ISSN 1750-9653, England, UK
International Journal of Management Science and Engineering Management
Vol. 4 (2009) No. 1, pp. 20-38

Balance conditions and control strategies for economic growth: Economic advance-retreat course analysis III

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(*Received December 23 2007, Revised May 22 2008, Accepted August 7 2008*)

Abstract. ARC (Advance-retreat course) is an analysis theory and method which aims at socio-economic development issues. The main viewpoint of ARC theory is that the human can design and implement his economic strategies initiatively, and the environment synthesized by various objective factors has to accept human choice passively, but it can influence human behaviors in a manner of pressure or resistance. This dynamic game between human and the economic environment is different from the traditional game problems. Based on the theoretical framework of ARC (Dai F, *et al.*, 2007), the conception and the analytical expression of environmental elasticity are put forward in this paper. ARC model based on the environmental elasticity is established and a general solving method for the model is given out, which is simple and elegant. Balance relations for economic growth are given respectively for the ARC model based on exponential innovation and that based on power-law innovation. Some basic control strategies are put forward, such as improving the contribution rate of the integrated innovation to economic growth and changing the proportion of the natural growth rate of the basic economic interests in the synthetical growth rate of the endogenous motivity in economic development under the balance condition. At the end, the empirical analysis results give an effective support to the conclusions gotten in this paper.

Keywords: economic growth; environmental elasticity; advance-retreat course (ARC); balance conditions

1 Introduction

The theoretical studies on economic development have gained many outstanding achievements up to now, such as the cycle theory[19], the real business cycle theory[16–18, 21], economic growth based on R & D [1, 15, 24, 25] and the new growth theory[22, 23]. In recent years, economists pay more attention to improving the efficiency of economic development[3], exploring ways to make the national or regional economic development more balanced by the government economic aid[14], and ensuring the economic growth through the formulation and choosing of policies[12], the continuously improvement of economic education system[13] and other methods. In addition, many economists study the issues of economic development in a variety of ways, such as using dynamics methods from genetics and ecology perspectives to study the economic growth[5, 11] and to choose rational economic measures[4], the application of game theory to solve economic momentum problems[2, 10, 20], and so on. These results are gained through the studies on economic development in the classical, industrial, process and choosing view, and they are very commendable.

However, one kind of problems exists aboard in reality, which is different from the traditional game (the two or more game actors are both wise and initiative), that is the game between the human who are able to choose strategies initiatively and the non-initiative environmental pressure. The result of this game will emerge only after some time. There are many examples of such a game, such as the games between economic

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development and environmental pollution, enterprise management and market environment, policies making and their implementation environment, energy use and the resource environment, transportation and the traffic environment, and so on. In these problems, the human as the intelligent and subjective actor will be free to choose and implement economic strategies and the environment as non-wisdom object only passively accept the human choices. But the environment can act to the human economic behaviors by pressure or resistance. Relatively speaking, the higher degree the economy develops to and the faster the economy grows, the greater the environmental pressure will be. This means that the process of economic development should have advance and retreat courses, or rise and fall courses. It is similar to the process of boating against the current. The socio-economic behaviors of human can be various and the environmental pressure is always changeable. Such processes known as advance-retreat courses (ARC)[6, 8] should not be overlooked.

The author has done some basic research on ARC and the model of discrete ARC has been discussed with the market countermeasures for enterprises in the development of new products[8], the result is analytically given that the quicker the economic growth is, the shorter the growth time is[9], and the economic growth and control strategies considering environment pressures are analyzed and the economic interest consumed by environmental pressure is computed[6]. In this paper, the impact of environmental factors on the economic growth and development will be discussed more deep. The core of this paper is to give the conception and the analytical expression of the environmental elasticity, based on which to build course models with one subject, and to give a specific method for solving the models.

This paper will gain the important conclusion that the exogenous pressure to an industrial growth will increase unlimitedly as the number of the similar enterprises in the industry become larger and larger.

2 Basic theories on advance-retreat course (ARC)

The growth and development process of anything includes two basic stages, advance (or rise) stage and retreat (or fall) stage, and the interactions between motivity and resistance must exist in the process. Based on this view, we call the process an advance-retreat course (ARC).

2.1 The concept system of advance-retreat analysis

Based on the abovementioned various cases, the basic elements, basic characteristics and types of countermeasures of advance-retreat analysis can be summed up. Then the basic concept system of advance- retreat course can be synthesized.

2.1.1 The basic elements of advance-retreat course

The advance-retreat course consists of two pair basic elements, the initiative actors (subject for short) and the integrative environment (object for short) in advance course, and the forward motivity for the subject and the pressure from the object. The correlated concepts are presented as following, and the more detailed explanations had given in literature[6, 8].

Definition 1. *(subject and object) The subject is the actor who takes the initiative to choose strategies and put the strategies into practice. The dominate behavior of the subject is moving in the direction of progressing and developing. The object refers to the objective things which hinder the subject from progress. The dominate behavior of the object is to passively resist on the subject's advancing.*

The various assets of the subject are called the benefit (or interest) of the subject (benefit for short), including the current and stock assets like capitals, estates, resources, and so on. The net benefit is the remaining benefit after the loss is subtracted, and is also called benefit for short if there is no confusion.

Definition 2. *(motivity) The motivity for the subject is the power pushing the subject to advance, which is also called as the power for the subject. The motivity includes endogenous motivity and exogenous motivity. The endogenous motivity is from the subject himself and his internal environment, and the exogenous motivity is from the related environment elements outside the subject.*

Definition 3. *(pressure) The resistance from the object is the force that hinders the subject from advancing, and it is also called as environmental pressure or resistance, including endogenous pressure and exogenous pressure. The pressure caused by the changes in the internal environment which is composed of the subject's own factors, is called as endogenous pressure, and the pressure as a result of changes in the external environment which is composed of factors outside the subject, is called as exogenous pressure.*

The net forward motivity is the remaining part after the pressure is subtracted from the motivity. That is the net motivity for short.

Assumption 1. (Basic assumption) Anything in socio-economic field have their influences to others, including the subject and object. Some related influences will make subject's interest increase or decrease, and can be distinguished as motivity or pressure. Their influence can be evaluated by fluctuation variance of state level of underlying things.

Based on assumption 1, the measurement methods for motivity and pressure can be gained.

The subject movement along with the net benefit increasing is defined as advance, and the movement along with the net benefit decreasing is defined as retreat. So the direction of the motivity is consisting with the advance direction, while the direction of the pressure is opposite to the advance direction.

2.1.2 The basic strategies for subject in advance-retreat course

In order to have an effective description on strategies in the advance-retreat course, the following marks are gived out.

u	:	The integrative motivity subject to advance.
$\overline{u}$	:	The integrative pressure encountered by the subject in advancing.
σ_L	:	The net motivity to the subject advancing. When $u > \overline{u}$, $\sigma_L = \sigma_L(u, \overline{u}) > 0$; When $u = \overline{u}$, $\sigma_L = \sigma_L(u, \overline{u}) = 0$; When $u < \overline{u}$, $\sigma_L = \sigma_L(u, \overline{u}) < 0$.
L	:	The net benefit. $L = L(\sigma_L) = L(u, \overline{u})$.

Normally, there are three countermeasures for the subject in the advace-retreat course.

- When $u - \overline{u} > 0$, the net forward motivity $\sigma_L = \sigma_L(u, \overline{u}) > 0$, then the subject can choose advance, and the net benefit $L(u, \overline{u})$ will increase.
- When $u - \overline{u} = 0$, the net forward motivity $\sigma_L = \sigma_L(u, \overline{u}) = 0$, then the subject generally chooses to stay here, and net benefit $L(u, \overline{u})$ will keep no change (here the cost for staying is ignored). The subject may also adopt measures to enhance motivity and to continue to advance, or choose temporary retreat under specific circumstances.
- When $u - \overline{u} < 0$, the net forward motivity $\sigma_L = \sigma_L(u, \overline{u}) < 0$, then it is feasible for the subject to choose retreat, and net benefit $L(u, \overline{u})$ will decrease. The subject may also adopt measures to enhance motivity and to continue to advance.

2.2 The advance-retreat course and the existence of its solution

The definition of the advance-retreat course (ARC), solution of ARC and the existence theorem of ARC's solution are given as follow.

Definition 4. *(advance-retreat course, ARC or course for short) The course means the changing process of things with development and recession. It can be divided into continuous course and discrete course, and the following is the analytical definition of each:*

(i) The net motivity $\sigma_L(t)$ and the accumulative net benefit $L[\sigma_L(t)]$ are continuous, $\sigma_L(0) > 0$ and $L[\sigma_L(0)] > 0$. If there is a moment $\overline{t}$ with $\overline{t} > 0$ or $\overline{t} = +\infty$, when $t \in [0, \overline{t})$ there is $L[\sigma_L(t)] > 0$ and $L[\sigma_L(\overline{t})] = 0$, $L[\sigma_L(t)](t \in [0, \overline{t}])$ will be called as a continuous course.

(ii) For the net motivity $\sigma_i (i = 0, 1, \ldots, n)$ and the accumulative net benefit $L(\sigma_i), \sigma_0 > 0$ and $L[\sigma_0] > 0$. If there is n_0

$n[0,n]$ at some moment, which satisfies $L[\sigma_i] > 0 (i = 0, 1, \ldots, n_{0-1})$ and $L[\sigma_{n_0}] = 0$, then $\{L[\sigma_i], i \in [0, n_0]\}$ will be called as a discrete course.

Based on the above discussions, the general model of course can be expressed as

$$L(t) = L(\sigma_L) = L(u, \overline{u}) \tag{1}$$

where, u is the motivity, $\overline{u}$ is the pressure, $\sigma_L = \sigma_L(u, \overline{u})$ is the forward net motivity, $L = L(\sigma_L) = L(u, \overline{u})$ is the net benefit.

Definition 5. *(solution of course) For the continuous course $L[\sigma_L(t)](t \in [0, \overline{t}]$, $\overline{t}$ is finite or $\overline{t} = +\infty$), if there is $T \in (0, \overline{t})$, which satisfies $L[\sigma_L(T)] = \max_{0<t<\overline{t}}\{L[\sigma_L(t)]\}$ and $\sigma_L(T) = 0$, then $L(T) = L[\sigma_L(T)]$ is a solution of the continuous course $L[\sigma_L(t)](t \in [0, \overline{t}])$. For the discrete course $L(\sigma_i)(i = 0, 1, \ldots, n)$, if there is $T \in (0, n)$, which satisfies $L(\sigma_T) = \max_{0<i<n}\{L(\sigma_i)\}$ and net motivity $\sigma_T = 0$, then $L(T) = L(\sigma_T)$ is a solution of the discrete course $L[\sigma_i](i = 0, 1, \ldots, n)$.*

Definition 6. *(equilibrium of net motivity) If $t > 0$ and the net motivity $\sigma_L(t) = \sigma_L[u(t), \overline{u}(t)] = 0$, the course is viewed as balanced in the net motivity, and t is called as an equilibrium point of the net motivity.*

According to definition 6, the solution of the course can be interpreted as the maximum of the net benefit for the subject under the equilibrium of the net motivity.

Theorem 1. *(theorem on the existence of course solution)*

(i) For the continuous course $L[\sigma_L(t)](t \in [0, \overline{t}])$, $L[\sigma_L(t)] \in C^{(2)}$ and $\sigma_L(t) \in C^{(1)}$. If there is $T \in (0, \overline{t})$, which satisfies $\sigma_L(T) = 0$ and $\frac{d^2L(T)}{dt^2} < 0$, the solution of the course $L[\sigma_L(t)]$ exists.

(ii) If the discrete course $L(\sigma_i)$ and $\sigma_i(i = 0, 1, \ldots, n)$ have the upper bound, that is there is $T \in [0, n]$ which satisfies $L(\sigma_T) = \max_{0<i<n}\{L(\sigma_i)\}$, the solution L(T) of the discrete course $L(\sigma_i)(i = 0, 1, \ldots, n)$ exists.

Proof. The proof is seen in the reference [6].

3 Environmental elasticity and the related analysis

In order to effectively describe the relation between the endogenous motivity and the environment, the conception of the environmental elasticity and the related analysis will be presented in the following. Firstly, the general model for describing course will be given out.

3.1 Marks and solution of the general course model

Here are the marks needed to be used:

For the starting of course needs some start-up capital (the initial basic benefit), course model (1) can be expressed more generally as

$$L = L(\mu_L, \sigma_L) \tag{2}$$

If the dynamic characteristics of the course are taken into account, there will be $L(t) = L[\mu_L(t), \sigma_L(t)]$, $t > 0$ in the model (2). And if $\mu_L(t), \sigma_L(t) = G[\sigma(t), r\kappa(t)]$ and $L(t) = L[\mu_L(t), \sigma_L(t)]$ are all differentiable, then the time T in the course solution $L(T)$ should satisfy $\sigma_L(T) = 0$ and $\frac{dL(t)}{dt} = L_1' \frac{d\mu_L(T)}{dt} + L_2' \frac{d\sigma_L(T)}{dt}$ at the same time. Integrating all these, the following expression can be derived.

$$\frac{d\mu_L}{d\sigma} = -\frac{L_2'}{L_1'} \left(G_1' + \theta G_2' \frac{g(\sigma)}{\sigma} \right) \tag{3}$$

where $\theta = \frac{d\kappa}{\kappa} \Big/ \frac{d\sigma}{\sigma}$, $r\kappa = g[\sigma(t)]$ is determined by $\sigma_L = G(\sigma, r\kappa) = 0$.

The expression (3) is the condition that the general solution of the course $L(T)$ should satisfy, and this is an expression with a simple form.

μ	:	All the current asset benefits owned by the subject, the basic benefit for short, $\mu > 0$.
σ	:	The motivity owned by the subject himself for advancing, the endogenous motivity for short, $\sigma > 0$.
δ	:	The endogenous pressure coefficient. $\delta\sigma$ is the endogenous pressure, and $\delta\mu$ is the basic benefit consumed by the endogenous pressure (endogenous cost for short), $\delta > 0$. the depreciation in asset benefits is regarded as the part of endogenous cost.
ϕ	:	The basic benefit consumed by the exogenous pressure, the exogenous cost for short, $\phi \geq 0$.
κ	:	The synthetical force from the external environment in advancing. It can be measured by the standard deviation of fluctuations of the total assets from the related external environment, $\kappa > 0$.
α	:	The exogenous motivity coefficient. $\alpha\kappa$ is the motivity from the external environment for the subject, the exogenous motivity for short, $0 < \alpha \leq 1$.
β	:	The exogenous pressure coefficient. $\beta\kappa$ is the pressure from the external environment for the subject, the exogenous pressure for short, $0 < \beta \leq 1$.
r^*	:	The net pressure coefficient from the external environment, the exogenous pressure coefficient for short, $r = \beta - \alpha$, $0 \leq r \leq 1$, $r\kappa$ is the external pressure.
$\overline{u}$	:	The basic benefit consumed by pressure, the basic cost for short. It consists of the endogenous cost $\delta\mu$ and the exogenous cost ϕ
μ_L	:	The basic net benefit (stock assets) of the subject, $\mu_L = \mu_L(\mu, \overline{\mu})$.
u	:	The synthetical motivity for the subject to advance, which consists of endogenous motivity σ and exogenous motivity $\alpha\kappa$, and the synthetical motivity for short.
$\overline{u}$	:	The synthetical pressure encountered by the subject in advancing. It consists of endogenous pressure $\delta\sigma$ and exogenous pressure $\beta\kappa$, and the synthetical pressure for short.
σ_L	:	The net motivity for the subject to advance, $\sigma_L = \sigma_L(u, \overline{u}) = G(\sigma, r\kappa)$.
L	:	The actual net benefit of the subject, $L = F(\mu_L, \sigma_L)$.

*According to assumption 1, the forces from the external environment can be divided into exogenous motivity and exogenous pressure, that is $\alpha + \beta = 1$. Generally, $r = \beta - \alpha > 0 (\alpha < \beta)$, this means that the pressure from the external environment is larger than the motivity from that in the advancing process for the subject.

3.2 The environmental elasticity and its expression

We have the following assumption on the impact of the external environment on the subject.

Assumption 2. The external environment includes all the relevant environmental factors. Whether these factors are favorable or unfavorable, they will fully act on and influence the subject.

Assumption 2 means that the impact of the environment on the subject is completely natural and uncontrolled, that is the environment does not unilaterally choose to have a favorable or disadvantage action on the subject. In the advancing process of the subject, if the endogenous motivities are different the effects of the environment will be different. Therefore, if the expression.

$$\theta = \frac{d\kappa}{\kappa} \Big/ \frac{d\sigma}{\sigma} \tag{4}$$

exists, where $\sigma > 0$ and $\kappa > 0$, it will be called as the environmental elasticity of the endogenous motivity, and the environmental elasticity for short.

Different environmental elasticities mean different exogenous pressure for the subject, namely different courses. At the same time, the environmental elasticity can also describe the memory status of the environment on the advance motivity. Based on the environmental elasticity θ courses can be divided into four categories:

- When $\theta \leq 0$, the course is called as a compensation course. Such a course requires reducing resources consumption and exogenous pressure at the same time of the subject's developing. In fact, this is often difficult to realize. The likely case is that the environment is restored to some degree at the cost of some interests. However, such a process has no features of compensation.
- When $0 < \theta \leq 1$, the course is called as a low-consumption course. Such a course consumes less environmental resources and faces a small exogenous pressure, and it is possible for the course to have a long-term, stable and continuing development. Usually, the low-consumption course is a high-efficiency and ideal development process.

- When $1 < \theta < +\infty$, the course is called as a consumption course. Such a course promotes socio-economic development at the cost of consuming more resources. Usually, a consumption course will end with the growing environment pressure and motivtity being less than pressure, and the recession will eventually arise. This is a common phenomenon in processes of socio-economic development, and it exists widely in every area of socio-economy.
- When $\theta = +\infty$, the course is called as a excessive-consumption course. With the infinity of resources consumption or the environmental pressure, such a course is the extreme phenomenon of a consumption course. It can be used to describe that many industries or products are short-term and washed out soon in socio-economic areas.

Because the socio-economic development is always at the cost of a variety of resources and capitals, so the socio-economic development is a process with the alternation of the consumption course, the low-consumption course, and the super-consumption course.

In most cases, all the industries or enterprises in socio-economy will perform with the characteristics of the consumption course, that is $\theta \in \Theta$, $\theta = (1, +\infty)$. This kind of cases will be mainly discussed in the following.

If noting $\sigma(0) = \sigma_0$, $\kappa(\theta, 0) = \kappa_0$, according to expression (4) the following expression can be derived

$$\kappa(\theta, t) = \kappa_0 \left(\frac{\sigma(t)}{\sigma_0}\right)^{\theta} \tag{5}$$

Expression (5) reflects that the memory of the environment on the advance motivity has nonlinear features and the degree of such nonlinear relationship is determined by the environmental elasticities θ. Usually, the larger the advance motivity is, the subject will be affected by the environment greater.

According to the characteristics of environmental elasticity to describe the relation between growth rate of external environment force and that of endogenous motivity, and considering that the exogenous cost will gradually change with the basic benefit and the environment forces, it is set that $\frac{d\phi}{\phi} = \theta\frac{d\mu}{\mu}$, i.e.,

$$\phi(t) = \phi_0 \left(\frac{\mu(t)}{\mu_0}\right)^{\theta} \tag{6}$$

3.3 Analysis on net motivity based on the environmental elasticity

If the the net motivity at time t is expressed as $\sigma_L(t) = u(\theta, t) - \overline{u}(\theta, t)$, where $u(\theta, t) = \sigma(t) + \alpha\kappa(\theta, t)$ and $\overline{u}(\theta, t) = \delta(t) + \beta\kappa(\theta, t),\ \theta \in \Theta$, then

$$\sigma_L(t) = (1 - \delta)\sigma(t) - r\kappa(\theta, t) \tag{7}$$

According to expression (5) and (7), there is

$$\sigma_L(t) = (1 - \delta)\sigma(t) - r\kappa_0 \left[\frac{\sigma(t)}{\sigma_0}\right]^{\theta} = \sigma(t)\left\{1 - \delta - \frac{r\kappa_0}{\sigma_0}\left[\frac{\sigma(t)}{\sigma_0}\right]^{\theta-1}\right\} \tag{8}$$

Usually, enterprises will gradually bring forward higher development requirements in the healthy development process. Therefore, an assumption on the endogenous motivity is given here.

Assumption 3. The endogenous motivity of enterprises will increase gradually with the enterprises development. Therefore, it can be assumed that $\sigma(t) > \sigma(0) = \sigma_0 (t > 0)$.

Conclusion 1. According to assumption 3 and expression (8), there are the following conclusions:

- When $\theta = 1$, the net motivity and the motivity changes synchronously. Whether the motivity $\sigma(t)$ for the subject increase or decrease, the net motivity $\sigma_L(t) > 0$ if $1 - \delta - \frac{r\kappa_0}{\sigma_0} > 0$, and $\sigma_L(t) \leq 0$ if $1 - \delta - \frac{r\kappa_0}{\sigma_0} \leq 0$.
- When $1 < \theta < +\infty$ and $1 - \delta - \frac{r\kappa_0}{\delta_0} > 0$, the initial net motivity $\sigma_L(0) > 0$. There is always a moment T, which satisfies that the advance net motivity $\sigma_L(t) < 0 (t > T)$.

- When $\theta = +\infty$ and $1 - \delta - \frac{r\kappa_0}{\sigma_0} > 0$, because the resources and the environment pressure are unlimited, the advance net motivity $\sigma_L(t) = -\infty$.

In expression (8), if $\sigma_L(T)$ is set to 0, the endogenous motivity at $t = T$ time can be derived

$$\sigma(T) = \sigma_0 \left(\frac{(1-\delta)\sigma_0}{r\kappa_0} \right)^{\frac{1}{\theta-1}} \tag{9}$$

If the net motivity $\sigma_L(t)$ is derivative, according to expression (8) and let $\frac{d\sigma_L(t)}{d\sigma(t)} = 0$, then $(1-\delta) - \frac{r\theta\kappa_0}{\sigma_0^\theta}[\sigma(t)]^{\theta-1} = 0$. Because $\frac{d^2\sigma_L(t)}{d[\sigma(t)]^2} = -\frac{r\theta(\theta-1)\kappa_0}{\sigma_0^\theta}[\sigma(t)]^{\theta-2} < 0$, when the endogenous motivity satisfies the following condition (10), the net motivity $\sigma_L(t)$ achieves the maximum value.

$$\sigma^*(t) = \sigma_0 \left(\frac{(1-\delta)\sigma_0}{r\theta\kappa_0} \right)^{\frac{1}{\theta-1}} \tag{10}$$

Integrating model (7) and expression (10), the following conclusion 2 can be derived.

Conclusion 2. If the solution of a course exists, then the marginal benefit increases by degree and diminishes separately before and after the endogenous motivity achieves $\sigma^*(t)$ determined by expression (10).

According to expression (8) and (10), conclusion 3 can be gotten.

Conclusion 3. If the initial value of the endogenous motivity σ_0 and the initial value of the exogenous pressure $r\kappa_0$ are known, the maximum value of the net motivity σ_L^* is only related with the endogenous pressure coefficient δ and the environmental elasticity θ, and

$$\sigma_L^* = (1-\delta)\left(1 - \frac{1}{\theta}\right)\sigma^* \tag{11}$$

where, σ^* is determined by expression (10), $\sigma_L^* = \max_{0<t<+\infty} \sigma_L(t)$.

According to expression (9) and (10), the endogenous motivity at the equilibrium point of net motivity and that at the largest net motivity have the following relation

$$\sigma(T) = \theta^{\frac{1}{\theta-1}}\sigma^* \tag{12}$$

In summary, there are the following recapitulatory conclusions:

- Conclusion 1 points out the basic relationship between the environmental elasticity and the characteristics of the net motivity changing. Especially, when θ nears to 1 the endogenous motivity is closer to the endogenous pressure. This makes the benefit of the subject have relatively small changes within the same time, thus it is more stable.
- Conclusion 2 points out that course theory can describe the law of diminishing marginal net benefit in analytical way.
- Conclusion 3 draws up the constant quantitive relationship between the maximum net motivity and the corresponding endogenous motivity.
- The relation expression (12) reveals the basic relationship between the endogenous motivity at the equilibrium point of net motivity and that at the largest net motivity.

These conclusions are very important to use the endogenous motivity and the net motivity to control economic development processes. According to expression (12), the proportion factor $\theta^{\frac{1}{\theta-1}}$, determined by the environmental elasticity, is only the difference between the endogenous motivity at the equilibrium point of net motivity and that at the largest net motivity. Further, $\sigma^* < \sigma(T) < e\sigma^*$ can be derived.

3.4 Estimation method on the environmental elasticity

In this section, the related marks in section 3.1 are still applicable, and besides, the basic net benefit of the subject at time t is $\mu_L(t) = (1-\delta)\mu(t) - \phi(t)$.

If there are samples $\mu_L(i), \phi(i), \mu(i), i = 1, \ldots, n$, the estimation of the endogenous pressure coefficient is

$$\hat{\delta} = 1 - \frac{1}{n}\sum_{i=1}^{n}\left(\frac{\mu_L(i) + \phi(i)}{\mu(i)}\right) \tag{13}$$

where, $\frac{\mu_L(i)}{\mu(i)}$ is the net benefit rate, $\frac{\phi(i)}{\mu(i)}$ is the exogenous cost rate. For $1 < M \leq l \leq n$, there is the following estimations

$$\hat{\sigma}_L(l) = \sqrt{\frac{1}{M-1}\sum_{m=l-M}^{l-1}[\mu_L(l-m) - \overline{\mu}_L(l)]^2}, \quad \overline{\mu}_L(l) = \frac{1}{M}\sum_{m=l-M}^{l-1}\mu_L(l-m)$$

$$\hat{\sigma}(l) = \sqrt{\frac{1}{M-1}\sum_{m=l-M}^{l-1}[\mu(l-m) - \overline{\mu}(l)]^2}, \quad \overline{\mu}(l) = \frac{1}{M}\sum_{m=l-M}^{l-1}\mu(l-m)$$

When the exogenous pressure coefficient $r > 0$ is constant, according to expression (5) and (7), there is $\frac{r\kappa_0}{\sigma_0^\theta} + \theta \ln \sigma(t) = \ln[(1-\delta)\sigma(t) - \sigma_L(t)]$. Using the least-square method, the following can be derived

$$\begin{cases} \hat{\theta} = \sum_{l=M+1}^{n}\left[\left(\ln \frac{[\hat{\sigma}(l)]^{n-M}}{\prod_{j=M+1}^{n}\hat{\sigma}(j)}\right) \cdot \ln\left(\frac{[(1-\hat{\delta})\hat{\sigma}(l) - \hat{\sigma}_L(l)]^{n-M}}{\prod_{j=M+1}^{n}[(1-\delta)\hat{\sigma}(j) - \hat{\sigma}_L(j)]}\right)\right] \Big/ \sum_{l=M+1}^{n}\left(\ln \frac{[\hat{\sigma}(l)]^{n-M}}{\prod_{j=M+1}^{n}\hat{\sigma}(j)}\right)^2 \\ \frac{r\kappa_0}{\sigma_0^\theta} = \frac{1}{n-M}\ln\left(\frac{\prod_{l=M+1}^{n}[(1-\hat{\delta})\hat{\sigma}(l) - \hat{\sigma}_L(l)]}{\left(\prod_{l=M+1}^{n}\hat{\sigma}(l)\right)^{\hat{\theta}}}\right) \end{cases} \tag{14}$$

So both δ and θ can be estimated by expression (13) and (14).

4 Economic ARC model based on the environmental elasticity

Based on the environmental elasticity, this section will give a course (ARC) model which considers factors on integrated innovation. The integrated innovation mentioned here includes the progress, innovation and the related investment in areas of technology, production, system, policy, management, education, market, and so on, all these will promote the growth of the socio-economic development motivity.

4.1 Linear model of economic course

If the general course model (2) can be expressed linearly as

$$L(t) = \mu_L(t) + h\sigma_L(t)^\star \tag{15}$$

$^\star L(t)$ can be supposed as $L(t) = e^{\mu_L(t) + h\sigma_L(t)}$ if it is never less than zero, for example, no overdraft.

where, $h (h > 0)$ is called as coefficient of benefit growth, $\mu_L(t) = (1-\delta)\mu(t) - \phi(t)$ and $\sigma_L(t) = (1-\delta)\sigma(t) - r\kappa(t)$. Based on the literature[7], when $h = 0.2869947990$ and $h = 0.618033989$, the course model in the manner of average and golden section (golden for short thereinafter) are respectively gotten. If $\mu_L(t)$ and $h\sigma_L(t)$ are respectively regarded as the per stock assets at the beginning and the per asset increments, $L(t)$ can be regarded as the per output.

When $\mu_L(t) > 0$, model (15) can be expressed as $L(t) = \mu_L(t)\left(1 + h\frac{\sigma_L(t)}{\mu_L(t)}\right)$, here $g = h\frac{\sigma_L(t)}{\mu_L(t)}$ is the benefit growth rate.

In order to use the course model (15) to analyze and solve practical economic problems, two specific assumptions on basic benefit and endogenous motivity, environmental forces and exogenous costs are given in the following based on the actual situation.

Assumption 4. If the basic benefit and the endogenous motivity are time-related, there are

(i) The basic benefit of the subject will gradually grow according to the level of the natural rate of return (determined by economic growth, assets accumulation and its value naturally rising, and other factors) in the absence of pressure. Therefore, it is assumed that the basic benefit $\mu(t) = \mu_0 e^{\lambda t}$, where μ_0 is the initial benefit, λ is the natural rate of return (return rate for short), $\lambda > 0$.

(ii) As the motivity will gradually increase with the growth of the basic benefit and the sustained integrated innovation, it is assumed that the endogenous motivity gradually grows in the manner of $\sigma(t) = \sigma_0 q(t) e^{\lambda t}$, where σ_0 is the initial motivity, $q(t)$ is the integrated innovation factor[2], $q(0) = 1$. This means that the integrated innovation is regard as a kind of basic motivity to push economy growth. $q(t)$ is continuous and increasing by degree.

Assumption 5. If the basic cost and the environmental force are time-related, there are the following assumptions

(i) In the advancing process of the subject, the integrated force from the social and economic environment will grow with the endogenous motivity. Therefore, it is assumed that the integrated environmental force gradually increases in a nonlinear way of $\kappa(t) = \kappa_0 \left[q(t)e^{\lambda t}\right]^{\theta}$, where the environmental elasticity $\theta > 1$.

(ii) According to expression (6), it is set that $\phi(t) = \phi_0 e^{\lambda\theta t}$,$\phi_0 \geq 0$ is the initial exogenous cost, $\psi = \lambda\theta$ expresses the growth rate of exogenous cost.

From assumption 5, there is $q(t)$ in $\kappa(t)$ and $r\kappa(t)$, these mean that integrated innovations will bring the environment forces, and exogenous pressure further more. For instance, the development of electronic technique, as a technical innovation, has been pushing economy and society headway, but the discard electronic products may cause the pollutions. In addition, automobiles will push economy and society headway, but they will consume energy and cause air pollution. Further more, $\phi(t) = \phi_0 e^{\lambda\theta t}$ means that $\frac{d\phi}{\phi} \Big/ \frac{d\mu}{\mu} = \theta = \frac{d\kappa}{\kappa} \Big/ \frac{d\sigma}{\sigma}$ is supposed. The essential meaning of this assumption is that the exogenous cost will be near to or larger than the basic benefits though the initial basic benefits is much larger than initial exogenous cost. So we need to offset the growth in exogenous cost by the growth in integrated innovation, this can be carried out based on the balance condition discussed behind. In generally, it is can be generally assumed that $\frac{d\phi}{\phi} \Big/ \frac{d\mu}{\mu} = \theta_1 \neq \theta(\theta_1 > 1)$.

Based on the above assumptions, the synthetical basic cost is $\delta\mu(t) + \phi(t)$, the synthetical motivity $\mu(t) = \sigma(t) + \alpha\kappa(t)$, and the synthetical pressure $\overline{u} = \delta\sigma(t) + \beta\kappa(t)$. Therefore, model (15) can be expressed as

$$L(t) = \mu_L(t) + h\sigma_L(t) \tag{16}$$

where, $\mu_L(t) = (1-\delta)\mu(t) - \phi(t)$, $\sigma_L(t) = u(t) - \overline{u}(t) = (1-\delta)\sigma(t) - r\kappa(t)$, $\mu(t) = \mu_0 e^{\lambda t}$ is the basic benefit, $\phi(t) = \phi_0 e^{\theta\lambda t}$ is the exogenous cost, $\sigma(t) = \sigma_0 q(t) e^{\lambda t}$ is the endogenous motivtiy, $r\kappa(t) = r\kappa_0 \left[q(t)e^{\lambda t}\right]^{\theta}$ is the exogenous pressure, and $r = \beta - \theta$ is the exogenous pressure coefficient. The model (16) show: economic development motivity can be translated into the basic asset benefit, and the exogenous pressure can translate into the exogenous cost. Endogenous cost $\delta\mu(t)$ has the same growth rate with basic asset benefit, and endogenous pressure $\delta\sigma(t)$ has the same growth rate with endogenous motivity.

In particular, if $q(t) = e^{st}$ (s is growth rate of integrated innovation, $s > 0$), then model (16) is called as the course based on the exponential innovation mode. And if $q(t) = (1 + st)^{\psi}(\psi > 0)$, model (16) is called as the course based on the power-law innovation mode, where s is the innovation coefficient, ψ is called as the power parameter of innovation factor (innovation parameter for short), $\psi > 0$.

4.2 The general solution of course model

In the process of economic development, as the endogenous motivity and the exogenous pressure always exist at the same time, usually the initial data which are gotten directly in model (15) and (16) are often the initial net benefit $\mu_L(0) = (1 - \delta)\mu_0 - \phi_0$ and the initial net motivity $\sigma_L(0) = (1 - \delta)\sigma_0 - r\kappa_0$. Therefore, there is

[2] It is assumed here that the exogenous innovation in economy and technology can be changed into endogenous motivity.

$$\mu_0 = \frac{\mu_L(0) + \phi_0}{1-\delta} \text{ and } \sigma_0 \frac{\sigma_L(0) + r\kappa_0}{1-\delta} \tag{17}$$

Setting $\sigma_L(T) = 0$ in model (16) and combining expression (17), the exogenous pressure coefficient $r = \frac{\sigma_L(0)+r\kappa_0}{\kappa_0}\left[q(T)e^{\lambda T}\right]^{1-\theta}$ can be derived, that is

$$r = \frac{\sigma_L(0)}{\kappa_0\left\{\left[q(T)e^{\lambda T}\right]^{\theta-1} - 1\right\}} \tag{18}$$

Generally, $L(T)$, the solution of the course (15), should satisfy equation (3), because $\sigma_L(T) = (1-\delta)\sigma(T) - r\kappa(T) = 0$ at this time, that is . So the solution of the course (16) $L(T)$ should satisfy expression (18) and the following expression

$$\frac{d\mu_L}{d\sigma} = h(1-\delta)(\theta-1) \tag{19}$$

Expression (19) has a concise and graceful form.

If $q(t)$ is differentiable, based on expression (19), the solution of the course model (16) $L(T)$ satisfies

$$T = V(T) \tag{20}$$

where, $V(T) = \frac{1}{(\theta-1)\lambda}\ln\left[\frac{(1-\delta)[\lambda\mu_0 - h(\theta-1)\sigma_0 q(T)(s(t)+\lambda)]}{\theta\lambda\phi_0}\right]$, $\phi_0 > 0$, $s(T) = \frac{q'(t)}{q(t)}$ is growth rate of integrated innovation, the average growth rate of integrated innovation is $\bar{s} = \frac{1}{T}\int_0^T \frac{q'(t)}{q(t)}dt = \frac{\ln q(T)}{T}$, the exogenous pressure coefficient r is determined by expression (18).

Based on expression (20), for an arbitrary $T_0 \geq 0$, the iterative formula for computing the value of T can be gotten: $T_{i+1} = V(T_i)$, $i = 0, 1, \ldots$. In addition, for the course (16), if $X(t) = L(t)$ is set up, T_1 which satisfies $L(T_1) = 0$ can be calculated by the following algorithm, and $t = T_1$ is the terminative time of the course life.

Algorithm 1. (extract the root of function $X(t) = 0$) Set $t_0 = 0$. Choose a larger $t_1 > 0$, which satisfies $X(t_1) < 0$, and give a full small $\varepsilon > 0$.

(i) assignment: $t \leftarrow t_1$.

(ii) compute $X(t)$.

(iii) if $|X(t)| < \varepsilon$, end. Else, if $X(t) < -\varepsilon$, set $t_1 = t$; if $X(t) > \varepsilon$, set $t_0 = t$.

(iv) assignment: $t \leftarrow \frac{t_0+t_1}{2}$, turn to (ii).

By the way, we also could get the T in $L(T)$ (the solution of the course (16)) by means of Algorithm 1.

4.3 Balance conditions for course and the related analysis

In the following, the corresponding balance conditions are respectively given out for economic course based on the exponential innovation mode and one based on the power-law innovation mode, and the related analysis is conducted.

4.3.1 Balance conditions and the analysis for course based on the exponential innovation mode

We get the course based on the exponential innovation mode if let $q(t) = e^{st}$ in model (17), and $(1-\delta)\lambda\mu_0 = \lambda\theta\phi_0 e^{\lambda(\theta-1)T} + h(s+\lambda)(\theta-1)(1-\delta)\sigma_0 e^{sT}$ can be derived according to expression (19). Therefore, $L(T)$, the solution of the course based on the exponential innovation mode, satisfies

$$T = V(T) \tag{21}$$

where,

$$V(T)=\begin{cases}\frac{1}{\lambda(\theta-1)}\ln\left[\frac{(1-\delta)\lambda\mu_0}{\lambda\theta\phi_0+h(s+\lambda)(\theta-1)(1-\delta)\sigma_0}\right], & \text{when } s=\lambda(\theta-1)\\ \frac{1}{\lambda(\theta-1)-s}\ln\left\{\frac{(1-\delta)\left[\lambda\mu_0 e^{-sT}-h(s+\lambda)(\theta-1)\sigma_0\right]}{\lambda\theta\phi_0}\right\}, & \text{when } s<\lambda(\theta-1)\\ \frac{1}{s-\lambda(\theta-1)}\ln\left\{\frac{\lambda(1-\delta)\mu_0 e^{-\lambda(\theta-1)T}-\theta\phi_0}{h(s+\lambda)(\theta-1)(1-\delta)\sigma_0}\right\}, & \text{when } s>\lambda(\theta-1)\end{cases}$$

In expression (21), when $s=\lambda(\theta-1)V(T)$ has no direct relation with T, and is determined only by some constants. Then the course (16) is called as balance course, and the solution of it is called as balance solution. Thereforely, the balance solution of ARC model (16) is expressed as

$$L(T)=\frac{(\theta-1)\left[\frac{(1-\delta)\mu_0}{\theta}\right]^{\frac{\theta}{\theta-1}}}{\left[\phi_0+h(\theta-1)(1-\delta)\sigma_0\right]^{\frac{1}{\theta-1}}} \tag{22}$$

Further, $s=\lambda(\theta-1)$ can be expressed as

$$s+\lambda=\lambda\theta^{\star} \tag{23}$$

*This means $\frac{d\sigma}{\sigma}=\frac{d\phi}{\phi}$, i.e., the growth rate of the endogenous motivity is equal to that of exogenous cost.

Formula (23) is called as the balance relation of the economic course if economic growth process is described by course (16). Based on the balance relation (21) and the model (16), $s+\lambda>\lambda\theta$ means that the growth rate of the endogenous motivity is larger than that of the exogenous cost. It is easy for the market value of basic benefit to increase too quickly, to cause the inflation, and the final result is that continuous time of economic growth is shortened. And $s+\lambda<\lambda\theta$ denotes that the growth rate of the endogenous motivity is less than that of the exogenous cost. This means that the exogenous cost grows too fast or the endogenous motivity grows too slowly. Such an economic course is easy for the market value of basic benefit to increase too slowly, and to cause the deflation. When $s+\lambda=\lambda\theta$ economy wil keep an appropriate and steady growth. Therefore, it can usually be assumed that the balance relation (23) is satisfied.

Based on the balance relation (23), there is the following conclusion 4.

Conclusion 4. When expression (23), the balance relation of economic course, comes into existence, i.e., economic growth process follows balance relation, then

- The sum of the natural growth rate of the basic benefit and that of integrated innovation motivity, $\lambda+s$ equals the growth rate of the exogenous cost $\lambda\theta$.
- The synthetical growth rate of the endogenous motivity, $\lambda+s$ is equal to the growth rate of the exogenous cost $\lambda\theta$.
- The growth rate of the synthetical exogenous pressure is equal to the sum of growth rate of integrated innovation, that of the natural growth rate of the endogenous motivity, and that of environmental pressures caused by the integrated innovation, i.e., $(s+\lambda)\theta=s+\lambda+s\theta$.

If the ratio of the growth rate of the integrated innovation and that of the endogenous motivity, $s/(\lambda+s)$ is used to express the contribution rate of the integrated innovation to economic growth (CRI). When the balance condition (23) is established, CRI is

$$CRI=\frac{s}{\lambda+s}=1-\frac{1}{\theta} \tag{24}$$

The computation formula (24) means that if the larger the exogenous pressure (the environmental elasticity) is, the larger the proportion of the growth rate of integrated innovation in the integrated growth rate of the endogenous motivity should be in the steady economic growth.

If the economic innovation rate (EIR) can be measured by the ratio of the growth rate of integrated economic innovation and the natural growth rate s/λ, then when the balance conditions (23) is established the economic innovation rate is

$$EIR=\frac{s}{\lambda}=\theta-1 \tag{25}$$

It will be known by formula (25) that the greater the exogenous pressure is, the higher degree of economic innovation will be required. This means that in the modern economic environment with intense competition if the natural growth rate keeps unchanged the balance economic growth needs to rely on the support of highly integrated economic innovation, and on the other hand, with the growth rate of integrated economic innovation becoming larger the environmental pressure will subsequently increase too. This also means that the input for R&D on integrated economic innovation will be included in exogenous cost.

The proportion of the natural growth rate of the basic economic benefit in the synthetical growth rate of the endogenous motivity in economic development can be measured by expression (26), that is

$$NR = \frac{\lambda}{\lambda + s} = \frac{1}{\theta} \tag{26}$$

Expression (26) means that the contribution of the natural growth to economic growth is in inverse proportion to the environmental pressure, i.e., in the balance economic course, the larger the environmental pressure is, the smaller the role played by the natural growth rate is, and vice versa.

Based on expressions (24), (25) and (26), the following control strategies can be designed.

Conclusion 5. (control strategies based on the balance condition) When the balance relation of economic course, expression (23) comes into existence, then we have the following strategies:

- When the natural growth rate λ and the environmental elasticity θ are all changed, the balance economic growth can be kept by adjusting the growth rate of integrated innovation.
- When the natural growth rate λ is unchanged and the environmental elasticity θ is enlarged (i.e., exogenous pressure increases), the growth rate of integrated innovation can be raised to conform to the change in environment.
- When the environmental elasticity θ is minish, the natural growth rate λ can be enlarged by marking down loan or tax rate in order to maintain the balance economic growth.

4.3.2 The balance conditions for course based on the power-law innovation mode

For the course based on power-law innovation mode, that is, $q(t) = (1+st)^{\psi}(\psi > 0)$ is admitted in course model (16), $s > 0$ is called as the innovation coefficient. Then the following expression can be derived according to expression (19):

$$(1-\delta)\lambda\mu_0 = \lambda\theta\phi_0 e^{\lambda(\theta-1)T} + h(\theta-1)(1-\delta)\sigma_0\left(\frac{\psi s}{1+sT+\lambda}\right)(1+sT)^{\psi} \tag{27}$$

In expression (27), if $\lambda\theta\phi_0 e^{\lambda\theta T} = \sigma_0\left(\frac{\psi s}{1+sT+\lambda}\right)$, i.e., $\lambda\theta = \lambda + \frac{1}{T}\ln\left[\frac{\sigma_0}{\lambda\theta\phi_0}\left(\frac{\psi s}{1+sT}+\lambda\right)(1+sT)^{\psi}\right]$. Generally, if

$$\lambda\theta = \lambda + \frac{1}{t}\ln\left[\frac{\sigma_0}{\lambda\theta\phi_0}\left(\frac{\psi s}{1+st}+\lambda\right)(1+st)^{\psi}\right](t > 0) \tag{28}$$

comes into existence, it will be regarded that course (16) is with balance characteristics, and expression (28) is called as the first power-law balance condition, where $\frac{\sigma'(t)}{\sigma(t)} = \frac{\psi s}{1+st} + \lambda$ is the growth rate of the endogenous motivity at time t. Balance condition (28) implies that the growth speed of the exogenous cost is equal to that of the endogenous motivity at time t.

When $t = T$, the time T in the solution $L(T)$ of model (16), and substituting expression (28) into (27), then T can be determined by expression (29)

$$T = \frac{1}{\lambda(\theta-1)}\ln\left\{\frac{(1-\delta)\mu_0}{\theta\phi_0[1+h(\theta-1)(1-\delta)]}\right\} \tag{29}$$

where the time T is determined entirely by the constants λ, θ, h, δ and some related initial values. Based on expression (28), the first balance condition can be expressed as

$$s = \begin{cases} \frac{\lambda\left[\theta\phi_0(1+st)^{1-\psi}e^{\lambda(\theta-1)t}-\sigma_0\right]}{\sigma_0(\phi+\lambda t)}, & 0 < \psi \le 1 \\ \frac{1}{t}\left[\left(\frac{\lambda\theta\phi_0(1+st)e^{\lambda(\theta-1)t}}{\sigma_0[\psi s+\lambda(1+st)]}\right)^{\frac{1}{\psi}} - 1\right], & \psi > 1 \end{cases} \quad (t > 0) \tag{30}$$

If $\psi = \frac{1}{2}$ is admitted in expression (30), then $s = \frac{2\lambda\left[\theta\phi_0\sqrt{1+st}e^{\lambda(\theta-1)t}-\sigma_0\right]}{\sigma_0(s+2\lambda t)}$. Noting $x = \sqrt{1+st} > 0$, $x^2 = \frac{2\lambda t\theta e^{\lambda(\theta-1)t}x+\sigma_0}{\sigma_0(1+2\lambda t)}$ is derived, and the obtained balance condition is

$$s = \frac{1}{t}\left[\left(\frac{\lambda t\theta e^{\lambda(\theta-1)t} + \sqrt{\left[\lambda t\theta e^{\lambda(\theta-1)t}\right]^2 + \sigma_0^2(1+2\lambda t)}}{\sigma_0(1+2\lambda t)}\right)^2 - 1\right] \tag{31}$$

If $\psi = 1$ in expression (30), then the first balance condition is

$$s = \frac{\lambda\left[\theta\phi_0 e^{\lambda(\theta-1)t} - \sigma_0\right]}{\sigma_0(1+\lambda t)} \tag{32}$$

It can be known from expression (31) and (32) that the innovation index s based on the power-law innovation mode can be determined by λ, θ and t (especially, $t = T$, at which the benefit achieve the highest).

Similarly to expression (23), if the growth rate of the exogenous cost is equal to the growth rate of the endogenous motivity, i.e. balance condition is $\frac{\sigma'(t)}{\sigma(t)} = \frac{\phi'(t)}{\phi(t)}$, then we have

$$\frac{\psi s}{1+st} + \lambda = \lambda\theta \quad (t > 0) \tag{33}$$

Expression (33) is called as the second power-law balance condition. Based on balance condition (33), $s = \frac{(\theta-1)\lambda}{\psi+(1+\theta)\lambda t}$ is obtained. Because $\theta > 1$, that means that the innovation coefficient s will become larger and larger as the time t is going when the second power-law balance condition is satisfied $t < \frac{\psi}{(\theta-1)\lambda}$.

By expression (30) and algorithm 1 or by expression (33), the innovation coefficient s can be determined for economic course to be controlled to achieve a steady operation.

Limited by the length of paper, we do not discuss more deeply the first and second power-law balance condition and the relation between them here.

4.4 Balance solutions of economic course

The solutions of course (16) under balance conditions are called as the balance solutions of economic course. For courses based on exponential innovation mode and power-law innovation mode, the corresponding balance solutions will be given respectively.

4.4.1 Balance solutions of course based on exponential innovation mode

When the innovation factor $q(t) = e^{st}$, expression (18) can be expressed as

$$r = \frac{\sigma_L(0)}{\kappa_0\left[e^{(s+\lambda)(\theta-1)T} - 1\right]} \tag{34}$$

Based on expression (17), (34) and (21), the solution of course based on exponential innovation mode, $L(T)$, satisfies

$$T = V(T) \tag{35}$$

where,

$$V(T)=\begin{cases}\frac{1}{\lambda(\theta-1)}\ln\left\{\frac{\lambda[\mu_L(0)+\phi_0]\left[1-e^{-(s+\lambda)(\theta-1)T}\right]+\lambda\theta\phi_0e^{-s(\theta-1)T}}{\lambda\theta\phi_0+h(s+\lambda)(\theta-1)\sigma_L(0)}\right\}, & \text{when } s=\lambda(\theta-1)\\ \frac{1}{\lambda(\theta-1)-s}\ln\left\{\frac{\lambda[u_L(0)+\phi_0]\left[e^{-st}-e^{[\lambda-(s+\lambda)\theta]T}\right]+\lambda\theta\phi_0e^{-s\theta T}-h(s+\lambda)(\theta-1)\sigma_L(0)}{\lambda\theta\phi_0}\right\}, & \text{when } s<\lambda(\theta-1)\\ \frac{1}{s-\lambda(\theta-1)}\ln\left\{\frac{\lambda[\mu_L(0)+\phi_0]\left[e^{-\lambda(\theta-1)T}-e^{-(s+2\lambda)(\theta-1)T}\right]+\lambda\theta\phi_0\left[e^{-(s+\lambda)(\theta-1)T}-1\right]}{h(s+\lambda)(\theta-1)\sigma_L(0)}\right\}, & \text{when } s>\lambda(\theta-1)\end{cases}$$

$L(T)$ is the balance solutions of course based on exponential innovation mode If it is obtained by the first one in expression (35). At the same time,

$$T=\frac{1}{\lambda(\theta-1)}\ln\left\{\frac{\lambda[\mu_L(0)+\phi_0]\left[1-e^{-\theta\lambda(\theta-1)T}\right]+\lambda\theta\phi_0e^{-\lambda(\theta-1)^2T}}{\lambda\theta[\phi_0+h(\theta-1)\sigma_L(0)]}\right\} \tag{36}$$

4.4.2 Balance solutions of course based on power-law innovation mode

If the innovation factor $q(t)=(1+st)^{\psi}$, based on expression (18) there is

$$r=\frac{\sigma_L(0)}{\kappa_0\left\{\left[(1+sT)^{\psi}e^{\lambda T}\right]^{\theta-1}-1\right\}} \tag{37}$$

Based on expression (37) and (17), we have

$$\sigma_0=\frac{\sigma_L(0)\left[(1+sT)^{\psi}e^{\lambda T}\right]^{\theta-1}}{(1-\delta)\left\{\left[(1+sT)^{\psi}e^{\lambda T}\right]^{\theta-1}-1\right\}} \tag{38}$$

When the first power-law balance relation (28) comes into existence, according to expression (17) and (29), the balance solutions of course based on power-law innovation mode, $L(T)$, satisfies

$$T=\frac{1}{\lambda(\theta-1)}\ln\left\{\frac{\mu_L(0)+\phi_0}{\theta\phi_0[1+h(\theta-1)(1-\delta)]}\right\} \tag{39}$$

where innovation coefficient s is determined by expression (30), σ_0 is determined by expression (38), and $t=T$. Especially, If $\psi=\frac{1}{2}$, s is determined by expression (31). If $\psi=1$, s is determined by expression (32).

Furthermore, the similar discussions can be conducted under the second power-law balance condition (33), and omitted here.

5 Empirical analysis based on american GDP

In the following empirical research on course analysis, the 1940-2006 GDP(chained) price index of US are used (Fiscal Year 2000=1.000)., which are published in the WhiteHouse website[3].

5.1 Data, formula and error

$G(t)$ is adopted to express the actual GDP index value of the tth year, $t=1940,1941,\ldots,2006$. Because $G(1940)=0.0978$, $G(1941)=0.1014$, the initial value is identified as $\mu_L(0)=G(1941)=0.1014$, $\sigma_L(0)=G(1941)-G(1940)=0.0036$[4]. For the lack of the whole statistical data of US economic losses of the past years, here it is assumed that[5]: $\delta=0.3$, $\phi_0=0.02$, $K_0=0.01$.

[3] Data sources: http://www.whitehouse.gov.

[4] $\sigma_L(0)$-the initial fluctuation value of the GDP, $\sigma_L(0)=|G(1941)-G(1940)|$. Because the year is the unit of the GDP data, the data have a higher reliability in describing economic development. Therefore, $\sigma_L(0)$ can be effective in describing the initial fluctuation range of US economic growth, that is the initial net motivity.

[5] Here the basis of the assumptions about the values of δ, ϕ_0 and κ_0 is reasonability, and this does not affect the validity of the analyzing methods. In fact, as long as the related actual data can be gotten, the analysis results will be more reliable.

5.2 Fitting analysis of american economic course

Balance course model based on exponential innovation mode is used to conduct the fitting analysis of US economic growth process, and this analysis is done in the averages manner and the golden manner respectively[7]. The calculating formula for the fitting error is

$$\varepsilon = \sqrt{\frac{\sum_{t=1941}^{2006}[L(t) - G(t)]^2}{2006 - 1940}}, \quad \text{here} \quad L(t) \text{is the estimated value.}$$

5.2.1 The course fitting for us economy under the average manner

In accordance with the economic course under the averages manner: $L(t) = \mu_L(t) + 0.2869947990\sigma_L(t)$, when the balance conditions (23) is established, according to the algorithm 1 based on expression (35), and through debugging λ and θ to make the error ε as small as possible, there are the following results.

When $\lambda = 0.048$[6] and the environmental elasticity $\theta = 1.412$, based on balance conditions (23), the expression (34) and (17), it can be derived by calculation that: the growth rate of integrated innovation $s = \lambda(\theta - 1) = 0.019776$, the exogenous pressure coefficient $r = 0.05474438496$, the initial basic benefit of the subject $\mu_0 = 0.1734285714$ and the initial endogenous motivity $\sigma_0 = 0.005924919786$. The solution of the course $L(T) = 1.217687406$, here $T = 2012.518567$ is the moment at which the economic benefit achieve the maximum. And the estimated error is $\varepsilon = 0.04801596261$.

Seting $X(t) = L(t)$ and using algorithm 1, $T_1 = 2029.407625$ can be derived, and $L(T_1) = 0$, i.e., T_1 is the economic benefit achieves zero (the moment of the course life ending).

The above calculation results and the fitting status can be expressed directly by Fig 1.

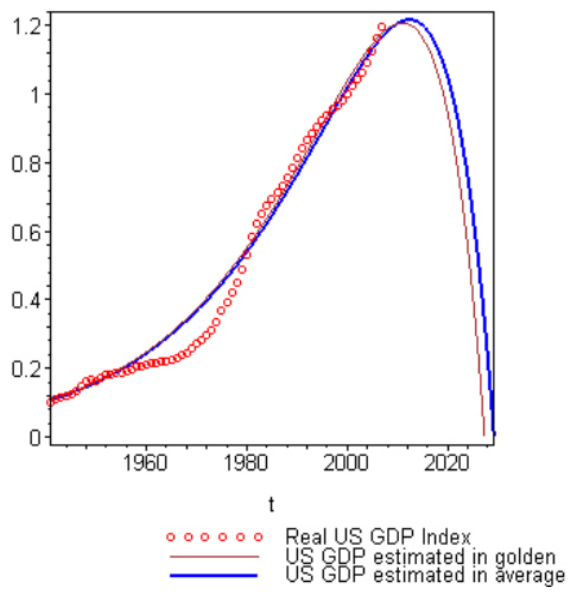

Fig. 1. Fitting of American GDP index and comparision

The average manner $L_A(t) = \mu_L(t) + 0.2869947990\sigma_L(t)$ and the golden manner $L_G(t) = \mu_L(t) + 0.618033989\sigma_L(t)$ are used respectively to fit the US GDP (chained) price index, and the fitting errors are $\varepsilon_A = 0.04801596261$ and $\varepsilon_G = 0.04966307820$ respectively. It can be seen by comparison that the fitting effect in the average manner is better than that in the golden manner. According to the fitting results in the average manner, US may achieve its highest economic level in GDP index in the period of 2012-2013.

[6] According to the Federal funds rate data given in Economic Report of the President: 2004 Report Spreadsheet Tables (http: //www.gpoaccess.gov/usbudget, B-73. Bond yields and interest rates, 1929-2003), it can be derived that the average of the Federal funds rate of 1941-2003 is 4.886825397%.

5.2.2 The course fitting for US economy under the golden manner

In accordance with the economic course under the golden manner: $L(t) = \mu_L(t) + 0.618033989\sigma_L(t)$, when the balance conditions (23) is established, according to the algorithm 1 based on expression (35), and through debugging λ and θ to make the error ε as small as possible, there are the following results.

When $\lambda = 0.048$ and the environmental elasticity $\theta = 1.412$, based on balance conditions (23), the expression (34) and (17), it can be derived by calculation that: the growth rate of integrated innovation $s = \lambda(\theta - 1) = 0.019776$, $r = 0.05726004992$, $\mu_0 = 0.173429$ and $\sigma_0 = 0.005960857856$. The solution of the course $L(T) = 1.208226815$, here $T = 2011.126162$ is the moment at which the economic benefit achieve the maximum. The estimated error $\varepsilon = 0.04966307820$.

Seting $X(t) = L(t)$ and using algorithm 1, $T_1 = 2027.534969$ can be derived, and $L(T_1) = 0$, i.e., T_1 is the economic benefit achieves zero.

The above calculation results and the fitting status can also be expressed directly in Fig 1.

5.3 The related analysis of US economic course

It can be known from the fitting results[7] above that the balance economic course based on the environmental elasticity tallies well with the US GDP index, so it can be considered that the US economic development process has significant features of the balance course.

In addition, the further analysis on the US economy and its development motivtiy can be conducted based on the calculation results in section 5.2.

5.3.1 Analysis on characteristics of american economy

Because the balance condition (24) has nothing to do with the economic growth mode (the average manner or the golden manner), that is the condition is not related with the growth coefficient h, the balance status analysis on US economy can be conducted from a general sense.

Through calculation in the above section, it can be derived that the growth rate of the integrated innovation under balance condition $s = 0.019776$. This shows that in the nearly 66 years of the US economic development, the average net promotion of the integrated innovation to economic growth rate is about 2%. In addition, a more detailed analysis and comparison between the average manner and the golden manner can be carried through, and the related data is shown in Tab 1.

Table 1. Comparison on fitting results

Growth mode	Estimated expression	Basic parameters		Time parameters of course (year)		Error
	The actual net benefit of the subject	The environmental elasticity	The exogenous pressure coefficient	The time with the maximum benefit	The end time of the course	
Average manner	$L(t) = \mu_L(t) + 0.28700\sigma_L(t)$	$\theta = 1.412$	$r = 0.05474438496$	$T = 2012.518567$	$T_1 = 2029.407625$	$\varepsilon = 0.04801596261$
Golden manner	$L(t) = \mu_L(t) + 0.61803\sigma_L(t)$	$\theta = 1.412$	$r = 0.05726004991$	$T = 2011.126162$	$T_1 = 2027.534969$	$\varepsilon = 0.04966307820$

The precision of the data in this table reaches the fifth digit after the radix point. The rate of natural ruturn $\lambda = 0.048$. It is assumed that: $\delta = 0.3$, $\phi_0 = 0.02$, $\kappa_0 = 0.01$, the growth rate of integrated innovation $s = 0.019776$.

(i) Comparative analysis on fitting status. From Tab. 1, the following comparative analysis on fitting status in the average mannerand that in the golden manner can be gotten:

[7] The author has also used the course model based on the power-law innovation mode to fit the US GDP data, and the fitting result is worse than that using course model based on exponential innovation mode.

- Error comparison. The estimated error in the average manner is less than that in the golden manner. It can be considered that US economic development process in nearly 65 years is more in line with the average way. The fitting results also show that in accordance with the current development trends, the GDP index of the American economy may achieve the highest point in the period of 2012-2013.
- Pressure comparison. The exogenous pressure coefficient in the average manner is less than that in the golden manner. This means that the faster growth speed in golden manner has caused greater environment pressure.
- Comparison on the maximum benefit. In the average manner, the course achieves the highest benefit (GDP index) $L(T) = 1.217687406$ at $T = 2012.518567$ (year). But in the golden manner, the course achieves the highest benefit $L(T) = 1.208226815$ at $T = 2011.126162$. This shows that although the growth speed in average way is slower than that in golden way, the sustained growth time of the benefit is longer, and the maximum benefit is even larger.

(ii) Analysis on the basic status of US economy. Based on formula (17) and the above results, the following analysis conclusions can also be made on the economic development process of America.

- Loss of GDP index. Known that the initial net benefit (GDP index value) $\mu_L(0) = 0.1014$, it can be derived by calculation that the initial value of the basic benefit $\mu_0 = 0.1734285714$. This shows that the existence of the endogenous and exogenous pressure makes the basic interests of US have a loss of $\mu_0 - \mu_L(0) = 0.0720285714$ at the beginning, it appears in the way of cost. Of course, the amount of annual loss can be readily estimated according to the actual GDP index value and $\mu_L(t) = (1-\delta)\mu_0 e^{\lambda t} - \phi_0 e^{\theta}\lambda t$.
- Loss of economic development motivity. Known that the initial net motivity $\sigma_L(0) = 0.0036$, the initial basic motivity in the average manner is $\sigma_0 = 0.005924919786$, and that in the golden manner is $\sigma_0 = 0.005960857856$. According to $\sigma_0 - \sigma_L(0)$ it can be gotten by calculation that
  The amount of the basic motivity consumption in the average manner is 0.002324919786.
  The amount of the basic motivity consumption in the golden manner is 0.002360857856.
  Comparatively speaking, the initial endogenous motivity required by economic development in average manner is smaller and the power consumption is also smaller, therefore this manner is more easier to adapt to the economic development environment.
- Comparison on the difference between environment pressure and environment motivity. Because $r = \beta - \alpha$ and $\beta + \alpha = 1$, the exogenous pressure coefficient in average way is $r = 0.05474438496$, it can be derived that $\beta = 0.5273721925$, $\alpha = 0.4726278075$. Therefore, in the past 65 years, the overall driving force from the international environmental changes on US economy takes a proportion of about 47.26%, and the hindering force takes a proportion of about 52.73%. In golden section way, according to $r = 0.05726004991$, it can be derived that $\beta = 0.528630025$ and $\alpha = 0.471369975$. Comparatively speaking, the environmental pressure faced in average manner is smaller.
- Analysis on the trend of US economic growth. It is indicated by ARC curve in Fig 1 that US economic growth will be decelerated at 2007. And 2008-2013 will be the important period for American economic growth, that is to say, American economic development course may turn from economic growth to economic recession between 2012-2013. To avoid the possible economic recession, the US government needs generally and effectively taking measures of economic reformation and economic promotion at this period or before, and in order to start a new course to drive economic growth.

5.3.2 Analysis on the endogenous motivity of US economy

Based on the above analysis it can be known that the average manner is closer to the actual situation of American economic development, so the following calculation and analysis will focus on the average way.

By using expression (10) and (11) respectively, it can be derived by calculation that: The endogenous motivity $\sigma^* = 0.3495999813$, the maximum of the net motivity $\sigma_L^* = 0.07140554858$.

If $\sigma(T^0) - \sigma^* = \frac{\sigma_L(0)+r\kappa_0}{1-\delta} e^{(s+\lambda)T^0} - \sigma^* = 0$ is set, it can be derived that: $T_0 = 2000.163221$.

The above calculation results show that the US economy has reached the maximum net motivity near 2000. In fact, it can be seen from Fig 1 that the actual curve of the US GDP index has a notable local high point of the growth rate near 2000.

6 Conclusions and comments

On the basis of the literature[6, 8], the basic theory on course and the related issues are further discussed in this paper, and the following work has been completed:

(i) The conception of the environmental elasticity and its analytical expression (4) are put forward, and consequently the socio-economic development can be divided into types of consumption course, low- consumption course and excessive-consumption course. And it is analyzed and pointed out that in most cases the socio-economic development processes perform with the characteristics of a consumption course.

(ii) The fundamental relationship (10) between the environmental elasticity and the maximum value of endogenous motivity, and the constant quantitative relation (11) among the environmental elasticity, the maximum net motivity and the corresponding endogenous motivity are obtained. These relationships play a very important role in controlling economic development by using the endogenous motivity and the net motivity.

(iii) The general course model (2) and the formula (3) which the solution of the course model satisfies. The linear model of course (16) which consider integrated innovation factor is established, and a general solving method (19) for the linear model is given out.

(iv) In virtue of the course model considering integrated innovation factor, the course model based on exponential innovation mode and that based on power innovation mode are given out respectively. And their solving method, expression (35) and (39) are also respectively presented under the balance relation (23).

(v) The balance relations (23) based on exponential innovation course and those (30) and (33) based on power innovation course are put forward, those relations are valuable to control the economic growth and development process.

Specifically speaking, the following interesting and important conclusions have been derived in this paper.

- The course theory can describe the laws that the marginal benefit increases by degree or diminishes from the analytical perspective.
- In the balance economic development the growth rate of integrated innovation added the natural growth rate of the basic economic benefit should be equal to the growth rate of the economic environmental costs.
- In the balance economies, a higher growth rate of integrated innovation will be the needed in order to maintain economic growth under the greater exogenous pressure, or it can be said that in modern economic environment with intense competition, the effective economic growth needs highly integrated innovation to support.

It is worth emphasizing that the core conclusions of this paper include that: the balance economic growth requires that the growth rate of the exogenous costs (the integrated consumption caused by economic environment pressure) should be equal to the growth rate (or the average growth rate) of the endogenous motivity (including the motivity from innovation and the natural growth), and some basic control strategies, such as, improving the contribution rate of the integrated innovation to economic growth and changing the proportion of the natural growth rate of the basic economic interests in the synthetical growth rate of the endogenous motivity in economic development in order to achieve economic balance growth. These results can provide a valuable analyzing and calculation tool for the state economic sector and the industrial development departments in major decision-makings to overcome various environment pressures like the lack of resources, environmental pollution, and temperature heating up in world, and so on.

Of course, the conclusions and methods in this paper need further empirical research.

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