

Stationary analysis for the bulk input Geom^[X]/Geom/1 queue with working vacation*

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Abstract. In this paper, we analyze a bulk input Geom^[X]/Geom/1 queue with single working vacation. Using the matrix analysis method, highly complicated PGF of the stationary queue size is firstly derived, from which we got the stochastic decomposition result for the PGF of the stationary queue size. It is important that we find the biparameter addition theorems of the conditional negative binomial distribution, with which we find the upper bound and the lower bound of the stationary waiting time in the moment generating function order. Furthermore, we gain the mean queue size and the upper bound and the lower bound of the mean waiting time. Finally, several numerical examples are presented.

Keywords: Geom^[X]/Geom/1 queue, bulk input, single working vacation, stochastic decomposition, the moment generating function order

1 Introduction

During the last two decades, queueing systems with server vacations have been well investigated and have extensively applied to various fields, such as computer systems, communication networks, production managing. In a vacation queue, the server completely stops service and may do some additional work during a vacation. The readers are referred to the surveys of Doshi^[2], the monographs of Takagi^[9] and Tian and Zhang^[10].

Lately, Servi and Finn^[6] introduced a class of semi-vacation policies: a server does not completely stop service but serves customers at a lower rate during a vacation. Such a vacation is called a working vacation(WV). The work of [6] rooted in performance analysis of gateway router in fiber communication networks, the authors studied an M/M/1 queue with working vacation, and gave distributions for the stationary queue length and waiting time of a customer. Subsequently, Kim^[3], Wu and Takagi^[11] generalized study of the model to an M/G/1 queue with working vacation. Baba^[12] discussed a GI/M/1 queue with working vacation. Banik et al.^[1] analyzed the GI/M/1/N queue with finite buffer and working vacations. Recently, Li and Tian^[5] studied the discrete-time GI/Geo/1 queue with working vacations and vacation interruption.

However, the bulk input working vacation queues are not involved in the existing literatures. On the other hand, the cells arrive in the computer networks and communication systems in batch, thus bulk input queue models have more extensive applications. Hence, we extend the studies in Li and Tian^[4] to bulk input model and consider here the Geom^[X]/Geom/1 queue system with single working vacation and it will be fitter for modeling analysis of the gateway router.

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2 Model description

Thereinafter, for any real number $x \in (0, 1)$, we denote $\bar{x} = 1 - x$. The $\text{Geom}^{[X]}/\text{Geom}/1/\text{WV}$ queue we studied here is as follows:

(1) Assume that potential customer arrivals occur at the end of slot $t = n^-, n = 0, 1, \dots$. Interarrival time is an independent and identically distributed random sequences and follow a geometric distribution

$$P\{T = k\} = p\bar{p}^{k-1}, \quad k \geq 1, \quad 0 < p < 1.$$

Furthermore, the actual number of customers in any arriving module is a random variable X , which may take on any positive integral value i less than ∞ with probability c_i , that is, $P\{X = i\} = c_i, i = 1, 2, \dots$. We assume that X has the probability generating function, expectation and second order moment respectively are $C(z), E(X), E(X^2)$, and

$$C(z) = \sum_{i=1}^{\infty} z^i c_i, \quad |z| \leq 1, \quad E(X) = \sum_{i=1}^{\infty} i c_i, \quad E(X^2) = \sum_{i=1}^{\infty} i^2 c_i.$$

(2) The beginning and ending of potential service occur at slot division point $t = n, n = 0, 1, \dots$. The distribution of service time S_b in a regular busy period is

$$P\{S_b = k\} = \mu_b \bar{\mu}_b^{k-1}, \quad k \geq 1, \quad 0 < \mu_b < 1.$$

and its PGF and second order moment respectively are

$$B(z) = \frac{\mu_b z}{1 - \bar{\mu}_b z}, \quad E(S_b^2) = \frac{2 - \mu_b}{\mu_b^2}.$$

The distribution of service time S_v in a working vacation period is

$$P\{S_v = k\} = \mu_v \bar{\mu}_v^{k-1}, \quad k \geq 1, \quad 0 < \mu_v < 1.$$

and its PGF is

$$B_1(z) = \frac{\mu_v z}{1 - \bar{\mu}_v z}.$$

(3) A server begins a working vacation at the epoch when the queue becomes empty, the distribution of vacation time V is

$$P\{V = k\} = \theta \bar{\theta}^{k-1}, \quad k \geq 1, \quad 0 < \theta < 1.$$

Suppose the beginning and ending of vacation occur at the epoch which is similar to $t = n^-$ in shape. During a working vacation arriving customers are served according to arrival order. The working vacation period is an operation period in a lower speed. At a vacation completion instant, if there are customers in the system, the server will come back to the normal working level. Otherwise, the server stays in a idle period. The server begin a new busy period immediately once there are customers arrived in the system.

We assume that interarrival times, service times, and working vacation times are mutually independent. In addition, the service discipline is first in first out (FIFO).

Based on the above description, our model here is referred to as a late arrival system with immediate entrance. Let Q_n be the number of customers in the system at time n^+ , and

$$J_n = \begin{cases} 0, & \text{the system is in a working vacation period at time } n^+, \\ 1, & \text{the system is in a regular busy period at time } n^+, \end{cases}$$

then $\{Q_n, J_n\}$ is a Markov chain with the state space

$$\Omega = \{(k, j) : k \geq 0, j = 0, 1\}.$$

Using the lexicographical sequence for the states, the transition probability matrix can be written as

$$\tilde{\mathbb{P}} = \begin{bmatrix} B_0 & B_1 & B_2 & B_3 & B_4 & \cdots \\ C_0 & A_0 & A_1 & A_2 & A_3 & \cdots \\ & C & A_0 & A_1 & A_2 & \cdots \\ & & C & A_0 & A_1 & \cdots \\ & & & \vdots & \vdots & \vdots \end{bmatrix},$$

where

$$\begin{aligned} B_0 &= \begin{bmatrix} \bar{\theta}\bar{p} & \theta\bar{p} \\ 0 & \bar{p} \end{bmatrix}, & B_k &= \begin{bmatrix} \bar{\theta}c_k p & \theta c_k p \\ 0 & c_k p \end{bmatrix}, k \geq 1, \\ C_0 &= \begin{bmatrix} \bar{p}\mu_v & 0 \\ \bar{p}\mu_b & 0 \end{bmatrix}, & A_0 &= \begin{bmatrix} \bar{\theta}(\bar{p}\bar{\mu}_v + c_1 p \mu_v) & \theta(\bar{p}\bar{\mu}_v + c_1 p \mu_v) \\ 0 & \bar{p}\bar{\mu}_b + c_1 p \mu_b \end{bmatrix}, \\ C &= \begin{bmatrix} \bar{\theta}\bar{p}\mu_v & \theta\bar{p}\mu_v \\ 0 & \bar{p}\mu_b \end{bmatrix}, & A_k &= \begin{bmatrix} \bar{\theta}(c_k p \bar{\mu}_v + c_{k+1} p \mu_v) & \theta(c_k p \bar{\mu}_v + c_{k+1} p \mu_v) \\ 0 & c_k p \bar{\mu}_b + c_{k+1} p \mu_b \end{bmatrix}, k \geq 1. \end{aligned}$$

3 Stationary queue size and its stochastic decomposition

Assume that (Q, J) be the stationary limit of $\{Q_n, J_n\}$, and its distribution is denoted as

$$\begin{aligned} \pi_{kj} &= \lim_{n \rightarrow \infty} P\{Q_n = k, J_n = j\} = P\{Q = k, J = j\}, \quad (k, j) \in \Omega, \\ \pi_k &= (\pi_{k0}, \pi_{k1}), \quad k \geq 0. \end{aligned}$$

Based on the stationary equations obtained directly by stochastic balance, we have

$$\pi_{00}\bar{\theta}\bar{p} + \pi_{10}\bar{p}\mu_v + \pi_{11}\bar{p}\mu_b = \pi_{00}, \quad (1)$$

$$\pi_{00}\bar{\theta}c_1 p + \pi_{10}\bar{\theta}(\bar{p}\bar{\mu}_v + c_1 p \mu_v) + \pi_{20}\bar{\theta}\bar{p}\mu_v = \pi_{10}, \quad (2)$$

$$\pi_{00}\bar{\theta}c_n p + \bar{\theta}\bar{p}\bar{\mu}_v \sum_{k=1}^{n-1} c_k \pi_{n-k,0} + \bar{\theta}\bar{p}\mu_v \sum_{k=1}^n c_k \pi_{n+1-k,0} + \pi_{n,0}(\bar{\theta}\bar{p}\bar{\mu}_v - 1) + \pi_{n+1,0}\bar{\theta}\bar{p}\mu_v = 0, \quad n \geq 2, \quad (3)$$

$$\pi_{00}\theta\bar{p} + \pi_{01}\bar{p} = \pi_{01}, \quad (4)$$

$$\pi_{00}\theta c_1 p + \pi_{01}c_1 p + \pi_{10}\theta(\bar{p}\bar{\mu}_v + c_1 p \mu_v) + \pi_{11}(\bar{p}\bar{\mu}_b + c_1 p \mu_b) + \pi_{20}\theta\bar{p}\mu_v + \pi_{21}\bar{p}\mu_b = \pi_{11}, \quad (5)$$

$$\begin{aligned} \pi_{00}\theta c_n p + \pi_{01}c_n p + \theta\bar{p}\bar{\mu}_v \sum_{k=1}^{n-1} c_k \pi_{n-k,0} + \theta\bar{p}\mu_v \sum_{k=1}^n c_k \pi_{n+1-k,0} + \bar{p}\bar{\mu}_b \sum_{k=1}^{n-1} c_k \pi_{n-k,1} \\ + \bar{p}\mu_b \sum_{k=1}^n c_k \pi_{n+1-k,1} + \pi_{n0}\theta\bar{p}\bar{\mu}_v + \pi_{n1}(\bar{p}\bar{\mu}_b - 1) + \pi_{n+1,0}\theta\bar{p}\mu_v + \pi_{n+1,1}\bar{p}\mu_b = 0, \quad n \geq 2. \end{aligned} \quad (6)$$

Denote

$$Q_0(z) = \sum_{k=0}^{\infty} z^k \pi_{k0}, \quad |z| \leq 1, \quad Q_1(z) = \sum_{k=1}^{\infty} z^k \pi_{k1}, \quad |z| \leq 1,$$

then the PGF of stationary queue size Q can be written as

$$Q(z) = Q_0(z) + Q_1(z), \quad |z| \leq 1.$$

In order to solve the PGF of stationary queue size, the following lemma is necessary.

Lemma 1. The equation $\bar{\theta}p\bar{\mu}_v zC(z) + \bar{\theta}p\mu_v C(z) = (1 - \bar{\theta}\bar{p}\bar{\mu}_v)z - \bar{\theta}\bar{p}\mu_v$ has the unique root $z = \gamma$ in the interval $(0,1)$.

Proof. We consider the equation system $f(z) = \bar{\theta}p\bar{\mu}_v zC(z) + \bar{\theta}p\mu_v C(z)$, $0 < z < 1$. It is found that

$$f(0) = 0 < f(1) = \bar{\theta}p.$$

On the other hand, for any $0 < z < 1$, we have

$$\begin{aligned} f'(z) &= \bar{\theta}p\bar{\mu}_v(C(z) + zC'(z)) + \bar{\theta}p\mu_v C'(z) > 0, \\ f''(z) &= \bar{\theta}p\bar{\mu}_v(2C'(z) + zC''(z)) + \bar{\theta}p\mu_v C''(z) > 0. \end{aligned}$$

Hence, the equation $f(z)$ is increasing and convex function in the interval $(0,1)$.

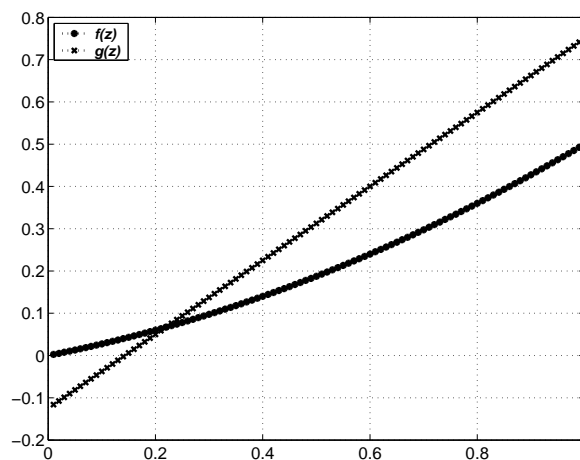


Fig. 1. The change curve of $f(z)$ and $g(z)$ versus z .

Similarly, we let $g(z) = (1 - \bar{\theta}\bar{p}\bar{\mu}_v)z - \bar{\theta}\bar{p}\mu_v$, $0 < z < 1$. Then we have

$$g(0) = -\bar{\theta}\bar{p}\mu_v < 0, \quad g(1) = 1 - \bar{\theta}\bar{p} > f(1).$$

Because the curves of $f(z)$ and $g(z)$ described in Fig.1 have unique crossover point in $(0,1)$, which mean that the equation $\bar{\theta}p\bar{\mu}_v zC(z) + \bar{\theta}p\mu_v C(z) = (1 - \bar{\theta}\bar{p}\bar{\mu}_v)z - \bar{\theta}\bar{p}\mu_v$ has the unique root $z = \gamma$ in the interval $(0,1)$.

Hence, we obtain the following theorem.

Theorem 1. If $\rho = pE(X)/\mu_b < 1$ and $\mu_v < \mu_b$, the stationary queue size Q can be decomposed into the sum of two independent random variables: $Q = L_0 + L_d$, where L_0 is the stationary queue size in corresponding classical $Geom^{[X]}/Geom/1$ queue without vacation and has the PGF

$$L_0(z) = \frac{(1 - \rho)(1 - C(z))\mu_b(pC(z) + \bar{p})}{E(X)\{\mu_b(pC(z) + \bar{p}) - [1 - \bar{\mu}_b(pC(z) + \bar{p})]z\}}.$$

The additional queue size L_d has the PGF

$$L_d(z) = \frac{E(X)\left\{(\bar{\theta}\bar{p} + p)f(z)z + g(z)(\bar{\mu}_b z + \mu_b) - \bar{\theta}h(z)(\bar{\mu}_v z + \mu_v)\right\}}{(1 - C(z))\delta p \left[\bar{\theta}p\bar{\mu}_v C(z) + \frac{1}{z}\bar{\theta}p\mu_v C(z) + (\bar{\theta}\bar{p}\bar{\mu}_v - 1) + \frac{1}{z}\bar{\theta}\bar{p}\mu_v\right]},$$

where

$$\begin{aligned} f(z) &= \bar{\theta}(pC(z) + \bar{p})(\bar{\mu}_b + \mu_b \frac{1}{z})(\bar{\mu}_v + \mu_v \frac{1}{z}) + 1, & g(z) &= \bar{\theta}p^2(C(\gamma) - C(z)) - (\bar{\theta}\bar{p} + p), \\ h(z) &= p(pC(\gamma) + \bar{p}) + \bar{\theta}\bar{p}(pC(z) + \bar{p}), & \delta &= \frac{\mu_b\theta + (\mu_b - \mu_v)\bar{\theta}p(1 - C(\gamma))}{\theta}. \end{aligned}$$

Proof. If Eq. (2) and each equation of Eq. (3) is multiplied by the appropriate z^n , and the resultant equations are summed, it is found that

$$Q_0(z) = \frac{\left[\bar{\theta} p \bar{\mu}_v C(z) + \frac{1}{z} \bar{\theta} p \mu_v C(z) + (\bar{\theta} \bar{p} \bar{\mu}_v - 1) + \frac{1}{z} \bar{\theta} \bar{p} \mu_v - \bar{\theta} p C(z) \right] \pi_{00} + \bar{\theta} \bar{p} \mu_v \pi_{10}}{\bar{\theta} p \bar{\mu}_v C(z) + \frac{1}{z} \bar{\theta} p \mu_v C(z) + (\bar{\theta} \bar{p} \bar{\mu}_v - 1) + \frac{1}{z} \bar{\theta} \bar{p} \mu_v} \quad (7)$$

Since $Q_0(z)$ is analytic function in the interval (0,1), wherever the denominator of the the right-side of Eq. (7) has zeros in that interval, so must the numerator. This fact is henceforth used to evaluate π_{10} . From lemma 1, the denominator of $Q_0(z)$ is equal to 0 if $z = \gamma$, so does the numerator. Substituting $z = \gamma$ into the numerator of the the right-side of Eq. (7), we have

$$\bar{\theta} \bar{p} \mu_v \pi_{10} = \bar{\theta} p C(\gamma) \pi_{00},$$

then we obtain

$$Q_0(z) = \frac{\left[\bar{\theta} p \bar{\mu}_v C(z) + \frac{1}{z} \bar{\theta} p \mu_v C(z) + (\bar{\theta} \bar{p} \bar{\mu}_v - 1) + \frac{1}{z} \bar{\theta} \bar{p} \mu_v - \bar{\theta} p C(z) + \bar{\theta} p C(\gamma) \right] \pi_{00}}{\bar{\theta} p \bar{\mu}_v C(z) + \frac{1}{z} \bar{\theta} p \mu_v C(z) + (\bar{\theta} \bar{p} \bar{\mu}_v - 1) + \frac{1}{z} \bar{\theta} \bar{p} \mu_v}. \quad (8)$$

Similarly, if Eq. (5) and each equation of Eq. (6) is multiplied by the appropriate z^n , and the resultant equations are summed, we have

$$\Phi(z)(Q_1(z) - \pi_{01}) = \bar{p} \mu_b \pi_{11} + \bar{\theta} \bar{p} \mu_v \pi_{10} - \bar{\theta} p C(z) \pi_{00} - p C(z) \pi_{01} - \frac{\theta}{\bar{\theta}} (\Psi(z) + 1)(Q_0(z) - \pi_{00}),$$

where

$$\begin{aligned} \Phi(z) &= p \bar{\mu}_b C(z) + \frac{1}{z} p \mu_b C(z) + (\bar{p} \bar{\mu}_b - 1) + \frac{1}{z} \bar{p} \mu_b, \\ \Psi(z) &= \bar{\theta} p \bar{\mu}_v C(z) + \frac{1}{z} \bar{\theta} p \mu_v C(z) + (\bar{\theta} \bar{p} \bar{\mu}_v - 1) + \frac{1}{z} \bar{\theta} \bar{p} \mu_v. \end{aligned}$$

with Eq. (1), Eq. (4) and Eq. (8), it is easy to get

$$\bar{p} \mu_b \pi_{11} = (1 - \bar{\theta} \bar{p} - p C(\gamma)) \pi_{00}, \quad p \pi_{01} = \bar{\theta} \bar{p} \pi_{00}.$$

Then

$$Q_1(z) = \frac{\pi_{00} \left[\bar{\theta} \bar{p} \Phi(z) \Psi(z) + p(1 - \bar{\theta} \bar{p} - p C(\gamma) - \bar{\theta} \bar{p} C(z)) \Psi(z) - \theta p^2 (C(\gamma) - C(z)) \right]}{p \Phi(z) \Psi(z)}.$$

Therefore, using $Q(z) = Q_0(z) + Q_1(z)$ and the normalization condition $Q(1) = 1$ and L'Hôspital's rule, we get

$$1 = Q(1) = \frac{\pi_{00} \left[\theta(\bar{\theta} \bar{p} + p) \mu_b + (\mu_b - \mu_v) \bar{\theta} p^2 (1 - C(\gamma)) \right]}{\theta p (\mu_b - p E(X))},$$

evidently, the numerator of the above expression is greater than 0 if $\mu_v < \mu_b$, so the denominator is also greater than 0, that is

$$\mu_b - p E(X) > 0,$$

hence

$$\rho = \frac{p E(X)}{\mu_b} < 1,$$

with $\rho < 1$ and $\mu_v < \mu_b$ sufficient for stationarity. Furthermore, we have

$$\pi_{00} = \frac{\mu_b(1 - \rho)}{\delta}, \tag{9}$$

where

$$\delta = \frac{\theta(\theta\bar{p} + p)\mu_b + (\mu_b - \mu_v)\bar{\theta}p^2(1 - C(\gamma))}{\theta p}.$$

Hence, after manipulating we obtain

$$\begin{aligned} Q(z) &= \frac{(1 - \rho)(1 - C(z))\mu_b(pC(z) + \bar{p})}{E(X)\{\mu_b(pC(z) + \bar{p}) - [1 - \bar{\mu}_b(pC(z) + \bar{p})]z\}} \\ &\quad \times \frac{E(X)\{(\theta\bar{p} + p)f(z)z + g(z)(\bar{\mu}_bz + \mu_b) - \bar{\theta}h(z)(\bar{\mu}_vz + \mu_v)\}}{(1 - C(z))\delta p\left[\bar{\theta}p\bar{\mu}_vC(z) + \frac{1}{z}\bar{\theta}p\mu_vC(z) + (\bar{\theta}\bar{p}\bar{\mu}_v - 1) + \frac{1}{z}\bar{\theta}\bar{p}\mu_v\right]} \\ &= L_0(z)L_d(z), \end{aligned}$$

where

$$\begin{aligned} f(z) &= \bar{\theta}(pC(z) + \bar{p})(\bar{\mu}_b + \mu_b\frac{1}{z})(\bar{\mu}_v + \mu_v\frac{1}{z}) + 1, \quad g(z) = \bar{\theta}p^2(C(\gamma) - C(z)) - (\theta\bar{p} + p), \\ h(z) &= p(pC(\gamma) + \bar{p}) + \theta\bar{p}(pC(z) + \bar{p}), \quad \delta = \frac{\mu_b\theta + (\mu_b - \mu_v)\bar{\theta}p(1 - C(\gamma))}{\theta}. \end{aligned}$$

After manipulating, we find that $L_d(1) = 1$, which indicates that $L_d(z)$ is a PGF. Thus, we get theorem 1.

Furthermore, we can obtain the mean of stationary queue size Q derived from

$$E(Q) = E(L_0) + E(L_d),$$

where

$$E(L_0) = \rho + \frac{E(X^2) - E(X) + p^2E(S_b^2) - E(X)\mu_b\rho^2}{2(1 - \rho)}.$$

Using L'Hôpital's rule, we get

$$\begin{aligned} E(L_d) &= \left. \frac{dL_d(z)}{dz} \right|_{z=1} \\ &= \frac{2\bar{\theta}E(X)[(\theta\bar{p} + p)\mu_b\mu_v - pE(X)(\theta\bar{p}\mu_b + p\mu_v)] - p\delta[\theta(E(X^2) - E(X)) - 2\bar{\theta}E(X)(pE(X) - \mu_v)]}{2\theta p\delta E(X)}, \end{aligned}$$

therefore, we obtain

$$\begin{aligned} E(Q) &= \rho + \frac{E(X^2) - E(X) + p^2E(S_b^2) - E(X)\mu_b\rho^2}{2(1 - \rho)} \\ &+ \frac{2\bar{\theta}E(X)[(\theta\bar{p} + p)\mu_b\mu_v - pE(X)(\theta\bar{p}\mu_b + p\mu_v)] - p\delta[\theta(E(X^2) - E(X)) - 2\bar{\theta}E(X)(pE(X) - \mu_v)]}{2\theta p\delta E(X)}. \end{aligned} \tag{10}$$

On the other hand, the probability $P\{J = 0\}$ that the system is in a working vacation period and the probability $P\{J = 1\}$ that the system is in a regular busy period can be derived, respectively.

Substituting the expression of π_{00} into Eq. (8), we have

$$Q_0(z) = \frac{\mu_b(1 - \rho)\left[\bar{\theta}p\bar{\mu}_vC(z) + \frac{1}{z}\bar{\theta}p\mu_vC(z) + (\bar{\theta}\bar{p}\bar{\mu}_v - 1) + \frac{1}{z}\bar{\theta}\bar{p}\mu_v - \bar{\theta}pC(z) + \bar{\theta}pC(\gamma)\right]}{\delta\left[\bar{\theta}p\bar{\mu}_vC(z) + \frac{1}{z}\bar{\theta}p\mu_vC(z) + (\bar{\theta}\bar{p}\bar{\mu}_v - 1) + \frac{1}{z}\bar{\theta}\bar{p}\mu_v\right]}. \tag{11}$$

thus

$$P\{J = 0\} = Q_0(1) = \frac{\mu_b(1 - \rho)[\theta + \bar{\theta}p(1 - C(\gamma))]}{\theta\delta},$$

then, we have

$$P\{J = 1\} = Q_1(1) = 1 - Q_0(1) = \frac{\mu_b\theta(\theta\bar{p} + p(1 + \rho)) + (\mu_b\rho - \mu_v)\bar{\theta}p^2(1 - C(\gamma))}{\theta\delta}.$$

4 Analysis for the stationary waiting time

Denoted the stationary waiting time of a customer in the system and its PGF by W and $W(z)$, respectively. Using several properties of conditional negative binomial distribution, we can obtain the upper bound and the lower bound of stationary waiting time W in the moment generating function order, furthermore, we get the upper bound and the lower bound of the mean waiting time $E(W)$.

Let X be a geometric distributed random variable with parameter α , and $X^{(\gamma)}$ be the independent sum of γ X 's, $\gamma \geq 0 (X^{(0)} \equiv 0)$. Then $X^{(\gamma)}$ follows a negative binomial distribution with parameters γ and α , that is,

$$P\{X^{(\gamma)} = k\} = C_{k-1}^{\gamma-1} \alpha^\gamma \bar{\alpha}^{k-\gamma}, \quad k = \gamma, \gamma + 1, \dots$$

Assume that Y is a geometric distributed random variable with parameter β , and Y and X are mutually independent. Then with the property of independence we found that

$$P\{X^{(\gamma)} < Y < X^{(\gamma+1)}\} = \frac{\beta \bar{\alpha}}{1 - \beta \bar{\alpha}} \left(\frac{\bar{\beta} \alpha}{1 - \bar{\beta} \bar{\alpha}} \right)^\gamma, \quad \gamma \geq 0. \tag{12}$$

Furthermore, it is easy to get two biparameter addition theorems of conditional negative binomial distribution.

Lemma 2. *If $X^{(\gamma)} < Y < X^{(\gamma+1)}, \gamma \geq 0, (V|X^{(\gamma)} < Y < X^{(\gamma+1)})$ follows a negative binomial distribution with parameters $\gamma + 1$ and $(1 - \bar{\beta} \bar{\alpha})$.*

Lemma 3. *If $X^{(\gamma)} < Y, \gamma \geq 1, (X^{(\gamma)}|X^{(\gamma)} < Y)$ follows a negative binomial distribution with parameters γ and $(1 - \bar{\beta} \bar{\alpha})$.*

Definition 1. *Let X and Y be two nonnegative random variables and such that*

$$E[z^X] \geq E[z^Y] \quad \text{for all } z \in (0, 1).$$

then X is said to be smaller than Y in the moment generating function order (denoted as $X \leq_{mgf} Y$)

Lemma 4. *Let X and Y be two nonnegative random variables. If $X \leq_{mgf} Y$, then it is found that*

$$X \leq_{mgf} Y \Rightarrow E(X) \leq E(Y),$$

provided the expectations exist.

The proof of the lemma 4 can be seen in Moshe Shaked and Shanthikumar [7].

Denote

$$\alpha_\gamma = \frac{\theta \bar{\mu}_v}{1 - \theta \bar{\mu}_v} \left(\frac{\bar{\theta} \mu_v}{1 - \bar{\theta} \bar{\mu}_v} \right)^\gamma, \quad \gamma \geq 0, \quad \tilde{f}_\gamma(z) = \left[\frac{(1 - \bar{\theta} \bar{\mu}_v)z}{1 - \bar{\theta} \bar{\mu}_v z} \right]^\gamma, \quad \gamma \geq 1.$$

Theorem 2. *If $\rho = pE(X)/\mu_b < 1$ and $\mu_v < \mu_b$, the stationary waiting time W of a customer has the upper bound and the lower bound in the moment generating function order, that is,*

$$W_2 \leq_{mgf} W \leq_{mgf} W_1,$$

where the PGF of the W_1, W and W_2 respectively are $W_1(z), W(z)$ and $W_2(z)$, and

$$\begin{aligned} W_1(z) &= \frac{1 - C(B(z))}{E(X)(1 - B(z))} \left\{ Q_1(B(z)) + \frac{\mu_b \theta \bar{\mu}_v z}{\mu_b(1 - \bar{\theta} \bar{\mu}_v z) - \bar{\theta} \mu_v(1 - \bar{\mu}_b z)} \left[Q_0(B(z)) - Q_0\left(\frac{\bar{\theta} \mu_v z}{1 - \bar{\theta} \bar{\mu}_v z}\right) \right] \right. \\ &\quad \left. + \left[\mu_v Q_0\left(\frac{(1 - \bar{\theta} \bar{\mu}_v)z}{1 - \bar{\theta} \bar{\mu}_v z}\right) + \bar{\mu}_v Q_0\left(\frac{\bar{\theta} \mu_v z}{1 - \bar{\theta} \bar{\mu}_v z}\right) \right] \right\}, \\ W_2(z) &= \frac{1 - C(B(z))}{E(X)(1 - B(z))} \left\{ Q_1(B(z)) + \frac{\mu_b \theta \bar{\mu}_v z}{\mu_b(1 - \bar{\theta} \bar{\mu}_v z) - \bar{\theta} \mu_v(1 - \bar{\mu}_b z)} \left[Q_0(B(z)) - Q_0\left(\frac{\bar{\theta} \mu_v z}{1 - \bar{\theta} \bar{\mu}_v z}\right) \right] \right\} \\ &\quad + \frac{1 - C(B_1(z))}{E(X)(1 - B_1(z))} \left[\mu_v Q_0\left(\frac{(1 - \bar{\theta} \bar{\mu}_v)z}{1 - \bar{\theta} \bar{\mu}_v z}\right) + \bar{\mu}_v Q_0\left(\frac{\bar{\theta} \mu_v z}{1 - \bar{\theta} \bar{\mu}_v z}\right) \right]. \end{aligned} \tag{13}$$

Proof. In order to compute the upper bound and lower bound of $W(z)$, we consider all possible cases as follows:

- case 1: A batch of customers (tagged customers) arrive in the state $(k, 1)$, $k \geq 1$, that is, there are k customers in front of this batch of customers in the system. In this case, a tagged customer's waiting time in this batch is the sum of the service times of k customers outside of his batch and a period of waiting time inside of his batch. Note that all the customers are served at the normal service level, using the result of the chapter 6.3.4 in Sun and Li^[8], we have

$$W_{k,1}(z) = \left(\frac{\mu_b z}{1 - \bar{\mu}_b z} \right)^k \frac{1 - C(B(z))}{E(X)(1 - B(z))}, \quad k \geq 1.$$

It is easy to get

$$\begin{aligned} \sum_{k=1}^{\infty} \pi_{k,1} W_{k,1}(z) &= \sum_{k=1}^{\infty} \pi_{k,1} \left(\frac{\mu_b z}{1 - \bar{\mu}_b z} \right)^k \frac{1 - C(B(z))}{E(X)(1 - B(z))} \\ &= Q_1(B(z)) \frac{1 - C(B(z))}{E(X)(1 - B(z))}. \end{aligned} \tag{14}$$

- case 2: A batch of customers (tagged customers) arrive in the state $(k, 0)$, $k \geq 1$. In this case, even though k customers outside of this batch are served, a tagged customer in this batch has to wait until part of other customers in front of him inside of his batch finish their services. If $S_v^{(\gamma)} < V < S_v^{(\gamma+1)}$, $0 \leq \gamma \leq k - 1$, by lemma 2, the conditional waiting time outside of the batch follows an Erlang distribution with parameters $\gamma + 1$ and $(1 - \bar{\theta} \bar{\mu}_v)$. If $V > X^{(k)}$, with lemma 3, the conditional waiting time outside of the batch follows an Erlang distribution with parameters k and $(1 - \bar{\theta} \bar{\mu}_v)$. Therefore, the total conditional waiting time is the sum of the conditional waiting time outside of his batch and the conditional waiting time inside of his batch. However, we can not confirm the accurate expression for the conditional waiting time inside of his batch, so we can not obtain the precise expression of the conditional waiting time, and we present its the upper bound and the lower bound of $W_{k,0}(z)$

$$\begin{aligned} &\sum_{\gamma=0}^{k-1} \alpha_{\gamma} \tilde{f}_{\gamma+1}(z) \left(\frac{\mu_b z}{1 - \bar{\mu}_b z} \right)^{k-\gamma} \frac{1 - C(B(z))}{E(X)(1 - B(z))} + \left(1 - \sum_{\gamma=0}^{k-1} \alpha_{\gamma} \right) \tilde{f}_k(z) \frac{1 - C(B_1(z))}{E(X)(1 - B_1(z))} \\ &\leq W_{k,0}(z) \\ &\leq \sum_{\gamma=0}^{k-1} \alpha_{\gamma} \tilde{f}_{\gamma+1}(z) \left(\frac{\mu_b z}{1 - \bar{\mu}_b z} \right)^{k-\gamma} \frac{1 - C(B(z))}{E(X)(1 - B(z))} + \left(1 - \sum_{\gamma=0}^{k-1} \alpha_{\gamma} \right) \tilde{f}_k(z) \frac{1 - C(B(z))}{E(X)(1 - B(z))} \end{aligned}$$

substituting α_{γ} , $\tilde{f}_{\gamma+1}(z)$ and $\tilde{f}_k(z)$, we have

$$\begin{aligned} \sum_{k=1}^{\infty} \pi_{k,0} W_{k,0}(z) &\leq \frac{1 - C(B(z))}{E(X)(1 - B(z))} \left\{ \frac{\mu_b \bar{\theta} \bar{\mu}_v z}{\mu_b (1 - \bar{\theta} \bar{\mu}_v z) - \bar{\theta} \bar{\mu}_v (1 - \bar{\mu}_b z)} \left[Q_0(B(z)) - Q_0\left(\frac{\bar{\theta} \bar{\mu}_v z}{1 - \bar{\theta} \bar{\mu}_v z}\right) \right] \right. \\ &\quad \left. + \left[\mu_v Q_0\left(\frac{(1 - \bar{\theta} \bar{\mu}_v) z}{1 - \bar{\theta} \bar{\mu}_v z}\right) + \bar{\mu}_v Q_0\left(\frac{\bar{\theta} \bar{\mu}_v z}{1 - \bar{\theta} \bar{\mu}_v z}\right) - \pi_{00} \right] \right\} \end{aligned} \tag{15}$$

$$\begin{aligned} \sum_{k=1}^{\infty} \pi_{k,0} W_{k,0}(z) &\geq \frac{1 - C(B(z))}{E(X)(1 - B(z))} \frac{\mu_b \bar{\theta} \bar{\mu}_v z}{\mu_b (1 - \bar{\theta} \bar{\mu}_v z) - \bar{\theta} \bar{\mu}_v (1 - \bar{\mu}_b z)} \left[Q_0(B(z)) - Q_0\left(\frac{\bar{\theta} \bar{\mu}_v z}{1 - \bar{\theta} \bar{\mu}_v z}\right) \right] \\ &\quad + \frac{1 - C(B_1(z))}{E(X)(1 - B_1(z))} \left[\mu_v Q_0\left(\frac{(1 - \bar{\theta} \bar{\mu}_v) z}{1 - \bar{\theta} \bar{\mu}_v z}\right) + \bar{\mu}_v Q_0\left(\frac{\bar{\theta} \bar{\mu}_v z}{1 - \bar{\theta} \bar{\mu}_v z}\right) - \pi_{00} \right] \end{aligned} \tag{16}$$

- case 3: A batch of customers (tagged customers) arrive in the state (0,0). A tagged customer's waiting time is equal to the waiting time inside of his batch. Thus

$$\frac{1 - C(B_1(z))}{E(X)(1 - B_1(z))} \leq W_{00}(z) \leq \frac{1 - C(B(z))}{E(X)(1 - B(z))},$$

$$\pi_{00} \frac{1 - C(B_1(z))}{E(X)(1 - B_1(z))} \leq \pi_{00} W_{00}(z) \leq \pi_{00} \frac{1 - C(B(z))}{E(X)(1 - B(z))} \quad (17)$$

From Eq. (14)~ Eq. (17), after manipulating, we obtain

$$W_1(z) \leq W(z) = \sum_{k=0}^{\infty} \pi_{k,0} W_{k,0}(z) + \sum_{k=1}^{\infty} \pi_{k,1} W_{k,1}(z) \leq W_2(z),$$

where

$$W_1(z) = \frac{1 - C(B(z))}{E(X)(1 - B(z))} \left\{ Q_1(B(z)) + \frac{\mu_b \theta \bar{\mu}_v z}{\mu_b(1 - \theta \bar{\mu}_v z) - \theta \mu_v(1 - \bar{\mu}_b z)} \left[Q_0(B(z)) - Q_0\left(\frac{\theta \mu_v z}{1 - \theta \bar{\mu}_v z}\right) \right] \right. \\ \left. + \left[\mu_v Q_0\left(\frac{(1 - \theta \bar{\mu}_v)z}{1 - \theta \bar{\mu}_v z}\right) + \bar{\mu}_v Q_0\left(\frac{\theta \mu_v z}{1 - \theta \bar{\mu}_v z}\right) \right] \right\},$$

$$W_2(z) = \frac{1 - C(B(z))}{E(X)(1 - B(z))} \left\{ Q_1(B(z)) + \frac{\mu_b \theta \bar{\mu}_v z}{\mu_b(1 - \theta \bar{\mu}_v z) - \theta \mu_v(1 - \bar{\mu}_b z)} \left[Q_0(B(z)) - Q_0\left(\frac{\theta \mu_v z}{1 - \theta \bar{\mu}_v z}\right) \right] \right\} \\ + \frac{1 - C(B_1(z))}{E(X)(1 - B_1(z))} \left[\mu_v Q_0\left(\frac{(1 - \theta \bar{\mu}_v)z}{1 - \theta \bar{\mu}_v z}\right) + \bar{\mu}_v Q_0\left(\frac{\theta \mu_v z}{1 - \theta \bar{\mu}_v z}\right) \right].$$

Assume that $W_1(z)$ and $W_2(z)$ are the PGF of the random variables W_1 and W_2 , respectively. Then from definition 1, we have

$$W_2 \leq_{mgf} W \leq_{mgf} W_1.$$

Utilizing lemma 4, we obtain the upper bound and lower bound of the mean waiting time $E(W)$, that is,

$$E(W_2) \leq E(W) \leq E(W_1),$$

where

$$E(W_2) = \frac{E(X^2) - E(X)}{2\mu_b E(X)} + \frac{E(Q)}{\mu_b} + \frac{\bar{\mu}_v(\mu_b - \theta \bar{\mu}_v)}{\mu_b \theta} \left[Q_0(1) - Q_0\left(\frac{\theta \mu_v}{1 - \theta \bar{\mu}_v}\right) \right] + \mu_v Q'_0(1) \frac{\theta \bar{\mu}_v - \bar{\mu}_b}{\mu_b(1 - \theta \bar{\mu}_v)},$$

$$E(W_1) = E(W_2) + \frac{E(X^2) - E(X)}{2E(X)} \left[\frac{\mu_b - \mu_v}{\mu_b} Q_0(1) + \left(\frac{1}{\mu_v} - \frac{1}{\mu_b} \right) \bar{\mu}_v Q_0\left(\frac{\theta \mu_v}{1 - \theta \bar{\mu}_v}\right) \right]. \quad (18)$$

Remark 1. If the server stops working during the vacation period, that is, $\mu_v = 0$, our model reduces to a bulk input Geom^[X]/Geom/1 queue with classical multiple vacations. Furthermore, if $\theta \rightarrow \infty$, our results here are the same as the results in a normal bulk input Geom^[X]/Geom/1 in Sun and Li^[8].

Remark 2. If the batch size of arrivals is always equal to 1, that is, $c_1 = P\{X = 1\} = 1$. Then the system reduces to a Geom/Geom/1 queue with working vacations considered in Li and Tian^[4].

5 Numerical examples

In this section, to demonstrate the numerical feasibility of the formulas (10) and (18), we present several numerical examples of the queue models with working vacations.

We consider a discrete time bulk input Geom^[X]/Geom/1 queue with working vacation, where the arrival batch size X follows a geometric distribution with parameter α , that is,

$$P\{X = i\} = \alpha\bar{\alpha}^{i-1}, \quad 0 < \alpha < 1, \quad i = 1, 2, \dots,$$

thus it is easy to get $E(X) = \frac{1}{\alpha}$, $E(X^2) = \frac{2 - \alpha}{\alpha^2}$, $C(z) = \frac{\alpha z}{1 - \bar{\alpha}z}$, $|z| \leq 1$.

Assume that the system parameter $p = 0.3, \mu = 0.8, \theta = 0.1$. The first one in Fig. 2 indicates the changing curve of the mean queue length $E(L)$ with the increasing of the service rate μ_v in the working vacation period when the parameter α of batch size equals 0.4, 0.6, 0.8, respectively. The second one shows that the changing curve of the upper bound $E(W_1)$ and the lower bound $E(W_2)$ of the mean waiting time with the increasing of the service rate μ_v when the parameter α of batch size equals 0.4, 0.5, respectively. Evidently, for a given α , $E(L)$, $E(W_1)$ and $E(W_2)$ will decrease with the increasing of the service rate μ_v ; Moreover, for a given μ_v , $E(L)$, $E(W_1)$ and $E(W_2)$ will also decrease with the increasing of the batch size parameter α . Hence, we can control $E(L)$ and $E(W)$ under the appropriate μ_v and α . On the other hand, we know that the curves for the upper bound $E(W_1)$ and the lower bound $E(W_2)$ of the mean waiting time are approaching with the increasing of μ_v which indicates that our results are accord with the practical cases.

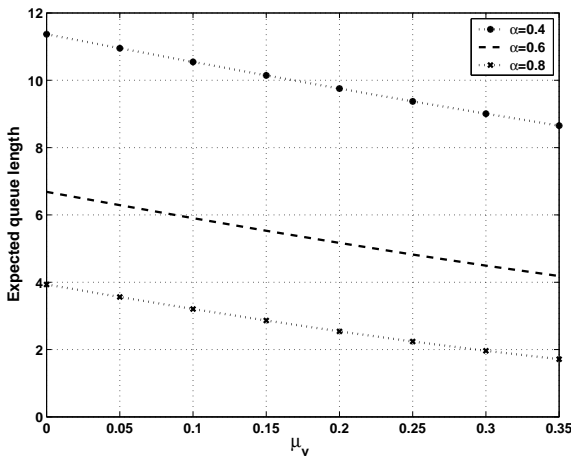


Fig. 2. $E(L), E(W_1)$ and $E(W_2)$ versus μ_v in different batch size, respectively.

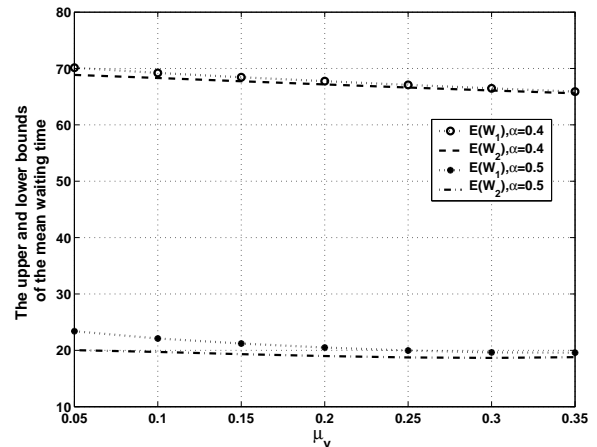


Fig. 3. $E(L), E(W_1)$ and $E(W_2)$ versus ρ in different batch size, respectively.

The first one in Fig. 3 indicates the changing curve of the mean queue length $E(L)$ with the increasing of the system workload ρ when the parameter α of batch size equals 0.4, 0.6, 0.8, respectively. The second one shows that the changing curve of the upper bound $E(W_1)$ and the lower bound $E(W_2)$ of the mean waiting time with the increasing of the system workload ρ when the parameter α of batch size equals 0.4, 0.5, respectively. Evidently, for a given α , $E(L)$, $E(W_1)$ and $E(W_2)$ will increase with the increasing of the system workload ρ ; Moreover, for a given ρ , $E(L)$, $E(W_1)$ and $E(W_2)$ will also decrease with the increasing of the batch size parameter α . Hence, we can control $E(L)$ and $E(W)$ under the appropriate ρ and α . On the other hand, we know that the curves for the upper bound $E(W_1)$ and the lower bound $E(W_2)$ of the mean waiting time are approaching with the increasing of ρ which shows that our results are correct.

On the other hand, from the second ones of Fig. 2 and Fig. 3, we know that the upper bound and the lower bound of the mean waiting time are nearly accordant, which is coincident with the practical instance.

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