Solving transportation problems with mixed constraints

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Abstract. In this paper we provide a heuristic algorithm for solving transportation problems with mixed constraints and extend the algorithm to find a more-for-less (MFL) solution, if one exists. Though many transportation problems in real life have mixed constraints, these problems are not addressed in the literature because of the rigor required to solve these problems optimally. The proposed algorithm builds on the initial solution of the transportation problem.

Keywords: transportation problem, mixed constraints, more-for-less paradox

1 Section heading

Much effort has been concentrated on transportation problems (TP) with equality constraints. In real life, however, most problems have mixed constraints accommodating many applications that go beyond transportation related problems to include job scheduling, production inventory, production distribution, allocation problems, and investment analysis. A literature search revealed no systematic method for finding an optimal solution or addressing more-for-less situations in transportation problems with mixed constraints.

The more-for-less (MFL) paradox in a TP occurs when it is possible to ship more total goods for less (or equal) total cost, while shipping the same amount or more from each origin and to each destination and keeping all the shipping costs non-negative. [4] discusses all possible situations in a TP with mixed constraints. Their approach is similar to [3] and is based on perturbation analysis. The coefficient of the perturbed parameters in the optimal parametric cost function is used to indicate the existence of a cheaper solution, by shipping more goods, than the current optimal one. Both papers obtain the MFL solution by relaxing the constraints and by introducing new slack variables. While yielding the best MFL solution, their method is tedious since it introduces more variables and requires solving sets of complex equations. Perturbed method was used extensively by [6], [5], and [7].

This paper introduces a heuristic method for solving the MFL paradoxical situation in a TP with mixed constraints. It uses the algorithm presented by [2] and [1] for a MFL solution in a TP with equality constraints. The algorithm is based on the theory of shadow prices. Both easy to understand and to apply, the method can serve as an effective tool for solving mixed constraints and paradoxical situations - while making the method accessible to managers.

2 Transportation problem with mixed constraints

In this section we keep the notations used by [3]. The mathematical model for the transportation problem with mixed constraints (MP) is as follows:

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$$\begin{aligned}
& \text{Minimize } \sum_I \sum_J c_{ij} x_{ij} \\
& \text{subject to} \\
& \sum_J x_{ij} = a_i \quad i \in \alpha_1 \\
& \sum_J x_{ij} \geq a_i \quad i \in \alpha_2 \\
& \sum_J x_{ij} \leq a_i \quad i \in \alpha_3 \quad (\text{MP}) \\
& \sum_I x_{ij} = b_j \quad j \in \beta_1 \\
& \sum_I x_{ij} \geq b_j \quad j \in \beta_2 \\
& \sum_I x_{ij} \leq b_j \quad j \in \beta_3 \\
& x_{ij} \geq 0 \quad (i, j) \in I * J
\end{aligned}$$

where $a_i > 0, \forall i \in I; b_j > 0, \forall j \in J$,

I = the index set of supply points = $\{1, 2, \dots, m\}$,

J = the index set of destinations = $\{1, 2, \dots, n\}$,

c_{ij} = the unit cost of transportation from the i th supply point to the j th destination.

Remark 1 (Gupta *et al.*^[4]): The transportation flow in every feasible solution of MP is at least $\max\{\sum a_i (i \in \alpha_1 \cup \alpha_2), \sum b_j (j \in \beta_1 \cup \beta_2)\}$. This observation is the basis for obtaining an initial solution of the algorithm presented in this paper.

[3] shows that the existence of a MFL situation in regular TP requires only one condition: the existence of a location with a negative plant-to-market shipping shadow price. These shadow prices, u_i and v_j , are calculated easily from the optimal solution matrix ([3]).

3 Proposed method

We will demonstrate that analyzing the shadow price (SP) matrix can yield good results for TP with mixed constraints, very often the optimal ones. We propose the following heuristic algorithm based on a procedure for MFL solution of a TP.

Step 1. Reformulate MP to problem LBP as follows:

$$\begin{aligned}
& \text{Minimize } \sum_I \sum_J c_{ij} x_{ij} \\
& \text{subject to} \\
& \sum_J x_{ij} = a_i \quad i \in \alpha_1 \\
& \sum_J x_{ij} = a_i \quad i \in \alpha_2 \\
& \sum_J x_{ij} \leq a_i \quad i \in \alpha_3 \quad (\text{LBP}) \\
& \sum_I x_{ij} = b_j \quad j \in \beta_1 \\
& \sum_I x_{ij} = b_j \quad j \in \beta_2 \\
& \sum_I x_{ij} \leq b_j \quad j \in \beta_3 \\
& x_{ij} \geq 0 \quad (i, j) \in I * J
\end{aligned}$$

Note that problem LBP is obtained from MP by changing all inequalities to equalities with the lowest possible feasible right-hand side values.

Step 2. Solve LBP with $a_i = 0$ for $i \in \alpha_3$ and $b_j = 0$ for $j \in \beta_3$ using the Vogel Approximation Method (VAM) to assign $\min\{\sum a_i (i \in \alpha_1 \cup \alpha_2), \sum b_j (j \in \beta_1 \cup \beta_2)\}$ loads to cells.

Step 3. Place the remaining $|\sum a_i - \sum b_j|$ load at the lowest cost feasible cells to obtain a feasible solution for the LBP.

Step 4. Create the SP matrix using the solution obtained in Step 3.

Step 5. Identify negative SPs and related columns and rows. If none exist, this is an optimal solution to MP (no MFL paradox is present). STOP.

Step 6. Pick out rows/columns with the most entries of negative SPs.

Step 7. Pick out rows/columns with a single loaded cell among those identified in Step 6.

Step 8. Pick out the cell (k, r) with the lowest cost coefficient among those identified in Step 7.

Step 9. Assign $X_{kr} = \min(a_k, b_r)$ if $r \in \beta_3$ and $b_r < a_k$, or if $k \in \alpha_3$ and $a_k < b_r$;
Otherwise $X_{kr} = \max(a_k, b_r)$.

Step 10. Delete the kth row and the rth column from the cost matrix and the related SP matrix.

Step 11. If the reduced SP matrix contains any negative entry, go to Step 6.

Step 12. Solve the reduced problem as a regular unbalanced TP to satisfy the remaining constraints.

4 Numerical examples

We explain the proposed method using examples from [4] to provide a point of common comparison. These TPs are small in size; hence many steps of the proposed method are not needed.

Example 1

Table 1. Cost matrix of Example 1

	b_1	b_2	b_3	Supply
a_1	10	1	4	= 5
a_2	5	7	1	≥ 6
a_3	8	9	2	≤ 9
Demand	= 8	≥ 10	≤ 5	

Step 1. Consider the TP

	b_1	b_2	b_3	Supply
a_1	10	1	4	= 5
a_2	5	7	1	≥ 6
a_3	8	9	2	$\leq 9 = 0$
Demand	= 8	= 10	$\leq 5 = 0$	

Step 2. Using VAM assign $X_{12} = 5, X_{21} = 6$.

Step 3. The solution to the unbalanced LBP is $X_{12} = 5, X_{21} = 8, X_{22} = 5$, and all other $X_{ij} = 0$, for a flow of 18 with a cost of 80.

Step 4. The SP matrix of solution obtained in Step 3 is given below.

Table 2. SP matrix for Example 1

	v_1	v_2	v_3	u_i
u_1	-1	1*	-5	-6
u_2	5*	7*	1**	0
u_3	6	8	2**	1
v_j	5	7	1	

where the cells loaded in the solution are marked by * and the cell assumed to be loaded (in order to obtain the remaining SP values) are marked by **.

Step 5-6. Row 1 has two negative shadow prices.

Step 7-8. Row 1 has single loading cell (1, 2).

Step 9. Assign $X_{12} = \max(a_1, b_2) = 10$.

Step 10. Delete row 1 and column 2 from Tab. 1 and Tab. 2.

Step 11-12. The reduced SP matrix has no negative SP. Solving the reduced 2×2 TP to satisfy the remaining constraints, we obtain an MFL solution as $X_{12} = 10$, $X_{21} = 8$, and all other $X_{ij} = 0$, for a flow of 18 with a cost of 50. This solution matches the solution obtained by [4].

Example 2

Table 3. Cost matrix of Example 2

	b_1	b_2	b_3	Supply
a_1	2	5	4	= 5
a_2	6	3	1	≥ 6
a_3	8	9	2	≤ 9
Demand	= 8	≥ 10	≤ 5	

Step 1. Consider the TP

	b_1	b_2	b_3	Supply
a_1	2	5	4	= 5
a_2	6	3	1	≥ 6
a_3	8	9	2	$\leq 9 = 0$
Demand	= 8	= 10	$\leq 5 = 0$	

Step 2. Using VAM assign $X_{11} = 5$, $X_{22} = 6$.

Step 3. The solution to the unbalanced LBP is $X_{11} = 5$, $X_{21} = 3$, $X_{22} = 10$, and all other $X_{ij} = 0$, for a flow of 18 units with a cost of 58.

Step 4. The SP matrix of solution obtained in Step 3 is given below.

Table 4. SP matrix for Example 2

	v_1	v_2	v_3	u_i
u_1	2*	-1	-3	-4
u_2	6*	3*	1**	0
u_3	7	4	2**	1
v_j	6	3	1	

Step 5-6. Row 1 has two negative shadow prices.

Step 7-9. Row 1 has a single loading cell (1, 1). Assign $X_{11} = \max(a_1, b_1) = 8$.

Step 10. Delete row 1 and column 1 from the cost matrix and the SP matrix.

Step 11-12. The reduced SP matrix has no negative SP. Solving the reduced 2×2 TP to satisfy the remaining constraints, we obtain an MFL solution of $X_{11} = 8$, $X_{22} = 10$, and all other $X_{ij} = 0$ with the same flow at a lower cost of 46. This solution is the same as obtained by [4].

Example 3

Table 5. Cost matrix of Example 3

	b_1	b_2	b_3	Supply
a_1	3	5	5	≤ 3
a_2	5	9	4	= 15
a_3	4	2	4	≤ 10
Demand	= 6	= 10	= 5	

Step 1. Consider the TP

	b_1	b_2	b_3	Supply
a_1	3	5	5	$\leq 3 = 0$
a_2	5	9	4	$= 15$
a_3	4	2	4	$\leq 10 = 0$
Demand	$= 6$	$= 10$	$= 5$	

Step 2. Using VAM assign $X_{21} = 6$, $X_{22} = 4$, and $X_{23} = 5$.

Step 3. The solution to the unbalanced LBP is $X_{21} = 6$, $X_{22} = 4$, $X_{23} = 5$, and $X_{32} = 6$ for a flow of 21 units with a cost of 98.

Step 4. The SP matrix of solution obtained in Step 3 is given below.

Table 6. SP matrix for Example 3

	v_1	v_2	v_3	u_i
u_1	1	5^{**}	0	-4
u_2	5^*	9^*	4^*	0
u_3	-2	2^*	-3	-7
v_j	5	9	4	

Step 5-6. Row 3 has two negative shadow prices.

Step 7-8. Row 3 has a single loading cell (3, 2) where $3 \in \alpha_3$ and $2 \in \beta_1$ with $a_3 = b_2 = 10$.

Step 9. Assign $X_{32} = 10$.

Step 10. Delete row 3 and column 2 from the cost matrix and the SP matrix.

Step 11-12. The reduced SP matrix has no negative SP. Solving the reduced 2×2 TP to satisfy the remaining constraints, we obtain an MFL solution as $X_{21} = 6$, $X_{23} = 9$, and $X_{32} = 10$, with an increased flow of 25 at a lower cost of 86. The solution matches the solution obtained by [4]

Note 1: In Step 4 of the proposed method we need to calculate the SP matrix for the LBP solution obtained in Step 3. To do so, we need $n + m - 1$ cost values as initial SP values for calculating the remaining SP values. By design, the LBP would often have less than $n + m - 1$ loaded cells. We recommend that in Step 2 to keep track of cells that would be loaded using the VAM method even with a load of 0. One can also assume any other appropriate cells to be loaded to solve for SP values for a degenerate solution of LBP.

Note 2: The MFL phenomenon is based on relaxing the constraints for a given TP. Since $\sum \sum x_{ij}$ in the MFL is greater than corresponding sum in nominal TP due to the constraints relaxation, the gain from moving the load from cells with a bigger cost coefficient c_{ij} to cells with smaller cost coefficients must offset the extra cost from the additional load. Usually, the MFL solution is decomposable with fewer loaded cells. We refer readers to [2] for more details on a resolution of the MFL situation.

5 Conclusion

We have provided a heuristic algorithm to find a solution for transportation problems with mixed constraints where more-for-less paradox exists. The proposed method solves the problem for a lower bound (which in some cases is the optimal solution) and provides a simple step-by-step approach to finding an MFL solution, if one exists. The occurrence of MFL in transportation problems is not a rare event, often vital to Decision Maker and existing literature has demonstrated the practicality of identifying cases where the paradoxical situation exists. Easy to understand and apply, the proposed algorithm can serve managers by providing the solution to a variety of distribution problems with mixed constraints.

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