Q}(T) = [-1.72 - 1.42 - 1.105]\dot{Q}(T) \tag{28}$$

which is apparently a function of all γ_i through $\dot{Q}(T)$. The computation of $\dot{q}_i(T)$ yields (recalling that $\alpha_{i,in} = 0, \forall i$)

$$\dot{q}_i(T) = -q_i(0)\frac{-\alpha_{i,fin}e^{\alpha_{i,fin}}}{T(1-e^{\alpha_{i,fin}})}\left(1 - \frac{\gamma_i e^{\gamma_i}}{1-e^{\gamma_i}}\right) \tag{29}$$

with each $\alpha_{i,fin}$ given by (27).

Moreover, since the human body can generate different maximum torques at different joints we consider the following constraints to be satisfied (see [4], [8])

$$|m(t)| \leq \begin{bmatrix} 160\,Nm \\ 90\,Nm \\ 30\,Nm \end{bmatrix} \qquad \forall t \in [0,T] \tag{30}$$

where all the torques are computed by mean of (18), (recall T=0.34). At this point can state the following

Optimization Problem: Find

$$\max_{\gamma,Q(0)} [-1.72 - 1.42 - 1.105]\dot{Q}(T) \tag{31}$$

s.to (29) and (27) for $i = 1,3$, and conditions (30).

The solution of the above problem can be carried out using a numerical gradient approach with the use of penalty functions to represent (30) (theoretically not the best method, but sufficient to show the effectiveness of the proposed structure of joint trajectories).

4.1. Fixed initial posture

Experiment 1: consider the initial posture fixed as follows:

$$Q(0) = [5\pi/6, \pi/12, \pi/2]^T \tag{32}$$

The optimization of the parameters lead to the following results

$$\gamma = \begin{bmatrix} 0.6 \\ 0.2 \\ 5.3 \end{bmatrix}; \quad \dot{Q}(T) = \begin{bmatrix} -12.83 \\ -1.22 \\ -17.3 \end{bmatrix}; \quad |m(t)| \leq \begin{bmatrix} 161.1 \\ 89.2 \\ 27.5 \end{bmatrix}; \quad v = 42.92\,m/s; \quad p = 5.71m$$

where p is the length of the path traveled by the club-head. The time behaviors are plotted on the set of Figure 4

It is interesting to note that as reported in many studies (see for instance [8]) the wrist angle is kept steady for most of the downswing. Moreover, the path traveled by the club-head is 5.71 m. Computing the velocity of an object covering such a distance with under an uniformly accelerated motion we have v=33.6 m/s. This proves that the head motion is not subject to an uniformly accelerated motion but instead is driven by some kind of higher order torque function.

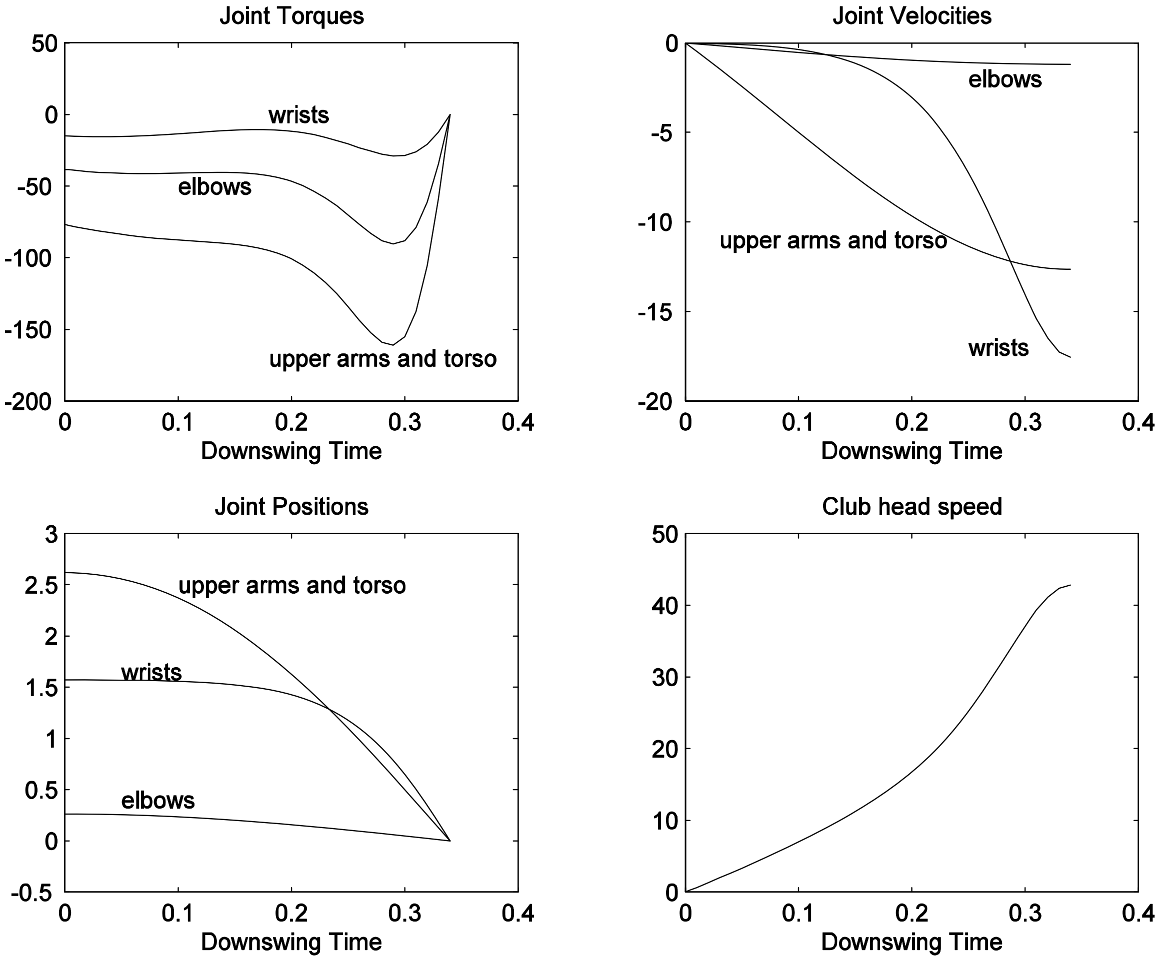

Figure 4. Position, velocities, club-head speed and applied torques for $q(0)=(5\pi/6,\pi/12,\pi/2)$

4.2. On the initial posture

Consider now the initial posture (i.e.the position at the beginning of the downswing) as a matter of optimization. To justify this point of view, we report the

Experiment 2: with the same parameters γ as in Experiment 1, but starting with $Q(0)=[5\pi/6,\pi/4,4\pi/10]^T$ we get a speed $v=42.5$ m/s (namely less than the previous one), but with a set or torques that do not exceed

$$|m(t)|\le\begin{bmatrix}149.1\\69.2\\19.2\end{bmatrix};\ v=42.5m/s;\ p=6.15m$$

which means that with different initial posture we obtain a swing where the club-head runs on a longer path, in the same time and with lower applied torques. Even if in this case we obtain a slightly lower head speed, such a result suggests that the influence of the initial posture has to be studied. Even if in the following, there will be no exhaustive analysis of such a point, some results will be reported to support such a statement.

Experiment 3: consider the ’reference’ model that have been mostly used in the previous literature i.e. the double pendulum. In the presented framework it is easy to replicate this setting since fixing $q_2(0)=0$ imply that the lower and upper arms remain aligned during the whole motion. So, taking

$$Q(0)=[\pi,0,\pi/2]^T$$

we obtain the following results

$$\gamma = \begin{bmatrix} 0.8 \\ 1 \\ 4.5 \end{bmatrix}; |m(t)| \le \begin{bmatrix} 159.2 \\ 82.3 \\ 17.5 \end{bmatrix}; v = 44.15\,m/s;\ p = 6.3m$$

Here we have obtained an higher head speed without violating the upper bound for the torques. Note that the torque applied at the elbow joint is the one needed to maintain such a joint steady at zero.

Experiment 4: in order to stress the influence of the initial posture consider two other set of values for $Q(0)$ computed in such a way that the position of the club-head at the beginning of the downswing does not change with respect to the double pendulum setting. This can be easily done by linearizing the cartesian position of the head with respect to the starting angles of the double pendulum and computing the null-space of this linear transformation. The null-space turns out to be

$$\delta Q = \xi \begin{bmatrix} 0.3983 \\ -0.8164 \\ 0.4182 \end{bmatrix} \text{for small}\, \xi \in \Re \qquad (33)$$

Even if there is not a dramatic difference, for such experiments the path length is not the same $p = 6.3$ as reported above since the swing develop different motions even if the club-head starts from the same position. The results are, (after a fine tuning of γ the applied torque are basically the same)

$$Q(0) = \begin{bmatrix} \pi & 0 & \pi/2 \end{bmatrix}^T - 0.2\delta Q \rightarrow v = 44.55\,m/s;$$

$$Q(0) = \begin{bmatrix} \pi & 0 & \pi/2 \end{bmatrix}^T - 0.3\delta Q \rightarrow v = 45.0\,m/s$$

where we have got higher club-head velocities without increasing the applied torques. At this point, we report the experiment that has given the best results in the analysis.

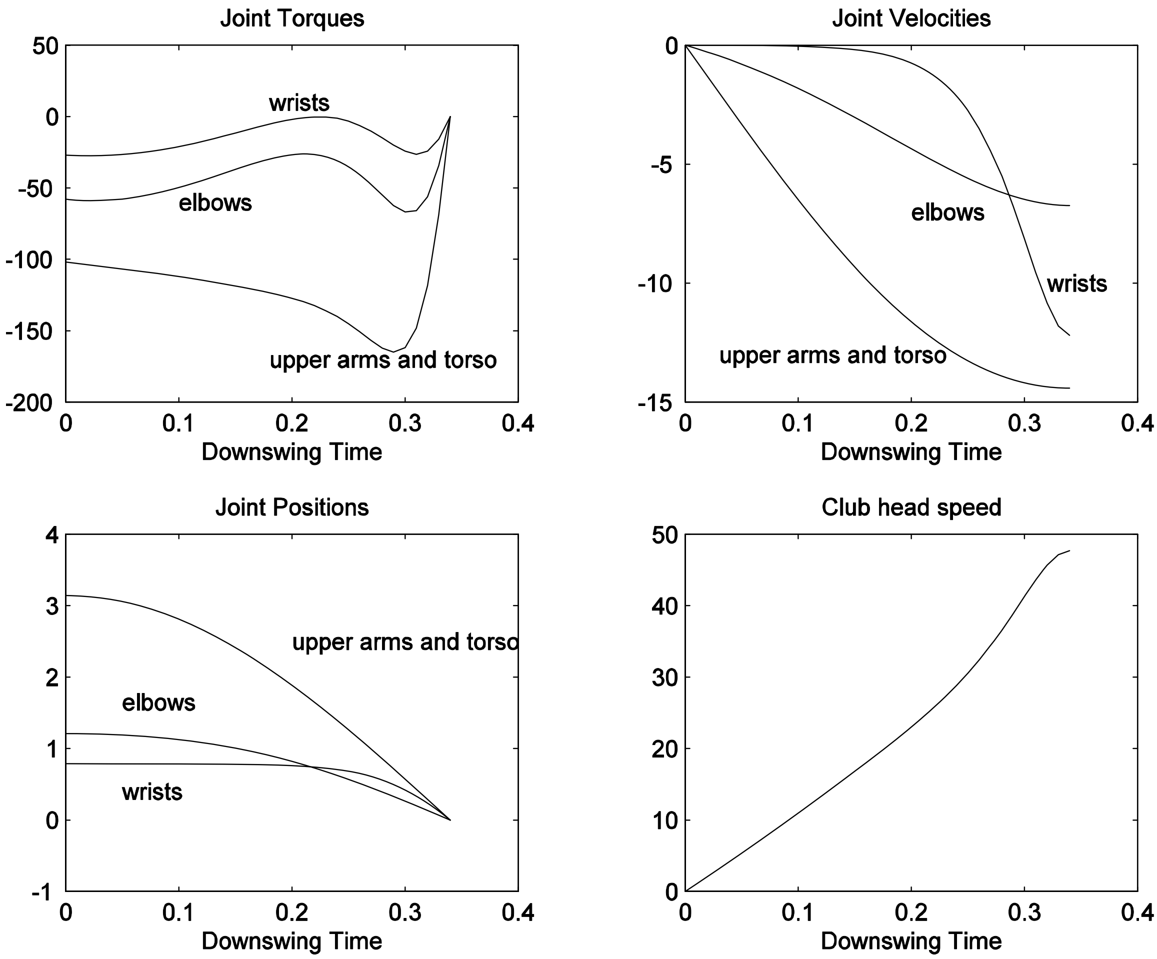

Figure 5. Position, velocities, club-head speed and applied torques for $q(0) = (\pi, 5\pi/13, \pi/4)$

Experiment5: starting with $Q(0)=\begin{bmatrix}\pi & 5\pi/13 & \pi/4\end{bmatrix}^T$,after the optimization of λ, we get

$$|m(t)|\le\begin{bmatrix}162\\66.6\\26.2\end{bmatrix};\ v=47.71m/s;\ p=7.15m$$

and the results are depicted in Figure 5.

The following figure is related to the path difference between three of the previous mentioned experiments:

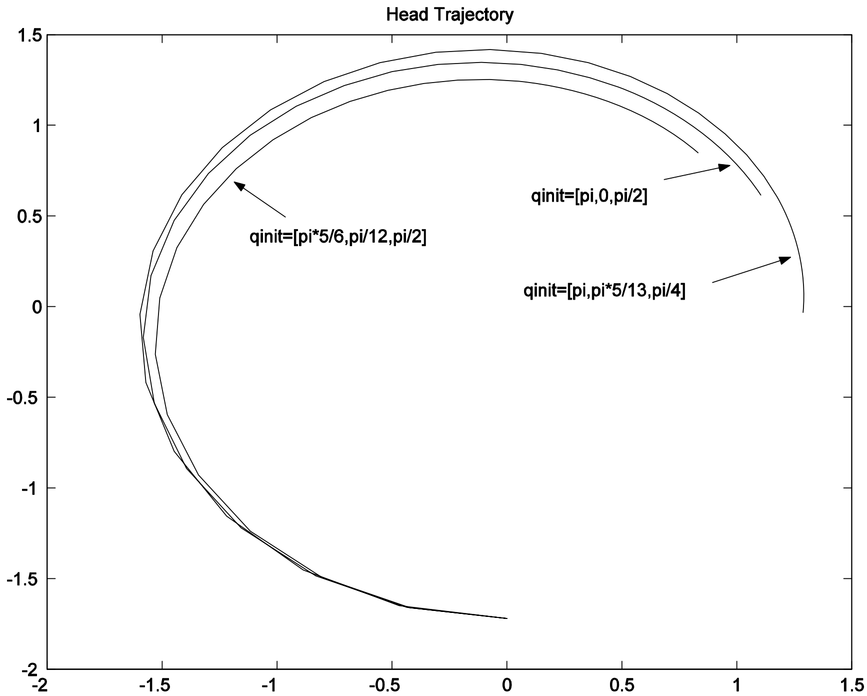

Figure 6. Different paths for different swings

As to the timing of the swing, the following figures show the differences between Experiment 1 and Experiment 5:

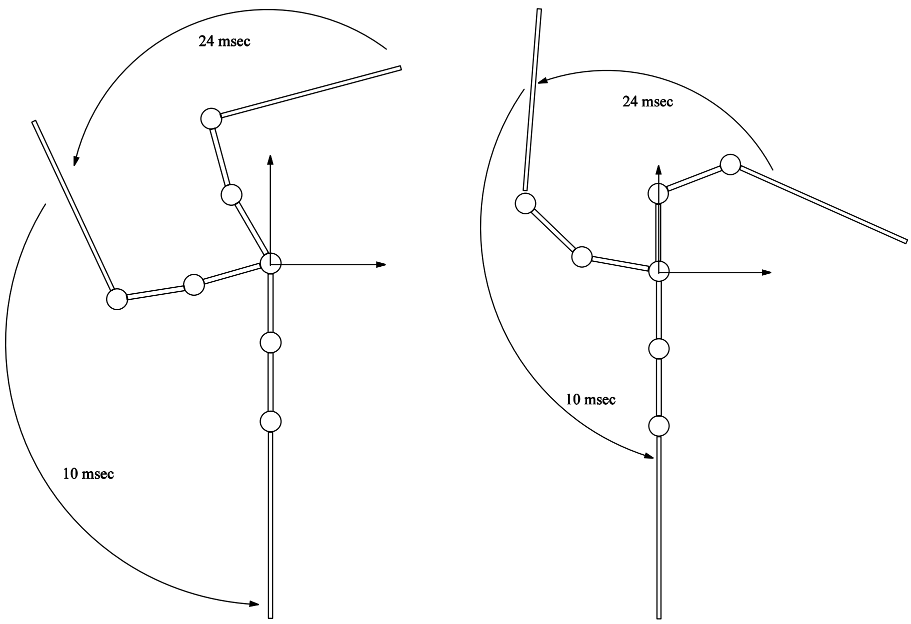

Figure 7. Timing for experiment 1 Figure 8. Timing for experiment 5

Summing up, we can say that either the joint trajectories and the initial postures have a measurable influence on the final result that is hitting the ball with the lowest effort and with the highest velocities of the club-head. From a mathematical point of view this is related with the 'equivalent moment of inertia' seen by the actuators and with the synchronization of the applied torques.

5. Conclusions

The paper has presented an analysis of the golf swing based on a triple pendulum model. Basically the approach relies on the analysis of robot motion with reference to simulated human tasks. For such problems, only boundary conditions on the trajectories are generally given whereas the actual time-behaviors are unknown and matter of optimization. A structure has been proposed with the advantage of not using fixed basis functions but functions whose shape in influenced by the parameters involved.

Moreover each parameter has an apparent effect on the shape so, for particular tasks it is possible to override safely some of them on the basis of qualitative considerations. After, the actual analysis of the swing has been carried out showing the effects of either the joints' trajectories and the initial posture.

6. References

[1] J.C. Latombe, et al. *Robot Motion Planing*, Boston: Kluwer Academic Publisher. MA. 1990.

[2] Y-C. Chen. Solving Robot Trajectory Planning Problems with Uniform Cubic B-Spline. *Optimal Control and Applications and Methods.* 1991, **12**: pp 247-262.

[3] B. Cao et al. Constrained time-efficient and smooth cubic spline trajectory generation for industrial robots, *IEE-Proceedings-Control Theory and Applications*. 1997, 144: 467-475.

[4] W. M. Pickering, Vickers. On the double pendulum model of the golf swing. *Sports Engineering.* Blackwell. , 1999, **2**: 165-172.

[5] J. T. Betts. Survey of Numerical Methods for Trajectory Optimization. *Journal of Guidance Control and Dynamics.* 1999, **21**: 193-207.

[6] J. E. Bobrow, et al. Optimal Robot Motions for Physical Criteria. *Journal of Robotic systems*. Wiley and Sons. **18**: 785-795.

[7] A. R. Penner. The physics of golf: The optimum loft of a driver. *American Journal of Physics.* 2001, 5: 563-568.

[8] E. J. Springings, Mackenzie. Examining the delayed release in the golf swing using computer simulation. *Sports Engineering.* Blackwell, 2002, **5**: 23-32.

[9] G. Bessonet, et al. Optimal motion synthesis-Dynamic modelling and Numerical Solving Aspects. *Multibody System Dynamics.* 2002, Kluwer, **8**: 257-278.

[10] A. R. Penner. The physics of golf. *Reports on Progress in Physics.* **66**: 131-171, 2003.

[11] S. M. Nesbit, Serrano. Work and power analysis of the golf swing. *Journal of Sports Science and Medicine.* www.jssm.org, vol. 4, 520-533, 2005.