ISSN 1750-9653, England, UK
International Journal of Management Science and Engineering Management
Vol. 4 (2009) No. 4, pp. 294-301

New exact solutions for the generalized Kawahara and modified Kawahara equation using the modified extended direct algebraic method

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(*Received February 26 2009, Revised May 4 2009, Accepted August 13 2009*)

Abstract. In this paper, we consider the nonlinear equations such as the generalized Kawahara equation and Modified Kawahara equation. By applied the modified extended direct algebraic (MEDA) method, the traveling wave solutions for these equations are presented. New exact traveling solutions are explicitly obtained with the help of symbolic computation, provides a very effective and powerful (mathematical tools) for discrete nonlinear evolution equations in mathematical physics. The obtained solutions include compactons, solutions, solitary patterns and periodic solutions.

Keywords: travelling wave solutions, MEDA, generalized Kawahara equation, Modified Kawahara equation

1 Introduction

In recent years, nonlinear evolution equations[1] in mathematical phisics play a major role in various fileds of sciences, especially in fluid mechanics, solid state physics, plasma physics, plasma wave and chemical physics. Nonlinear wave phenomena of dispersion, dissipation, diffusion, reaction and convection are very important in nonlinear wave equations. Concepts like solutions, compactions, breathers, cusps, kinks and peakons have now been thoroughly investigated in the scientific literature [18–20]. Several direct methods for obtaining nonlinear evolution equations have been recently proposed, such as the homogeneous balance method[9], the modified tanh-function method[6], the sine-cosine method[23], the extendnd tanh-function method[25], the Jacobi elliptic function expansion method[12], the tanh-function method[14] and the auxiliary equation method[14, 16], and so on. have been proposed for obtaining exact and approximate analytic solutions. By using the solutions of an auxilary ordinary differential equation, a direct algebraic method is described to construct the exact traveling wave solutions for nonlinear evolution equations. By this method the Kawahara and modified Kawahara equations are investigated[14]. Some doubly periodic (Jacobi elliptic function) solutions of the modified Kawahara equations are presented in closed form [26]. Existence and uniqueness of solutions to nonlinear Kawahara equations are obtained in [13]. The Kawahara equation (KE) and the modified Kawahara equation (mKE) have been the subject of extensive research work in recent decades in [2, 4, 7, 9, 11, 14, 21, 24, 25, 29]. Recently, the direct algebraic method has been suggested to obtain the exact complex solutions of nonlinear partial differential equations[5, 27, 28]. The aim of this paper is to extend the modified extended direct algebraic method to solve two different types of nonlinear differential equations such as the generalized Kawahara equation (gKE) and modified Kawahara equation (mKE), which is been presented first in this paper.

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2 Analysis of the MEDA

We now describe the MEDA method for given partial differential equations. The motivation of the algebraic method will be described starting from the description of the tanh method[10]. We first consider a general form of PDE

$$F(u, u_x, u_t, u_{xx}, \cdots) = 0, \tag{1}$$

Let us consider its complex solutions $u(x,\, t) = U(z)$, $z = i(x - ct)$ or $z = i(x + ct), i = \sqrt{-1}$, then Eq. (1) becomes an ordinary differential equation

$$N(u, iu', -icu', -u', \cdots) = 0, \tag{2}$$

In order to seek the solutions of Eq. (1) under the above transformation, we assume that the solution of Eq. (2) is of the following form

$$u(z) = a_0 + \sum_{j=1}^{M} (a_j \Phi^j + b_j \Phi^{-j}), \tag{3}$$

$$\Phi'(z) = b + \Phi(z)^2 \tag{4}$$

where $b_j, a_j (j = 1, 2, \cdots, M), b$ and a_0 are a parameter to be determined later, $\Phi = \Phi(z)$, $\Phi(z) = \frac{\mathrm{d}\Phi}{\mathrm{d}z}$. The parameter M can be found by balancing the highest - order derivative term with the nonlinear terms. Substituting (3) into (2) with (4) will yield a system of algebraic equations with respect to a_j, b_j, b and c (where $j = 1, 2, \cdots, M$) because all the coefficients of Φ^j have to vanish. we can then determine a_0, a_j, b_j, b and c. Eq. (4) has the general solutions: (I) If $b < 0$ then $\Phi = -\sqrt{-b}\tanh(\sqrt{-b}z)$, or $\Phi = -\sqrt{-b}\coth(\sqrt{-b}z)$, it depends on initial conditions. (II) If $b > 0$ then $\Phi = \sqrt{b}\tanh(\sqrt{b}z)$, or $\Phi = \sqrt{b}\coth(\sqrt{b}z)$, it depends on initial conditions. (III) If $b = 0$ then

$$\Phi = -\frac{1}{z} \tag{5}$$

solving the algebraic equations and substituting the results into Eq. (3), then we obtain the exact traveling wave solutions of Eq. (1).

3 Exact traveling wave solutions for nonlinear Kawahara equation

In this section, we present new exact solutions for the generalized Kawahara and modified Kawahara nonlinear equations by using the modified extended direct algebraic method.

3.1 Example[1]

Let us consider the generalized Kawahara equation which has the form [3]:

$$u_t + \alpha u^n u_x + p u_{xxx} + q u_{xxxxx} = 0 \tag{6}$$

where α, p and q are some arbitrary constants and $n > 0$, we use the wave transformation $u(x, t) = U(z)$ with wave complex variable $z = i(x - ct)$, Eq. (1) takes the form of an ordinary differential equation as

$$-ciU - ipU^{(3)} - iqU^{(5)} = 0 \tag{7}$$

Integrating Eq. (7) once with respect to z and setting the constant of integration to be zero, we obtain

$$-ciU + \alpha \frac{i}{n+1} U^{n+1} - ipU^{(2)} - iqU^{(4)} = 0 \tag{8}$$

Balancing the order of U^{n+1}, with the order of $U^{(4)}$ in Eq. (8), we find $M = \frac{4}{n}$. To get a closed form analytic solution, the parameter M should be an integer. A transformation formula $U = v^{\frac{1}{n}}$ should be used to obtain this analytic solution[17]. So Eq. (8) takes the form

$$-cin^4v^4 + \alpha\frac{i}{\alpha+1}n^4v^5 - ip\left(\left(n^2-n^3\right)v^2(v)^2 + n^3v^3v''\right) + iq\left\{\left(11n^2-6n^3-6n+1\right)(v)^4 \right.$$
$$\left. +2(3-3n)(-2n+1)nv\left(v-(3n+3)\,n^2v^2\left(v''\right)\right)^2 + \left(-4n^3+4n^2\right)v^2v'v'' + n^3v^3v^{(4)}\right\} = 0 \quad (9)$$

Balancing the order of v^5 with the order of $v^3v^{(4)}$ in Eq. (9), gives $M = 4$. So the solution takes the form

$$v(z) = a_0 + a_1\Phi(z) + a_2\Phi(z)^2 + a_3\Phi(z)^3 + a_4\Phi(z)^4 + b_1\Phi(z)^{-1} + b_2\Phi(z)^{-2} + b_3\Phi(z)^{-3} + b_4\Phi(z)^{-4} \quad (10)$$

Inserting Eq. (10) into Eq. (9) and making use Eq. (4), using the Maple Package, we get a system of algebraic equations, for $a_0, a_1, a_2, a_3, a_4, b_1, b_2, b_3, b_4, b$ and q. We solve the obtained system of algebraic equations give the following three cases:

Case 1.

$$a_1 = a_2 = a_3 = a_4 = 0, a_0 = a_0, p = p, b = b, b_1 = b_1, b_2 = b_2, b_3 = b_3, b_4 = b_4, q = q, c = c \quad (11)$$

Substituting Eq. (11) in Eq. (9), we obtain

$$v(z) = a_0 - I\sqrt{-b}\eta\left[\frac{b_1}{b\eta^2} - \frac{Ib_2}{(-b)^{\frac{3}{2}}\eta^3} + \frac{b_3}{b^2\eta^4} + \frac{Ib_4}{(-b)^{\frac{5}{2}}\eta^5}\right] \quad (12)$$

So, the travelling wave solution is given by

$$U(x,t) = \left\{a_0 - I\sqrt{-b}\eta\left[\frac{b_1}{b\eta^2} - \frac{Ib_2}{(-b)^{\frac{3}{2}}\eta^3} + \frac{b_3}{b^2\eta^4} + \frac{Ib_4}{(-b)^{\frac{5}{2}}\eta^5}\right]\right\}^{\frac{1}{n}} \quad (13)$$

where $\eta = \tan(\sqrt{-b}(x+ct))$.

Case 2.

$$v(z) = a_0 - I\sqrt{-b}\eta\left[\frac{b_1}{b\eta^2} - \frac{Ib_2}{(-b)^{\frac{3}{2}}\eta^3} + \frac{b_3}{b^2\eta^4} + \frac{Ib_4}{(-b)^{\frac{5}{2}}\eta^5}\right]; b_2 = b_2, b_4 = b_4, q = 0, c = c \quad (14)$$

Substituting Eq. (14) in Eq. (9), we obtain

$$v(z) = a_0 - I\sqrt{-b}\eta\left[-\frac{2Ip(n\alpha+n+2+2\alpha)\sqrt{-b}\eta}{n^2\alpha} - \frac{Ib_2}{(-b)^{\frac{3}{2}}\eta^3} + \frac{Ib_4}{(-b)^{\frac{5}{2}}\eta^5}\right] \quad (15)$$

So, the travelling wave solution is given by

$$U(x,t) = \left\{a_0 - I\sqrt{-b}\eta\left[-\frac{2Ip(n\alpha+n+2+2\alpha)\sqrt{-b}\eta}{n^2\alpha} - \frac{Ib_2}{(-b)^{\frac{3}{2}}\eta^3} + \frac{Ib_4}{(-b)^{\frac{5}{2}}\eta^5}\right]\right\}^{\frac{1}{n}} \quad (16)$$

where $\eta = \tan(\sqrt{-b}(x+ct))$.

Case 3.

$$a_1 = a_2 = a_3 = a_4 = q = 0, a_0 = a_0, a_2 = \frac{2p(n\alpha + n + 2 + 2\alpha)}{n^2\alpha}, p = p,$$
$$b = b, b_1 = b_2 = 0, b_4 = b_4, c = \frac{6a_0n^2\alpha - 8pbn\alpha - 16pb\alpha - 16pb - 8pbn}{n^2(n\alpha + n + 2 + 2\alpha)} \tag{17}$$

Substituting Eq. (17) in Eq. (9), we obtain

$$v(z) = a_0 - I\sqrt{-b}\eta\left[-\frac{Ib_2}{(-b)^{\frac{3}{2}}\eta^3} + \frac{Ib_4}{(-b)^{\frac{5}{2}}\eta^5}\right] \tag{18}$$

So, the travelling wave solution is given by

$$U(x,t) = \left\{a_0 - I\sqrt{-b}\eta\left[-\frac{Ib_2}{(-b)^{\frac{3}{2}}\eta^3} + \frac{Ib_4}{(-b)^{\frac{5}{2}}\eta^5}\right]\right\}^{\frac{1}{n}} \tag{19}$$

where $\eta = \tan(\sqrt{-b}(x + \frac{6a_0n^2\alpha - 8pbn\alpha - 16pb\alpha - 16pb - 8pbn}{n^2(n\alpha + n + 2 + 2\alpha)}t))$.

3.2 Example[2]

Let us consider the following modified Kawahara equation[8]:

$$u_t + \alpha u^2 u_x + pu_{xxx} + qu_{xxxxx} = 0 \tag{20}$$

where α, p and q are some arbitrary constants. Eq. (20) was proposed first by Kawahara[9] as an important dispersive equation. This equation is also called singularly perturbed KdV equation. The Eq. (20) is a model equation for shallow water waves. Note that the exact solution of Eq. (20) when $\alpha = 1$is given by homotopy perturbation method as form[14]

$$\begin{aligned} u(x,t) = &\frac{3p}{\sqrt{-10q}}\text{sech}^2[kx] - \frac{27p^3kt}{5q\sqrt{-10q}}\text{sech}^6[kx]\tanh[kx] \\ &- \frac{24p^2k^3t}{\sqrt{-10q}}\text{sech}^2[kx]\tanh[kx](2 - 3\tanh^2[kx]) \\ &- \frac{pqk^5t}{\sqrt{-10q}}\text{sech}^2[kx]\tanh[kx](-2160\tanh^4[kx] + 2880\tanh^2[kx] - 816) \end{aligned} \tag{21}$$

making the transformation $u(x,t) = U(z)$, $z = i(x - ct)$, Eq. (20) takes the form of an ordinary differential equation as

$$-ciU - ipU^{(3)} + iqU^{(5)} = 0 \tag{22}$$

and integrating once with respect to z, Eq. (22) becomes

$$-ciU + \alpha\frac{i}{3}U^3 - ipU^{(2)} + iqU^{(4)} = 0 \tag{23}$$

balancing the highest order derivative term $U^{(4)}$ with the highest power nonlinear term U^3, yields $M = 2$. Therefore, can write the solution of Eq. (22) in the form

$$U(z) = a_0 + a_1\Phi(z) + a_2\Phi(z)^2 + b_1\Phi(z)^{-1} + b_2\Phi(z)^{-2} \tag{24}$$

Inserting Eq. (24) into Eq. (23) and making use Eq. (4), we have a system of algebraic equations, for a_0, a_1, a_2, b_1 and b_2. Solving the obtain system with the Maple Package, we have the follwing three cases of solutions.

Case 4.

$$U(z) = a_0 + a_1\Phi(z) + a_2\Phi(z)^2 + b_1\Phi(z)^{-1} + b_2\Phi(z)^{-2}; b = b, p = p, q = q, c = \frac{\alpha a_0^2}{3} \tag{25}$$

Substituting Eq. (25) in Eq. (24), the traveling wave solution is given by

$$U(x,t) = a_0 + b\tan\left(\sqrt{-b}\left(x + \frac{1}{3}\alpha a_0^2 t\right)\right)^2 a_2 \tag{26}$$

Case 5.

$$\begin{aligned} &a_1 = b_1 = b_2 = 0, a_0 = \frac{18p + 2\alpha a_2^2 b}{3\alpha a_2}, b = b, p = p, q = -\frac{\alpha a_2^2}{360}, a_2 = a_2, \\ &c = \frac{135 p a_2^4 b^2 \alpha^2 + 11\alpha^3 a_2^6 b^3 + 14580 p^3 + 4860 p^2 \alpha a_2^2 b}{135[\alpha a_2^2(9p + \alpha a_2^2 b)]} \end{aligned} \tag{27}$$

Substituting Eq. (27) in Eq. (24), the traveling wave solution is given by

$$U(x,t) = \left\{a_0 - I\sqrt{-b}\eta\left[-\frac{2Ip(n\alpha + n + 2 + 2\alpha)\sqrt{-b}\eta}{n^2\alpha} - \frac{Ib_2}{(-b)^{\frac{3}{2}}\eta^3} + \frac{Ib_4}{(-b)^{\frac{5}{2}}\eta^5}\right]\right\}^{\frac{1}{n}} \tag{28}$$

Case 6.

$$a_1 = b_1 = 0, a_2 = a_2, a_0 = \frac{18p + \alpha a_2^3 b}{3\alpha a_2}, b = \frac{9p}{2\alpha a_2^2}, p = p, b_2 = \frac{81p^2}{4\alpha^2 a_2^3}, q = -\frac{\alpha a_2^2}{360}, c = -\frac{288p^2}{5\alpha a_2^2} \tag{29}$$

Substituting Eq. (29) in Eq. (24), the traveling wave solution is given by

$$\begin{aligned} U(x,t) = \frac{9p}{\alpha a_2} - \frac{3}{2}I\eta\tan\left(\frac{3}{2}\eta\left(x + \frac{288p^2}{5\alpha a_2^2}t\right)\right)\Bigg[-\frac{3}{2}I\eta a_2\tan\left(\frac{3}{2}\eta\left(x + \frac{288p^2}{5\alpha a_2^2}t\right)\right) \\ - \frac{6Ip^2}{\alpha^2 a_2^3\left(-\frac{2p}{\alpha a_2^2}\right)^{\frac{3}{2}}\tan\left(\frac{3}{2}\eta\left(x + \frac{288p^2}{5\alpha a_2^2}t\right)\right)^3}\Bigg] \end{aligned} \tag{30}$$

where $\eta = \sqrt{-2\frac{p}{\alpha a_2^2}}$.

4 Numerical results for the modified Kawahara equation

In this section, we consider the modified Kawahara equation for numerical comparisons. Based on the modified extended direct algebraic method, we construct the solution $U(x,t)$ as Eq. (26) and Eq. (28) with $b = -0.0001, \alpha = 1, p = 0.001$ and $q = -1$. Comparison results are obtained by modified extended direct algebraic method and the exact solution of the Eq. (20) show the difference between the exact solutions Eq. (26) and the exact solution Eq. (21).

From these two numerical examples, we can conclude from the numerical results that the method provide high accuracy for the modified kawahara equation. There for, both methods can be seen as efficient methods for solving the modified kawahara equation.

5 Conclusions

We have proposed an approach for finding exact travelling wave solutions for nonlinear evolution equations by using the modified extended direct algebraic method. In this work, by this method and computerized

Table 1. The result for the exact solutions Eq. (26) obtained by modified extended direct algebraic method in comparison with the exact solutions Eq. (21)

x	t	MEDM	Exact solution	|MEDM-Exact solution|
-5.0	0.02	1.99999×10^{-4}	9.47498×10^{-4}	7.475×10^{-4}
	0.04	1.99999×10^{-4}	9.47498×10^{-4}	7.475×10^{-4}
	0.06	1.99999×10^{-4}	9.47498×10^{-4}	7.475×10^{-4}
	0.08	1.99999×10^{-4}	9.47498×10^{-4}	7.475×10^{-4}
	0.1	1.99999×10^{-4}	9.47498×10^{-4}	7.475×10^{-4}
-2.5	0.02	2×10^{-4}	9.48387×10^{-4}	7.48387×10^{-4}
	0.04	2×10^{-4}	9.48387×10^{-4}	7.48387×10^{-4}
	0.06	2×10^{-4}	9.48387×10^{-4}	7.48387×10^{-4}
	0.08	2×10^{-4}	9.48387×10^{-4}	7.48387×10^{-4}
	0.1	2×10^{-4}	9.48387×10^{-4}	7.48387×10^{-4}
0.0	0.02	2×10^{-4}	9.48683×10^{-4}	7.48683×10^{-4}
	0.04	2×10^{-4}	9.48683×10^{-4}	7.48683×10^{-4}
	0.06	2×10^{-4}	9.48683×10^{-4}	7.48683×10^{-4}
	0.08	2×10^{-4}	9.48683×10^{-4}	7.48683×10^{-4}
	0.1	2×10^{-4}	9.48683×10^{-4}	7.48683×10^{-4}
2.5	0.02	2×10^{-4}	9.48387×10^{-4}	7.48387×10^{-4}
	0.04	2×10^{-4}	9.48387×10^{-4}	7.48387×10^{-4}
	0.06	2×10^{-4}	9.48387×10^{-4}	7.48387×10^{-4}
	0.08	2×10^{-4}	9.48387×10^{-4}	7.48387×10^{-4}
	0.1	2×10^{-4}	9.48387×10^{-4}	7.48387×10^{-4}
5.0	0.02	1.99999×10^{-4}	9.47498×10^{-4}	7.475×10^{-4}
	0.04	1.99999×10^{-4}	9.47498×10^{-4}	7.475×10^{-4}
	0.06	1.99999×10^{-4}	9.47498×10^{-4}	7.475×10^{-4}
	0.08	1.99999×10^{-4}	9.47498×10^{-4}	7.475×10^{-4}
	0.1	1.99999×10^{-4}	9.47498×10^{-4}	7.475×10^{-4}
7.5	0.02	1.99997×10^{-4}	9.4602×10^{-4}	7.46023×10^{-4}
	0.04	1.99997×10^{-4}	9.4602×10^{-4}	7.46023×10^{-4}
	0.06	1.99997×10^{-4}	9.4602×10^{-4}	7.46023×10^{-4}
	0.08	1.99997×10^{-4}	9.4602×10^{-4}	7.46023×10^{-4}
	0.1	1.99997×10^{-4}	9.4602×10^{-4}	7.46023×10^{-4}

symbolic computation, we have found some new types of exact travelling wave solutions for the generalized Kawahara and modified Kawahara equations. This solutions may be important of significance for the explanation of some practical physical problems. More importantly, our method is actually applicable to fined new solutions to various kinds of nonlinear evolution equations. The results show that the modified extended direct algebraic method is a powerful mathematical tool and has been successfully applied to solve the generalized Kawahara and modified Kawahara equations. That we could obtain more kinds of exact traveling wave solutions, if we choose various auxiliary ordinary differential equations in our method.

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Table 2. The result for the exact solutions Eq. (28) obtained by modified extended direct algebraic method in comparison with the exact solutions Eq. (21)

x	t	MEDM	Exact solution	| MEDM-Exact solution |
	0.02	-9.53435×10^{-4}	9.47498×10^{-4}	1.90093×10^{-3}
	0.04	-9.53435×10^{-4}	9.47498×10^{-4}	1.90093×10^{-3}
-5.0	0.06	-9.53435×10^{-4}	9.47498×10^{-4}	1.90093×10^{-3}
	0.08	-9.53435×10^{-4}	9.47498×10^{-4}	1.90093×10^{-3}
	0.1	-9.53435×10^{-4}	9.47498×10^{-4}	1.90093×10^{-3}
	0.02	-9.4987×10^{-4}	9.48387×10^{-4}	1.89826×10^{-3}
	0.04	-9.4987×10^{-4}	9.48387×10^{-4}	1.89826×10^{-3}
-2.5	0.06	-9.4987×10^{-4}	9.48387×10^{-4}	1.89826×10^{-3}
	0.08	-9.4987×10^{-4}	9.48387×10^{-4}	1.89826×10^{-3}
	0.1	-9.4987×10^{-4}	9.48387×10^{-4}	1.89826×10^{-3}
	0.02	-9.48683×10^{-4}	9.48683×10^{-4}	1.89737×10^{-3}
	0.04	-9.48683×10^{-4}	9.48683×10^{-4}	1.89737×10^{-3}
0.0	0.06	-9.48683×10^{-4}	9.48683×10^{-4}	1.89737×10^{-3}
	0.08	-9.48683×10^{-4}	9.48683×10^{-4}	1.89737×10^{-3}
	0.1	-9.48683×10^{-4}	9.48683×10^{-4}	1.89737×10^{-3}
	0.02	-9.4987×10^{-4}	9.48387×10^{-4}	1.89826×10^{-3}
	0.04	-9.4987×10^{-4}	9.48387×10^{-4}	1.89826×10^{-3}
2.5	0.06	-9.4987×10^{-4}	9.48387×10^{-4}	1.89826×10^{-3}
	0.08	-9.4987×10^{-4}	9.48387×10^{-4}	1.89826×10^{-3}
	0.1	-9.4987×10^{-4}	9.48387×10^{-4}	1.89826×10^{-3}
	0.02	-9.53435×10^{-4}	9.47498×10^{-4}	1.90093×10^{-3}
	0.04	-9.53435×10^{-4}	9.47498×10^{-4}	1.90093×10^{-3}
5.0	0.06	-9.53435×10^{-4}	9.47498×10^{-4}	1.90093×10^{-3}
	0.08	-9.53435×10^{-4}	9.47498×10^{-4}	1.90093×10^{-3}
	0.1	-9.53435×10^{-4}	9.47498×10^{-4}	1.90093×10^{-3}
	0.02	-9.59396×10^{-4}	9.4602×10^{-4}	1.90542×10^{-3}
	0.04	-9.59396×10^{-4}	9.4602×10^{-4}	1.90542×10^{-3}
	0.06	-9.59396×10^{-4}	9.4602×10^{-4}	1.90542×10^{-3}
	0.08	-9.59396×10^{-4}	9.4602×10^{-4}	1.90542×10^{-3}
	0.1	-9.59396×10^{-4}	9.4602×10^{-4}	1.90542×10^{-3}

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