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A multiobjective conditional Value-at-Risk model in time interval for loan portfolios*

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Abstract. Within the past several years, important advances have been made in conditional value-at-risk (CVaR) model in finance engineering. CVaR has superior properties in many respects. But, some practical risk problems, such as lending plan of bank, are multiobjective decision making. So, it is key question to study multiobjective CVaR model. This paper proposes a general multiobjective CVaR model in time interval for loan portfolios. First, we introduce the concept of α-VaR and α-CVaR for the case of multiple losses with random variable under the multiple confidence level vector α in time interval. Then, we propose a multiobjective CVaR model in time interval and prove two equality theorems. The solution of solving multiobjective CVaR model can be obtained by finding out solution to another nonlinear optimal problem. Next, we build multiobjective CVaR model to find out the best period and proportion for lending plan. Finally, the numerical results of the best period and proportion for lending plan are given in time interval.

Keywords: credit risk, loss functions, α-CVaR, Pareto efficient solutions

1 Introduction

Since Rockafell and Uryasev (2000)[14] proposed CVaR model, theory and application of CVaR have be rapidly developed. Many researchers (such as Anderson, Mausser, Rosen and Uryasev(2001)[3], Rockafell and Uryasev (2002)[15], Krokhmal, Palmquist and Uryasev (2002),[12]) applied CVaR model to study portfolio and showed that CVaR model is efficient for computing hundred or thousand of stocks as portfolio. CVaR has many good properties, such as computable, convex etc. Especially, CVaR is more efficient than value-at-risk (VaR) for portfolio. Kibzun, Evgeniy and Kuznetsov (2006)[11] analyzed many important properties for VaR and CVaR. Alexander, Coleman and Li (2006)[2] demonstrated that it is possible to compute an optimal CVaR derivative investment portfolio with significantly fewer instruments and comparable CVaR and VaR. Trindade and Zhu (2007)[16] compared parametric and nonparametric estimators of VaR and CVaR under random sampling from the asymmetric Laplace distribution. Alexander, Baptista and Yan (2007)[1] showed that a CVaR constraint is more effective than a VaR constraint to curtail large losses in the meanCvariance model, because the impact of adding either a VaR or a CVaR constraint to the meanCvariance model when security returns are assumed to have a discrete distribution with finitely many jump points. Huang, Zhu, Fabozzi and Fukushima (2007)[6] studied the worst-case CVaR methodology for specifying the uncertain information on the distribution of the exit time associated with exogenous and endogenous incentives.

The CVaR model has been applied to many other fields, such as an electric energy market(Jabr and Rabih (2005)[7]), credit markets (Norbert, Gautam and Stavros (2006)[10]), enterprise risks (John and Hafize

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(2006)[13]), water resources development (Webby, Adamson, Boland, Howlett, Metcalfe and Piantadosi (2007)[17]), newsvendor problem (Gotoh and Takano (2007)[5]), multi-product ordering problem (Zhou, Chen and Wang (2007)[18]), pool-based electricity markets (Javier, Ernesto and Alicia (2007)[8]) and so on.

Many risk models in financial institutions, such as means-variance model, Downside-Risk model, CAPM model, VaR model and CVaR model, are single objective model. However, many risk problems in practical trades, such as bank, estate and supply chain, are multiple decision making. For example, bank regulators consider finding out lending period and loans portfolios in lending plan of bank. Bank possesses all kinds records of many historical loan data such as commerce, estate, automobile and so on, in past several years. It is necessary to evaluate risk for multiple objective, such as surplus loss and insufficient loss, in lending plan. An the other hand, financial institutions have developed a variety of sophisticated models of value-at-risk for market risk in trading portfolios. Dietsch and Petey (2002)[4] presented a value at risk model of the SME credit risk dealing with the specific methodological problems which arise in the modeling of small commercial loans portfolios. The objective of any credit risk model is to build the probability density function of future losses in a loans portfolio under the condition od distributional assumptions. But, it is difficult that distribution of losses for risk factors is often accurately known in applying risk model. Especially, sample data limitations create a serious difficulty for validation to bank regulators. VaR suffers from being unstable and difficult to work with numerically when losses are not 'normally' distributed which in fact is often the case, because loss distributions tend to exhibit 'fat tails' or empirical discreteness. CVaR has superior properties in many respects. Hence, it is very significant to develop a new CVaR model to deal with commercial loans portfolios. Jiang, Hu and Meng (2004)[9] studied a method on solving multiple conditional value-at-risk based on weights for portfolio. In this paper, we will introduce general multiobjective CVaR model to solve the best period of loan proportion in lending plan.

The remainder of this paper is organized as follows. In section 2, we introduce the concept of α-VaR, α-CVaR for multiobjective loss in time interval. Then, we discuss multiobjective CVaR model to find out the best period and proportion in time interval. In section 3, we build multiobjective CVaR model for lending plan of bank. The multiobjective CVaR model based on period can be transformed into linear programming. We obtain numerical results to find out the best period and proportion for lending plan. Section 4 draws the conclusions.

2 Multiobjective CvaR model in time interval

We study multiobjective CVaR model in time interval $[\tau_0, \tau_T]$. Let $[\tau_0, \tau_T]$ be divided into T periods, where T is integer. Let $\tau_0 < t\tau_1 < \cdots < \tau_T$, $[\tau_0, \tau_T] = \bigcup_{t=1}^{T} [\tau_{t-1}, \tau_t]$. For each T, our idea is to find out the best dividing period and proportion such that risk of CVaR value is the smallest in time interval.

Assume that losses functions $f_i(\boldsymbol{x}_t, \boldsymbol{\xi}_{ti}) \in R^n \times R^{m_i} \to R^1 (i = 1, 2, \cdots, I)$ is continuous with respective to decision variable $\boldsymbol{x}_t \in X_t \subset R^n$ in a time interval $[\tau_{t-1}, \tau_t]$ $(t = 1, 2, \cdots, T)$, where $\boldsymbol{\xi}_i$ is continuous random vector, which has a continuous density function $p_i(\boldsymbol{z}_i)$. In loan market, $\boldsymbol{x}_t$ denotes a loans portfolios in the time interval $[\tau_{t-1}, \tau_t]$, then X_t denotes a set of all loans portfolios. $\boldsymbol{\xi}_{ti}$ is a random factors that affect all loss functions in $[\tau_{t-1}, \tau_t]$. $[\tau_{t-1}, \tau_t]$ is called the period t. For each $\boldsymbol{x}_t$, we denote by $\Psi_i(\boldsymbol{x}_t, \cdot)$ the resulting distribution function for the loss $f_i(\boldsymbol{x}_t, \boldsymbol{\xi}_i)$ i.e.,

$$\Psi_i(\boldsymbol{x}_t, y_{ti}) = P\{f_i\ (\boldsymbol{x}_t, \boldsymbol{\xi}_{ti}\ ) \le y_{ti}\} = \int_{f_i\ (\boldsymbol{x}_t, \boldsymbol{z}_{ti}) \le y_{ti}} p_i(\boldsymbol{z}_{ti}) d\boldsymbol{z}_{ti},\ i = 1, 2, \cdots, I. \quad (1)$$

Given some confidence level, the VaR associated with the portfolio $\boldsymbol{x}_t$ is defined as follows.

Definition 1. *Given confidence level $\alpha_i \in (0, 1)$, $i = 1, 2, \cdots, I$ and $\boldsymbol{x}_t$. Denotes*

$$\boldsymbol{y}^*_{t\boldsymbol{\alpha}}(\boldsymbol{x}_t) = \min\{(y_{t1}, y_{t2}, \cdots, y_{tI}) \mid \Psi_i(\boldsymbol{x}_t, y_{ti}) \ge \alpha_i,\ i = 1, 2, \cdots, I\}, \quad (2)$$

i.e., there is not $(y_{t1}, y_{t2}, \cdots, y_{tI})$ to satisfy $\Psi_i(\boldsymbol{x}_t, y_{ti}) \ge \alpha_i,\ i = 1, 2, \cdots, I$ such that

$$y_{ti} < y^*_{ti}(\boldsymbol{x}_t)\ , i = 1, 2, \cdots, I,$$

then $\boldsymbol{y}^*_{t\boldsymbol{\alpha}}(\boldsymbol{x}_t) = (y^*_{t1}(\boldsymbol{x}_t), y^*_{t2}(\boldsymbol{x}_t), \cdots, y^*_{tI}(\boldsymbol{x}_t))$ *is called an* $\boldsymbol{\alpha}$*-VaR loss vector with confidence level* $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \cdots, \alpha_I)$ *to* $\boldsymbol{x}_t$ *at period* t*. All of* $\boldsymbol{\alpha}$*-VaR loss vector is denoted by* $E(\boldsymbol{\alpha}, \boldsymbol{x}_t)$ *at period* t*.*

Define

$$y^*_{i\alpha_i}(\boldsymbol{x}_t) = \min\{y_{ti} \mid \Psi_i(\boldsymbol{x}_t, y_{ti}) \geq \alpha_i\},\ i = 1, 2, \cdots, I, \tag{3}$$

then $y^*_{i\alpha_i}(\boldsymbol{x}_t)$ is called an α_i-VaR loss value with confidence level α_i to $\boldsymbol{x}_t$ at period t. We easy get the following property.

Property 1. (1) $(y^*_{1\alpha_1}(\boldsymbol{x}_t), y^*_{2\alpha_2}(\boldsymbol{x}_t), \cdots, y^*_{I\alpha_I}(\boldsymbol{x}_t)) \in E(\boldsymbol{\alpha}, \boldsymbol{x}_t)$,;

(2) For each $(y^*_{t1}(\boldsymbol{x}_t), y^*_{t2}(\boldsymbol{x}_t), \cdots, y^*_{tI}(\boldsymbol{x}_t)) \in E(\boldsymbol{\alpha}, \boldsymbol{x}_t)$, there is at least one $y^*_{ti}(\boldsymbol{x}_t)$ to be equal to $y^*_{i\alpha_i}(\boldsymbol{x}_t)$.

When $I = 1, T = 1$, the above definitions are of α-VaR the same as Rockfellar and Uryasev(2000)[14]. The $\boldsymbol{y}^*_{t\boldsymbol{\alpha}}(\boldsymbol{x}_t)$ is a Pareto efficient amount $\boldsymbol{y}$ such that the loss will not exceed $\boldsymbol{y}$ with probability $\boldsymbol{\alpha}$.

Let weight set $\Lambda = \{\boldsymbol{\lambda} = (\lambda_1, \lambda_2, \cdots, \lambda_I) \mid \lambda_i \in [0, 1], i = 1, 2, \cdots, I, \sum_{i=1}^{I} \lambda_i = 1\}$.

Now, we define $\boldsymbol{\alpha}$-VaR loss value based on weight vector.

Definition 2. *Given confidence level* $\alpha_i \in (0, 1)$*,* $i = 1, 2, \cdots, I$ *and* $\boldsymbol{x}_t$ *with* $\boldsymbol{\lambda} = (\lambda_1, \lambda_2, \cdots, \lambda_I) \in \Lambda$*. If*

$$\boldsymbol{y}^*_t(\boldsymbol{x}_t, \boldsymbol{\lambda}) = \min\{\boldsymbol{y} \mid \sum_{i=1}^{I} \lambda_i (1 - \alpha_i)^{-1} \Psi_i(\boldsymbol{x}_t, y_{ti}) \geq \sum_{i=1}^{I} \lambda_i \alpha_i (1 - \alpha_i)^{-1}\}, \tag{4}$$

i.e., there is not $(y_{t1}, y_{t2}, \cdots, y_{tI})$ *to satisfy* $\sum_{i=1}^{I} \lambda_i (1 - \alpha_i)^{-1} \Psi_i(\boldsymbol{x}_t, y_{ti}) \geq \sum_{i=1}^{I} \lambda_i \alpha_i (1 - \alpha_i)^{-1}$ *such that* $y_{ti} < y^*_{ti}(\boldsymbol{x}_t, \boldsymbol{\lambda}),\ i = 1, 2, \cdots, I$*, then* $\boldsymbol{y}^*_t(\boldsymbol{x}_t, \boldsymbol{\lambda})$ *is called* $\boldsymbol{\alpha}$*-VaR loss vector with confidence level* $\boldsymbol{\alpha}$ *to* $\boldsymbol{x}_t$ *based on weight* $\boldsymbol{\lambda}$ *at period* t*.*

When $I = 1, T = 1$, the above $\boldsymbol{\alpha}$-VaR loss vector is α-VaR of single loss function.

When $\alpha_1 = \alpha_2 = \cdots = \alpha_I = \alpha$, (4) is $\boldsymbol{y}^*_t(\boldsymbol{x}_t, \boldsymbol{\lambda}) = \min\{\boldsymbol{y} \mid \sum_{i=1}^{I} \lambda_i \Psi_i(\boldsymbol{x}_t, \boldsymbol{y}) \geq \alpha\}$. Then $\boldsymbol{y}^*_t(\boldsymbol{x}_t, \boldsymbol{\lambda})$ is $\boldsymbol{\alpha}$-VaR under the same confidence level.

Property 2. $\boldsymbol{y}^*_t(\boldsymbol{x}_t, \boldsymbol{\lambda}) \in E(\boldsymbol{\alpha}, \boldsymbol{x}_t)$, i.e., $\boldsymbol{y}^*_t(\boldsymbol{x}_t, \boldsymbol{\lambda})$ is also an $\boldsymbol{\alpha}$-VaR loss vector with confidence level $\boldsymbol{\alpha}$ to $\boldsymbol{x}_t$ at period t.

Consider function

$$\Phi_{ti,\alpha_i}(\boldsymbol{x}_t, y_{ti}) = (1 - \alpha_i)^{-1} \int_{f_i(\boldsymbol{x}_t, \boldsymbol{z}_{ti}) \geq y_{ti}} f_i(\boldsymbol{x}_t, \boldsymbol{z}_{ti}) p_i(\boldsymbol{z}_{ti}) d\boldsymbol{z}_{ti},\ i = 1, 2, \cdots, I. \tag{5}$$

Furthermore, we introduce the new conception of multiple CVaR as follows.

Definition 3. *Given confidence* $\alpha_i \in (0, 1)$*,* $i = 1, 2, \cdots, I$ *and* $\boldsymbol{x}_t$*.*

$$\boldsymbol{\Phi}_{t,\boldsymbol{\alpha}}(\boldsymbol{x}_t, \boldsymbol{y}^*_{t\boldsymbol{\alpha}}(\boldsymbol{x}_t)) = (\Phi_{t1,\alpha_1}(\boldsymbol{x}_t, y^*_{t1}(\boldsymbol{x}_t)), \Phi_{t2,\alpha_2}(\boldsymbol{x}_t, y^*_{t2}(\boldsymbol{x}_t)), \cdots, \Phi_{tI,\alpha_I}(\boldsymbol{x}_t, y^*_{tI}(\boldsymbol{x}_t)))$$

is called $\boldsymbol{\alpha}$*-CVaR loss vector with confidence level* $\boldsymbol{\alpha}$ *to* $\boldsymbol{x}_t$ *at period* t*.And*

$$\bar{\boldsymbol{\Phi}}_{T,\boldsymbol{\alpha}}(\boldsymbol{x}, \boldsymbol{y}^*_{\boldsymbol{\alpha}}(\boldsymbol{x})) = \sum_{t=1}^{T} \boldsymbol{\Phi}_{t,\boldsymbol{\alpha}}(\boldsymbol{x}_t, \boldsymbol{y}^*_{t\boldsymbol{\alpha}}(\boldsymbol{x}_t)) \tag{6}$$

is called $\boldsymbol{\alpha}$*-CVaR loss vector with confidence level* $\boldsymbol{\alpha}$ *to* $\boldsymbol{x}_t$ *at period* t*, where* $\boldsymbol{x} = (\boldsymbol{x}_1, \boldsymbol{x}_2, \cdots, \boldsymbol{x}_T)$*,* $\boldsymbol{y}^*_{\boldsymbol{\alpha}}(\boldsymbol{x}) = (\boldsymbol{y}^*_{1\boldsymbol{\alpha}}(\boldsymbol{x}_1), \boldsymbol{y}^*_{2\boldsymbol{\alpha}}(\boldsymbol{x}_2), \cdots, \boldsymbol{y}^*_{T\boldsymbol{\alpha}}(\boldsymbol{x}_T))$*.*

It is clear that we extend concept of CVaR of single loss [1-4]. In order to compute each function $\Phi_{ti,\alpha_i}(\boldsymbol{x}_t, y^*_{ti}(\boldsymbol{x}_t))$, we define a function

$$F_{ti,\alpha_i}(\boldsymbol{x}_t, y_{ti}) = y_{ti} + (1-\alpha_i)^{-1} \int\limits_{\boldsymbol{z}_{ti}\in R^m} (f_i(\boldsymbol{x}_t, \boldsymbol{z}_{ti}) - y_{ti})^+ p_i(\boldsymbol{z}_{ti}) d\boldsymbol{z}_{ti},\ i = 1, 2, \cdots, I. \quad (7)$$

Denote

$$\boldsymbol{F}_{t,\alpha}(\boldsymbol{x}_t, \boldsymbol{y}_t) = (F_{t1,\alpha_1}(\boldsymbol{x}_t, y_{t1}), F_{t2,\alpha_2}(\boldsymbol{x}_t, y_{t2}), \cdots, F_{tI,\alpha_I}(\boldsymbol{x}_t, y_{tI})), \bar{\boldsymbol{F}}_{T,\boldsymbol{\alpha}}(\boldsymbol{x}, \boldsymbol{y}) = \sum_{t=1}^{T} \boldsymbol{F}_{t,\alpha}(\boldsymbol{x}_t, \boldsymbol{y}_t).$$

Between Φ_{ti,α_i} and F_{ti,α_i}, we have the following formula.

$$\begin{aligned} F_{ti,\alpha_i}(\boldsymbol{x}_t, y_{ti}) &= y_{ti} + (1-\alpha_i)^{-1} \int\limits_{f_i(\boldsymbol{x}_t,\boldsymbol{z}_i)\geq y_{ti}} [f_i(\boldsymbol{x}_t, \boldsymbol{z}_i) - y_{ti}] p_i(\boldsymbol{z}_i) d\boldsymbol{z}_i \\ &= y_{ti} - (1-\alpha_i)^{-1} E\{f_i(\boldsymbol{x}_t, \boldsymbol{\xi}) \geq y_{ti}\} y_{ti} + \Phi_{ti,\alpha_i}(\boldsymbol{x}_t, y_{ti}) \\ &= \Phi_{ti,\alpha_i}(\boldsymbol{x}_t, y_{ti}) + y_{ti}(1-\alpha_i)^{-1}[\Psi_{i,\alpha_i}(\boldsymbol{x}_t, y_{ti}) - \alpha_i]. \end{aligned}$$

When $1-\alpha_i = P\{f_i(\boldsymbol{x}_t, \boldsymbol{\xi}) \geq y_{ti}\}$, we have $F_{ti,\alpha_i}(\boldsymbol{x}_t, y_{ti}) = \Phi_{ti,\alpha_i}(\boldsymbol{x}_t, y_{ti})$.

We will assume that the following condition is always true.

$$P\{f_i(\boldsymbol{x}_t, \boldsymbol{\xi}) = y\} = \int\limits_{f_i(\boldsymbol{x}_t,z)=y} p_i(z) dz = 0,\ y \in R,\ i = 1, 2, \cdots, I.$$

From Rockfellar and Uryasev (2000)[14], we have the following lemma concerned on each period $t = 1, 2, \cdots, T$.

Lemma 1. $F_{ti,\alpha_i}(\boldsymbol{x}_t, y_{ti})$ *is continuous, differentiable and convex on* y_{ti}, *and*

$$\min_{y_{ti}\in R} F_{ti,\alpha_i}(\boldsymbol{x}_t, y_{ti}) = \Phi_{ti,\alpha_i}(\boldsymbol{x}_t, y^*_{i\alpha_i}(\boldsymbol{x}_t)),\ i = 1, 2, \cdots, I, \quad (8)$$

$\frac{\partial F_{ti,\alpha_i}(\boldsymbol{x}_t, y_{ti})}{\partial y_{ti}} = (1-\alpha_i)^{-1}[\Psi_i(\boldsymbol{x}_t, y_{ti}) - \alpha_i],\ i = 1, 2, \cdots, I,$ *where* $y^*_{i\alpha_i}(\boldsymbol{x}_t)$ *an* α_i*-VaR with confidence level* α_i *to* $\boldsymbol{x}_t$ *at period* t.

From (8), we easily get

$$\min_{(y_{t1},\cdots,y_{tI})\in R^T} \sum_{t=1}^{T} F_{ti,\alpha_i}(\boldsymbol{x}_t, y_{ti}) = \sum_{t=1}^{T} \Phi_{ti,\alpha_i}(\boldsymbol{x}_t, y^*_{i\alpha_i}(\boldsymbol{x}_t)),\ i = 1, 2, \cdots, I, \quad (9)$$

Now, we study multiobjective CVaR problem on period. We divide time interval $[\tau_0, \tau_T]$ into T parts $[\tau_{t-1}, \tau_t], t = 1, 2, \cdots, T$. We want to find out the minimum CVaR loss value on $[\tau_0, \tau_T]$. Let decision set $\boldsymbol{X} = X_1 \times X_2 \times \cdots \times X_T$. Consider the following multiobjective CVaR problem on $[\tau_0, \tau_T]$.

$$\begin{aligned} \text{(P2-1)} \quad \min \quad & \bar{\boldsymbol{\Phi}}_{T,\boldsymbol{\alpha}}(\boldsymbol{x}, \boldsymbol{y}^*_{\boldsymbol{\alpha}}(\boldsymbol{x})) = \sum_{t=1}^{T} \boldsymbol{\Phi}_{t,\boldsymbol{\alpha}}(\boldsymbol{x}_t, \boldsymbol{y}^*_{t\boldsymbol{\alpha}}(\boldsymbol{x}_t)) \\ \text{s.t.} \quad & \boldsymbol{x}_t \in X_t,\ t = 1, 2, \cdots, T. \end{aligned}$$

If $\boldsymbol{x}^*$ is a weak Pareto-efficient solution to (P2-1), then $\bar{\Phi}_{T,\boldsymbol{\alpha}}(\boldsymbol{x}^*, \boldsymbol{y}^*_{\boldsymbol{\alpha}}(\boldsymbol{x}^*))$ is a weak Pareto-$\boldsymbol{\alpha}$-CVaR loss vector with confidence level $\boldsymbol{\alpha}$ to $\boldsymbol{x}^*$. $\boldsymbol{x}^*$ is called weak Pareto-$\boldsymbol{\alpha}$-CVaR solution.

In order to solve (P2-1), we consider the following multiobjective programming problem.

$$\begin{aligned} \text{(P2-2)} \quad \min \quad & \bar{F}_{T,\boldsymbol{\alpha}}(\boldsymbol{x}, \boldsymbol{y}) = \sum_{t=1}^{T} \boldsymbol{F}_{t,\alpha}(\boldsymbol{x}_t, \boldsymbol{y}_t) \\ \text{s.t.} \quad & \boldsymbol{x}_t \in \boldsymbol{X}_t, \boldsymbol{y}_t \in R^I,\ t = 1, 2, \cdots, T. \end{aligned}$$

We can obtian a weak Pareto-α-CVaR solution by finding out Pareto-efficient solution to (P2-2). From lemma 1, we have the following theorem.

Theorem 1. *If $(\overline{\boldsymbol{x}}, \overline{\boldsymbol{y}})$ is a Pareto-efficient solution to (P2-2), then $\overline{\boldsymbol{x}}$ is weak Pareto-$\boldsymbol{\alpha}$-CVaR solution. If for each $i = 1, 2, \cdots, I$, $\arg \min_{y_{ti} \in R} F_{ti,\alpha_i}(\overline{\boldsymbol{x}}_t, y_{ti})$ is only one , then $\overline{\boldsymbol{y}}_t$ is an $\boldsymbol{\alpha}$-VaR loss vector with $\boldsymbol{\alpha}$ to $\overline{\boldsymbol{x}}_t$ at period t.*

Proof. We first show that for each $i = 1, 2, \cdots, I$, $(\overline{y}_{1i}, \cdots, \overline{y}_{Ti})$ is an optimal solution to $\min_{(y_{1i}, \cdots, y_{Ti}) \in R^T} \sum_{t=1}^{T} F_{ti,\alpha_i}(\overline{\boldsymbol{x}}_t, y_{ti})$. Assume that there is some j and $\overline{y}_{tj}(t = 1, 2, \cdots, T)$ that is not an optimal solution, then there is some $y'_{tj}(t = 1, 2, \cdots, T)$ such that

$$\sum_{t=1}^{T} F_{tj,\alpha_j}(\overline{\boldsymbol{x}}_t, y'_{tj}) < \sum_{t=1}^{T} F_{tj,\alpha_j}(\overline{\boldsymbol{x}}_t, \overline{y}_{tj}), \tag{10}$$

We choose $(\overline{\boldsymbol{x}}_t, \overline{y}_{t1}, \cdots, \overline{y}_{tj-1}, y'_{tj}, \overline{y}_{tj+1}, \cdots, \overline{y}_{tI})$ for $t = 1, 2, \cdots, T$. By (9), $(\overline{\boldsymbol{x}}_t, \overline{\boldsymbol{y}}_t)(t = 1, 2, \cdots, T)$ is not a Pareto efficient solution to (P2-2). Hence, we get a contradiction. According to (9), we have

$$\sum_{t=1}^{T} F_{ti,\alpha_i}(\overline{\boldsymbol{x}}_t, \overline{y}_{ti}) = \sum_{t=1}^{T} \Phi_{ti,\alpha_i}(\overline{\boldsymbol{x}}_t, y^*_{ti}(\overline{\boldsymbol{x}}_t)),\ i = 1, 2, \cdots, I, \tag{11}$$

where $(y^*_{t1}(\overline{\boldsymbol{x}}_t), y^*_{t2}(\overline{\boldsymbol{x}}_t), \cdots, y^*_{ti}(\overline{\boldsymbol{x}}_t))$ is an $\boldsymbol{\alpha}$-VaR loss vector with confidence level $\boldsymbol{\alpha}$ to $\overline{\boldsymbol{x}}_t$ at period t. Assume that $\overline{\boldsymbol{x}}$ is not weak Pareto-$\boldsymbol{\alpha}$-CVaR solution, then there is some $\boldsymbol{x}' \in \boldsymbol{X}$ such that

$$\sum_{t=1}^{T} \Phi_{ti,\alpha_i}(\boldsymbol{x}'_t, y^*_{ti}(\boldsymbol{x}'_t)) < \sum_{t=1}^{T} \Phi_{ti,\alpha_i}(\overline{\boldsymbol{x}}_t, y^*_{ti}(\overline{\boldsymbol{x}}_t)),\ i = 1, 2, \cdots, I. \tag{12}$$

From (9), (11) and (12), we have

$$\min_{(y_{1i}, \cdots, y_{Ti}) \in R^T} \sum_{t=1}^{T} F_{ti,\alpha_i}(\boldsymbol{x}'_t, y_{ti}) = \sum_{t=1}^{T} \Phi_{ti,\alpha_i}(\boldsymbol{x}'_t, y^*_{ti}(\boldsymbol{x}'_t)) < \sum_{t=1}^{T} F_{ti,\alpha_i}(\bar{\boldsymbol{x}}_t, \bar{y}_{ti}),\ i = 1, 2, \cdots, I.$$

So, from the above formula, there is some $\boldsymbol{y}''$ such that

$$\sum_{t=1}^{T} F_{ti,\alpha_i}(\boldsymbol{x}'_t, y''_{ti}) = \min_{(y_{1i}, \cdots, y_{Ti}) \in R^T} \sum_{t=1}^{T} F_{ti,\alpha_i}(\boldsymbol{x}'_t, y_{ti}) < \sum_{t=1}^{T} F_{ti,\alpha_i}(\overline{\boldsymbol{x}}_t, \overline{y}_{ti}),\ i = 1, 2, \cdots, I.$$

Therefore, we know that $(\overline{\boldsymbol{x}}, \overline{\boldsymbol{y}})$ is not an Pareto-efficient solution to (P2-2).This is a contradiction. Henece, $\overline{\boldsymbol{x}}$ is weak Pareto-$\boldsymbol{\alpha}$-CVaR solution.

Furthermore, if $\min_{y_{ti} \in R} F_{\alpha_i}(\overline{\boldsymbol{x}}_t, y_{ti})$ has only one optimal solution, it is clear that

$$\overline{y}_{ti} = y^*_{ti}(\overline{\boldsymbol{x}}_t),\ i = 1, 2, \cdots, I.$$

Then $(\overline{y}_{t1}, \overline{y}_{t2}, \cdots, \overline{y}_{tI})$ is an $\boldsymbol{\alpha}$-VaR loss vector with $\boldsymbol{\alpha}$ to $\overline{\boldsymbol{x}}_t$ at period t. □

We consider the following single objective programming problem.

$$\text{(P2-3)} \quad \min \quad \hat{F}_{T,\boldsymbol{\lambda}}(\boldsymbol{x}, \boldsymbol{y}) = \sum_{t=1}^{T} \sum_{i=1}^{I} \lambda_i F_{ti,\alpha_i}(\boldsymbol{x}_t, y_{ti})$$
$$\text{s.t.} \quad (\boldsymbol{x}_1, \boldsymbol{x}_2, \cdots, \boldsymbol{x}_T) \in \boldsymbol{X}, (y_{t1}, y_{t2}, \cdots, y_{ti}) \in R^I, t = 1, 2, \cdots, T,$$

where $0 < \lambda_i < 1, i = 1, 2, \cdots, I$ are given. (P2-3) is denoted by $\min_{\boldsymbol{x},\boldsymbol{y}} \hat{F}_{T,\boldsymbol{\lambda}}(\boldsymbol{x}, \boldsymbol{y})$. The optimal objective value is denoted by $FCVaR(T, \boldsymbol{\lambda})$. From theorem 1, we have the following results.

Theorem 2. *If $(\overline{\boldsymbol{x}}, \overline{\boldsymbol{y}})$ is an optimal solution to (P2-3), then $\overline{\boldsymbol{x}}$ weak Pareto-$\boldsymbol{\alpha}$-CVaR solution. If for each $i = 1, 2, \cdots, I$, arg $\min\limits_{y_{ti} \in R} F_{ti,\alpha_i}(\overline{\boldsymbol{x}}_t, y_{ti})$ is only one , then $(\overline{y}_{t1}, \overline{y}_{t2}, \cdots, \overline{y}_{tI})$ is an $\boldsymbol{\alpha}$-VaR loss vector with $\boldsymbol{\alpha}$ to $\overline{\boldsymbol{x}}_t$ based on weight $\boldsymbol{\lambda}$ at period t.*

Now, it is important to find out best period such that CVaR value is minimum by solving (P2-3). In practice, T can not be too big, because time $[\tau_0, \tau_T]$ will be divided into too small interval. For example, each loan period in plan of bank is considered to be usually one month or one year. Let $\tilde{T}$ be the biggest number in all period. We need to find out a T^* as follows:

$$T^* = \arg\min_{T}\{\min_{\boldsymbol{x},\boldsymbol{y}} \hat{F}_{T,\boldsymbol{\lambda}}(\boldsymbol{x}, \boldsymbol{y}), T = 1, 2, \cdots, \tilde{T}\}. \tag{13}$$

There is a very difficult problem; i.e. give suitable each weight λ_i. We can find out the minimum risk in all weight set as follows.

$$\begin{aligned}
\text{(P2-4)} \quad \min \quad & \hat{F}_{T,\boldsymbol{\lambda}}(\boldsymbol{x}, \boldsymbol{y}) = \sum_{t=1}^{T}\sum_{i=1}^{I} \lambda_i F_{ti,\alpha_i}(\boldsymbol{x}_t, y_{ti}) \\
\text{s.t.} \quad & (\boldsymbol{x}_1, \boldsymbol{x}_2, \cdots, \boldsymbol{x}_T) \in \boldsymbol{X}, (y_{t1}, y_{t2}, \cdots, y_{ti}) \in R^I, t = 1, 2, \cdots, T, \\
& \sum_{i=1}^{I} \lambda_i = 1, 0 < \lambda_i < 1, i = 1, 2, \cdots, I,
\end{aligned}$$

where weight $\lambda_i < 1, i = 1, 2, \cdots, I$ are decision variable too. The objective value of (P2-4) is denoted by $MCVaR(T)$. (P2-4) is simply rewritten by $MCVaR(T) = \min\limits_{\boldsymbol{\lambda},\boldsymbol{x},\boldsymbol{y}} \hat{F}_{T,\boldsymbol{\lambda}}(\boldsymbol{x}, \boldsymbol{y})$. Similar to (P2-4), we can consider the worst risk problem for all weight set. $WCVaR(T) = \max\limits_{\boldsymbol{\lambda}} \min\limits_{\boldsymbol{x},\boldsymbol{y}} \hat{F}_{T,\boldsymbol{\lambda}}(\boldsymbol{x}, \boldsymbol{y})$, i.e. consider problem (P2-5).

$$\begin{aligned}
\text{(P2-5)} \quad \max_{\boldsymbol{\lambda}} \min_{\boldsymbol{x},\boldsymbol{y}} \quad & \hat{F}_{T,\boldsymbol{\lambda}}(\boldsymbol{x}, \boldsymbol{y}) = \sum_{t=1}^{T}\sum_{i=1}^{I} \lambda_i F_{ti,\alpha_i}(\boldsymbol{x}_t, y_{ti}) \\
\text{s.t.} \quad & (\boldsymbol{x}_1, \boldsymbol{x}_2, \cdots, \boldsymbol{x}_T) \in \boldsymbol{X}, (y_{t1}, y_{t2}, \cdots, y_{ti}) \in R^I, t = 1, 2, \cdots, T, \\
& \sum_{i=1}^{I} \lambda_i = 1, 0 < \lambda_i < 1, i = 1, 2, \cdots, I,
\end{aligned}$$

This is a maxmin problem. Usually, we have $MCVaR(T) \leq FCVaR(T, \boldsymbol{\lambda}) \leq WCVaR(T)$.

We can approximately compute problem (P2-2), (P2-3), (P2-4) and (P2-5). Let take sample points $\boldsymbol{z}_{tj}^i, j = 1, 2, \cdots, q_t$, we have

$$(1 - \alpha_i)^{-1} \int\limits_{\boldsymbol{z} \in R^m} (f_i(\boldsymbol{x}_t, \boldsymbol{z}) - y_{ti})^+ p_i(\boldsymbol{z}) d\boldsymbol{z} \approx (1 - \alpha_i)^{-1} q^{-1} \sum_{j=1}^{q_t} (f_i(\boldsymbol{x}_t, \boldsymbol{z}_{tj}^i) - y_{ti})^+.$$

Problem (P2-2) is approximately denoted by

$$\begin{aligned}
\text{(P2-2)}' \quad \min \quad & (\sum_{t=1}^{T} (y_{t1} + \frac{1}{q_t(1 - \alpha_1)} \sum_{j=1}^{q_t} (f_1(\boldsymbol{x}_t, \boldsymbol{z}_{tj}^1) - y_{t1})^+), \\
& \cdots, \sum_{t=1}^{T} y_{tI} + \frac{1}{q_t(1 - \alpha_I)} \sum_{j=1}^{q_t} (f_I(\boldsymbol{x}_t, \boldsymbol{z}_{tj}^I) - y_{tI})^+) \\
\text{s.t.} \quad & (y_{t1}, y_{t2}, \cdots, y_{tI}) \in R^I, \boldsymbol{x}_t \in X_t, t = 1, 2, \cdots, T.
\end{aligned}$$

For given weight $(\lambda_1, \lambda_2, \cdots, \lambda_I)$, Problem (P2-3) is approximately denoted by

$$\text{(P2-3)}' \quad \min \quad \sum_{t=1}^{T}\sum_{i=1}^{I}\lambda_i\{y_{ti}+\frac{1}{q_t(1-\alpha_i)}\sum_{j=1}^{q_t}(f_i(\boldsymbol{x}_t,\boldsymbol{z}_{tj}^i)-y_{ti})^+\}$$
$$\text{s.t.} \quad (y_{t1},y_{t2},\cdots,y_{tI})\in R^I,\boldsymbol{x}_t\in X_t, t=1,2,\cdots,T.$$

Hence, we can obtain approximate period by finding out (P2-3)′, when we take $T = 1,2,\cdots,\tilde{T}$. Then, we can find out period T^* of the minimum risk value. We need solve programming problems repeatedly in $\tilde{T}$ times.

Problem (P2-4) is approximately denoted by

$$\text{(P2-4)}' \quad \min \quad \sum_{t=1}^{T}\sum_{i=1}^{I}\lambda_i\{y_{ti}+\frac{1}{q_t(1-\alpha_i)}\sum_{j=1}^{q_t}(f_i(\boldsymbol{x}_t,\boldsymbol{z}_{tj}^i)-y_{ti})^+\}$$
$$\text{s.t.} \quad (y_{t1},y_{t2},\cdots,y_{tI})\in R^I,\boldsymbol{x}_t\in X_t, t=1,2,\cdots,T.$$
$$\sum_{i=1}^{I}\lambda_i=1,\ 0<\lambda_i<1, i=1,2,\cdots,I,$$

Problem (P2-5) is approximately denoted by

$$\text{(P2-5)}' \quad \max_{\boldsymbol{\lambda}}\min_{\boldsymbol{x},\boldsymbol{y}} \quad \sum_{t=1}^{T}\sum_{i=1}^{I}\lambda_i\{y_{ti}+\frac{1}{q_t(1-\alpha_i)}\sum_{j=1}^{q_t}(f_i(\boldsymbol{x}_t,\boldsymbol{z}_{tj}^i)-y_{ti})^+\}$$
$$\text{s.t.} \quad (y_{t1},y_{t2},\cdots,y_{tI})\in R^I,\boldsymbol{x}_t\in X_t, t=1,2,\cdots,T.$$
$$\sum_{i=1}^{I}\lambda_i=1,\ 0<\lambda_i<1, i=1,2,\cdots,I,$$

3 A multiobjective cvar model for loan portfolios

In this section, we first introduce multiobjective CVaR model for lending plan within a given time horizon. Then, we will give experiment results about the best period of lending plan with the model.

We consider dividing time interval $[\tau_0,\tau_T]$ into equivalent interval. Let $t_0 < t_1 < \cdots < t_{T-1} < t_T$, $t_i - t_{i-1} = \frac{t_T - t_0}{T},\ i = 1,2,\cdots,T$. We will study to find the best period T, i.e., the length $t_i - t_{i-1}$ under the lowest risk.

We build a CVaR model for loan problem of bank. Assume that each consumer will be able to pay off all loans, then we only consider two losses: surplus loss and insufficient loss. The problem is how to find out the best period T of loan plan in given time horizon $[\tau_0,\tau_T]$. There are n kinds of loans portfolios in the loan plan. Let x_{tk} be a proportion of the k'th loan in all loan at period t. Let ξ_{tk} be random variable of loan demand at period t. Then, let surplus loss function of lending at period t be

$$f_1(\boldsymbol{x}_t,\boldsymbol{\xi}_t)=\sum_{k=1}^{n}\gamma_{tk}(x_{tk}-\xi_{tk})^+\ t=1,2,\cdots,T, \tag{14}$$

where γ_{tk} is lending rate of the kth loan at period t. Let insufficient loss function of lending at period t be

$$f_2(\boldsymbol{x}_t,\boldsymbol{\xi}_t)=\sum_{k=1}^{n}\gamma_{tk}(\xi_{tk}-x_{tk})^+\ t=1,2,\cdots,T. \tag{15}$$

Let one unit of lending need be allocated at each period t, then all lending satisfy constraint

$$\sum_{t=1}^{T}\sum_{k=1}^{n}x_{tk}=1, x_{tk}\geq 0,\ k=1,2,\cdots,n, t=1,2,\cdots,T. \tag{16}$$

If we have lending sample data $\boldsymbol{z}_{tj}^i = (z_{tj1}, z_{tj2}, \cdots, z_{tjn}), j = 1, 2, \cdots, q_t$, then from (P2-2), we give a multiobjective CVaR model of lending in period T as follows.

$$\begin{aligned}
\text{(P3-1)} \quad \min \quad & \sum_{t=1}^{T}\{\lambda_1[y_{t1} + \frac{1}{q_t(1-\alpha_1)}\sum_{j=1}^{q_t}(\sum_{k=1}^{n}\gamma_{tk}(x_{tk} - z_{tjk})^+ - y_{t1})^+] \\
& +\lambda_2[y_{t2} + \frac{1}{q_t(1-\alpha_2)}\sum_{j=1}^{q_t}(\sum_{k=1}^{n}\gamma_{tk}(z_{tjk} - x_{tk})^+ - y_{t2})^+]\} \\
\text{s.t.} \quad & \sum_{t=1}^{T}\sum_{k=1}^{n} x_{tk} = 1, (y_{t1}, y_{t2}) \in R^2, t = 1, 2, \cdots, T, \\
& x_{tk} \geq 0,\ k = 1, 2, \cdots, n, t = 1, 2, \cdots, T.
\end{aligned}$$

where weight λ_1 and λ_2 are given.

In order to find out the best period of lending, let take $T = 1, 2, \cdots, \tilde{T}$ to solve (P3-1) respectively.

(P3-1) can be equal to

$$\begin{aligned}
\text{(P3-2)} \quad \min \quad & \sum_{t=1}^{T}[\lambda_1(y_{t1} + \frac{1}{q_t(1-\alpha_1)}\sum_{j=1}^{q_t} v_{tj}) + \lambda_2(y_{t2} + \frac{1}{q_t(1-\alpha_2)}\sum_{j=1}^{q_t} v'_{tj})] \\
\text{s.t.} \quad & x_{tk} - z_{tjk} - u_{tjk} \leq 0, u_{tjk} \geq 0, k = 1, 2, \cdots, n, t = 1, 2, \cdots, T, j = 1, 2, \cdots, q_t, \\
& \sum_{k=1}^{n}\gamma_{tk}u_{tjk} - y_{t1} - v_{tj} \leq 0, v_{tj} \geq 0, t = 1, 2, \cdots, T, j = 1, 2, \cdots, q_t, \\
& z_{tjk} - x_{tk} - u'_{tjk} \leq 0, u'_{tjk} \geq 0, k = 1, 2, \cdots, n, t = 1, 2, \cdots, T, j = 1, 2, \cdots, q_t, \\
& \sum_{k=1}^{n}\gamma_{tk}u'_{tjk} - y_{t2} - v'_{tj} \leq 0, v'_{tj} \geq 0, t = 1, 2, \cdots, T, j = 1, 2, \cdots, q_t, \\
& \sum_{t=1}^{T}\sum_{k=1}^{n} x_{tk} = 1, (y_{t1}, y_{t2}) \in R^2, t = 1, 2, \cdots, T, \\
& x_{tk} \geq 0,\ k = 1, 2, \cdots, n, t = 1, 2, \cdots, T.
\end{aligned}$$

(P3-2) is linear programming and exists optimal solution.

Now, we give numerical results of finding out the best period for lending plan, where we use loan data of some bank in China about thirty months from 2004 to 2006 for (P3-1). These data include lending date, lending quantity and interest rate etc. The sample data of lending quantity have be transformed into proportion data in numerical experiment. We suppose that all clients can return their loan in future. We obtain the best period and CVaR approximate value and lending proportion under different weight and confidence level in Tab. 1,Tab. 2 and Tab. 3 for $T = 1, 2, 3, 4, 5, 6, 7, 8, 9, 10$.

In Tab. 1, we get results of CVaR approximate value from $T = 1$ to 10 for combining two confidence level from $0.80, 0.90$ and 0.99 when weight $\lambda_1 = \lambda_2 = 0.5$. We find out the best period $T^* = 3$, i.e., the best length of lending is approximate 10 months in one plan. Change of confidence level do not sensitivity to the best period of loan. Results show that CVaR approximate value increase when T increase. The bigger risk loan is, the shorter length loan time is, when T is too big. If λ_1 and λ_2 are variable too in (P3-1), (P3-1) is rewritten from (P2-4)' as follows.

Table 1. He best period t^* and cvar of $(\lambda_1, \lambda_2) = (0.50, 0.50)$

(α_1, α_2)	T^*	1	2	3	4	5	6	7	8	9	10
(0.99,0.99)	3	0.195098	0.153700	0.083298	0.091191	0.103723	0.108497	0.106224	0.112610	0.112610	0.129362
(0.99,0.90)	3	0.188788	0.153606	0.083299	0.091188	0.103719	0.108495	0.106222	0.112612	0.112608	0.129358
(0.99,0.80)	3	0.181934	0.138104	0.056947	0.076668	0.093663	0.108500	0.106226	0.112608	0.112609	0.129358
(0.90,0.99)	3	0.188236	0.153701	0.083298	0.091189	0.103723	0.108495	0.106225	0.112614	0.112615	0.129358
(0.90,0.90)	3	0.182639	0.153606	0.083298	0.091191	0.103723	0.108495	0.106225	0.112616	0.112615	0.129358
(0.90,0.80)	3	0.175861	0.138104	0.056947	0.076667	0.093663	0.108496	0.106223	0.112610	0.112610	0.129367
(0.80,0.99)	3	0.169038	0.144559	0.082942	0.090300	0.103379	0.108499	0.106225	0.112609	0.112608	0.129358
(0.80,0.90)	3	0.161886	0.144061	0.082940	0.090302	0.103380	0.108495	0.106224	0.112612	0.112609	0.129368
(0.80,0.80)	3	0.155086	0.127885	0.056644	0.075682	0.093138	0.108495	0.106222	0.112612	0.112617	0.129365

$$
\begin{aligned}
\text{(P4-3)} \quad \min \quad & \sum_{t=1}^{T}\{\lambda_1[y_{t1} + \frac{1}{q_t(1-\alpha_1)}\sum_{j=1}^{q_t}(\sum_{k=1}^{n}\gamma_{tk}(x_{tk}-z_{tjk})^+ - y_{t1})^+] \\
& +\lambda_2[y_{t2} + \frac{1}{q_t(1-\alpha_2)}\sum_{j=1}^{q_t}(\sum_{k=1}^{n}\gamma_{tk}(z_{tjk}-x_{tk})^+ - y_{t2})^+]\} \\
\text{s.t.} \quad & (y_{t1}, y_{t2}) \in R^2, \sum_{k=1}^{n} x_{tk} = 1, t = 1, 2, \cdots, T, \\
& x_{tk} \geq 0,\ k = 1, 2, \cdots, n, t = 1, 2, \cdots, T. \\
& \lambda_1 + \lambda_2 = 1, \underline{\epsilon} \leq \lambda_1, \lambda_2 \leq \overline{\epsilon}.
\end{aligned}
$$

where $\underline{\epsilon} > 0$ and $\overline{\epsilon} < 1$ are given. Problem (P4-3) is not linear programming. We give an approximate method to find out the best period:

(1) Given confidence level (α_1, α_2), $N-1$ weight value,

$$
\begin{aligned}
\lambda_1^i &= \frac{i}{N}, i = 1, 2, \cdots, N-1, \\
\lambda_2^i &= 1 - \frac{i}{N}, i = 1, 2, \cdots, N-1,
\end{aligned}
$$

(2) For each $(\lambda_1^i, \lambda_2^i)$, compute linear programming problem (P3-1), then we get

$$
FCVaR(T, (\lambda_1^i, \lambda_2^i)),\ i = 1, 2, \cdots, N-1.
$$

(3) Find out the best period T^*:

$$
\begin{aligned}
MCVaR(T) &\approx \min FCVaR(T, (\lambda_1^i, \lambda_2^i)) : i = 1, 2, \cdots, N-1, \\
WCVaR(T) &\approx \max FCVaR(T, (\lambda_1^i, \lambda_2^i)) : i = 1, 2, \cdots, N-1, \\
T_M^* &= \min\{T \mid \min MCVaR(T) : T = 1, 2 \cdots, \widetilde{T}\}, \\
T_W^* &= \min\{T \mid \min WCVaR(T) : T = 1, 2 \cdots, \widetilde{T}\},
\end{aligned}
$$

By the above approximate method, we get results of CVaR approximate value from $T = 1$ to 10 for confidence level $(\alpha_1, \alpha_2) = (0.99, 0.99)$ when $N = 9 (N = 99)$ in Tab. 2 and Tab. 3. CVaR approximate value is different under different weight values. In the last two line of Tab. 2 and Tab. 3, we give the minimum and worst CVaR approximate value with respective to column at same period. The best period T^* ($\lambda_1 = 0.9$ or $\lambda_1 = 0.99$) of the minimum CVaR approximate value is 3, i.e., the best length of lending is approximate 10 months. All loans will be supplied to borrowers such that lending loss may decrease in one loan period. The best period T^* ($\lambda_1 = 0.5$ or $\lambda_1 = 0.51$) of the worst CVaR approximate value is 3, i.e., the best length of lending is approximate 10 months.

Table 2. The best period t^* and cvar of $(\alpha_1, \alpha_2) = (0.99, 0.99)$ and $n = 9$

(λ_1, λ_2)	T^*	1	2	3	4	5	6	7	8	9	10
(0.10,0.90)	4	0.039020	0.031258	0.018556	0.018493	0.022147	0.021950	0.021389	0.022841	0.022834	0.026225
(0.20,0.80)	4	0.078039	0.062516	0.037113	0.036985	0.044287	0.043898	0.042778	0.045669	0.045669	0.052450
(0.30,0.70)	3	0.117059	0.093774	0.055472	0.055478	0.066431	0.065847	0.064169	0.068503	0.068503	0.078675
(0.40,0.60)	3	0.156078	0.124102	0.070272	0.073800	0.086094	0.087625	0.085386	0.091167	0.091167	0.104900
(0.50,0.50)	3	0.195098	0.153700	0.083298	0.091191	0.103723	0.108497	0.106224	0.112610	0.112610	0.129362
(0.60,0.40)	3	0.234118	0.182798	0.070530	0.077119	0.086908	0.091246	0.088958	0.094069	0.094068	0.105635
(0.70,0.30)	3	0.268578	0.192613	0.053004	0.057900	0.066191	0.069524	0.067761	0.071594	0.071594	0.079312
(0.80,0.20)	3	0.291940	0.201283	0.035336	0.038599	0.044128	0.046353	0.045174	0.047732	0.047729	0.052870
(0.90,0.10)	3	0.315302	0.208899	0.017668	0.019299	0.022064	0.023177	0.022587	0.023865	0.023865	0.026437
MCVaR(T)	3	0.039020	0.031258	0.017668	0.018493	0.022064	0.021950	0.021389	0.022841	0.022834	0.026225
WCVaR(T)	3	0.315302	0.208899	0.083298	0.091191	0.103723	0.108497	0.106224	0.112610	0.112610	0.129362

Table 3: The best period T^* and CVaR of $(\alpha_1, \alpha_2) = (0.99, 0.99)$ and $N = 99$

(λ_1, λ_2)	T^*	1	2	3	4	5	6	7	8	9	10
(0.01,0.99)	4	0.003902	0.003126	0.001858	0.001849	0.002217	0.002200	0.002139	0.002284	0.002283	0.002622
(0.02,0.98)	4	0.007804	0.006252	0.003711	0.003699	0.004433	0.004390	0.004278	0.004567	0.004568	0.005245
(0.03,0.97)	4	0.011706	0.009377	0.005570	0.005551	0.006644	0.006585	0.006417	0.006850	0.006852	0.007868
(0.04,0.96)	4	0.015608	0.012503	0.007423	0.007397	0.008858	0.008781	0.008561	0.009141	0.009136	0.010495
(0.05,0.95)	4	0.019510	0.015629	0.009280	0.009249	0.011072	0.010975	0.010698	0.011417	0.011420	0.013120
(0.06,0.94)	4	0.023412	0.018755	0.011136	0.011096	0.013287	0.013172	0.012833	0.013701	0.013701	0.015741
(0.07,0.93)	4	0.027314	0.021881	0.012989	0.012945	0.015502	0.015368	0.014977	0.015984	0.015984	0.018362
(0.08,0.92)	4	0.031216	0.025007	0.014845	0.014794	0.017715	0.017563	0.017112	0.018267	0.018267	0.020980
(0.09,0.91)	4	0.035118	0.028132	0.016701	0.016643	0.019932	0.019755	0.019252	0.020553	0.020551	0.023603
(0.10,0.90)	4	0.039020	0.031258	0.018556	0.018493	0.022147	0.021950	0.021389	0.022841	0.022834	0.026225
(0.11,0.89)	4	0.042922	0.034384	0.020412	0.020345	0.024362	0.024145	0.023528	0.025118	0.025118	0.028850
(0.12,0.88)	4	0.046823	0.037510	0.022268	0.022191	0.026574	0.026340	0.025667	0.027401	0.027401	0.031474
(0.13,0.87)	4	0.050725	0.040635	0.024123	0.024040	0.028787	0.028534	0.027806	0.029685	0.029685	0.034101
(0.14,0.86)	4	0.054627	0.043761	0.025979	0.025893	0.031001	0.030730	0.029946	0.031975	0.031973	0.036715
(0.15,0.85)	4	0.058529	0.046887	0.027835	0.027739	0.033220	0.032925	0.032087	0.034256	0.034256	0.039337
(0.16,0.84)	4	0.062431	0.050013	0.029692	0.029588	0.035430	0.035120	0.034225	0.036541	0.036542	0.041960
(0.17,0.83)	4	0.066333	0.053140	0.031546	0.031438	0.037644	0.037316	0.036361	0.038818	0.038823	0.044582
(0.18,0.82)	4	0.070235	0.056266	0.033402	0.033287	0.039859	0.039512	0.038500	0.041102	0.041107	0.047205
(0.19,0.81)	4	0.074137	0.059390	0.035260	0.035136	0.042073	0.041707	0.040639	0.043385	0.043393	0.049827
(0.20,0.80)	4	0.078039	0.062516	0.037113	0.036985	0.044287	0.043898	0.042778	0.045669	0.045669	0.052450
(0.21,0.79)	4	0.081941	0.065642	0.038971	0.038835	0.046502	0.046093	0.044917	0.047952	0.047954	0.055073
(0.22,0.78)	4	0.085843	0.068767	0.040824	0.040684	0.048716	0.048287	0.047061	0.050235	0.050235	0.057695
(0.23,0.77)	4	0.089745	0.071894	0.042680	0.042533	0.050930	0.050482	0.049200	0.052521	0.052519	0.060322
(0.24,0.76)	4	0.093647	0.075020	0.044538	0.044382	0.053145	0.052677	0.051338	0.054802	0.054802	0.062944
(0.25,0.75)	4	0.097549	0.078145	0.046369	0.046232	0.055360	0.054872	0.053473	0.057086	0.057093	0.065566
(0.26,0.74)	4	0.101451	0.081271	0.048187	0.048081	0.057574	0.057067	0.055612	0.059369	0.059378	0.068189
(0.27,0.73)	4	0.105353	0.084396	0.050007	0.049930	0.059788	0.059262	0.057756	0.061660	0.061653	0.070807
(0.28,0.72)	4	0.109255	0.087522	0.051829	0.051779	0.062003	0.061457	0.059896	0.063943	0.063936	0.073430
(0.29,0.71)	4	0.113157	0.090648	0.053650	0.053629	0.064218	0.063652	0.062034	0.066226	0.066219	0.076052
(0.30,0.70)	3	0.117059	0.093774	0.055472	0.055478	0.066431	0.065847	0.064169	0.068503	0.068503	0.078675
(0.31,0.69)	3	0.120961	0.096885	0.057288	0.057328	0.068645	0.068041	0.066306	0.070786	0.070787	0.081297
(0.32,0.68)	3	0.124863	0.099985	0.059108	0.059177	0.070860	0.070239	0.068445	0.073070	0.073070	0.083920
(0.33,0.67)	3	0.128765	0.103032	0.060800	0.061026	0.073073	0.072431	0.070584	0.075353	0.075353	0.086542
(0.34,0.66)	3	0.132667	0.106042	0.062336	0.062858	0.075122	0.074610	0.072706	0.077619	0.077619	0.089165
(0.35,0.65)	3	0.136568	0.109054	0.063657	0.064682	0.076952	0.076780	0.074821	0.079879	0.079885	0.091787
(0.36,0.64)	3	0.140470	0.112062	0.064982	0.066505	0.078782	0.078948	0.076938	0.082135	0.082135	0.094410
(0.37,0.63)	3	0.144372	0.115072	0.066305	0.068329	0.080609	0.081117	0.079046	0.084398	0.084399	0.097032
(0.38,0.62)	3	0.148274	0.118082	0.067626	0.070153	0.082438	0.083286	0.081165	0.086651	0.086651	0.099655
(0.39,0.61)	3	0.152176	0.121092	0.068951	0.071977	0.084266	0.085460	0.083278	0.088909	0.088909	0.102277

Table 3: Continued

(0.40,0.60)	3	0.156078	0.124102	0.070272	0.073800	0.086094	0.087625	0.085386	0.091167	0.091167	0.104900
(0.41,0.59)	3	0.159980	0.127113	0.071598	0.075627	0.087923	0.089795	0.087499	0.093425	0.093425	0.107522
(0.42,0.58)	3	0.163882	0.130122	0.072918	0.077448	0.089753	0.091970	0.089613	0.095683	0.095683	0.110147
(0.43,0.57)	3	0.167784	0.133094	0.074242	0.079269	0.091581	0.094130	0.091728	0.097942	0.097943	0.112765
(0.44,0.56)	3	0.171686	0.136038	0.075563	0.081083	0.093409	0.096295	0.093846	0.100199	0.100199	0.115383
(0.45,0.55)	3	0.175588	0.138981	0.076882	0.082898	0.095227	0.098456	0.095953	0.102457	0.102457	0.117990
(0.46,0.54)	3	0.179490	0.141925	0.078201	0.084711	0.097044	0.100615	0.098066	0.104713	0.104714	0.120604
(0.47,0.53)	3	0.183392	0.144869	0.079520	0.086525	0.098861	0.102768	0.100178	0.106951	0.106956	0.123219
(0.48,0.52)	3	0.187294	0.147813	0.080837	0.088240	0.100601	0.104828	0.102289	0.108933	0.108934	0.125629
(0.49,0.51)	3	0.191196	0.150757	0.082134	0.089780	0.102247	0.106716	0.104317	0.110827	0.110827	0.127761
(0.50,0.50)	3	0.195098	0.153700	0.083298	0.091191	0.103723	0.108497	0.106224	0.112610	0.112610	0.129362
(0.51,0.49)	3	0.199000	0.156645	0.083551	0.091601	0.103634	0.108751	0.106079	0.112340	0.112340	0.128725
(0.52,0.48)	3	0.202902	0.159588	0.083337	0.091339	0.102624	0.107744	0.104965	0.111098	0.111101	0.126420
(0.53,0.47)	3	0.206804	0.162532	0.082688	0.090516	0.100951	0.105969	0.103303	0.109308	0.109316	0.124031
(0.54,0.46)	3	0.210706	0.165476	0.080951	0.088603	0.098946	0.103868	0.101273	0.107157	0.107159	0.121404
(0.55,0.45)	3	0.214608	0.168420	0.079214	0.086690	0.096943	0.101763	0.099234	0.104981	0.104981	0.118780
(0.56,0.44)	3	0.218510	0.171363	0.077477	0.084776	0.094933	0.099659	0.097177	0.102802	0.102804	0.116147
(0.57,0.43)	3	0.222412	0.174307	0.075740	0.082861	0.092927	0.097556	0.095123	0.100618	0.100621	0.113517
(0.58,0.42)	3	0.226313	0.177251	0.074005	0.080947	0.090920	0.095455	0.093070	0.098437	0.098439	0.110889
(0.59,0.41)	3	0.230215	0.180195	0.072266	0.079033	0.088914	0.093352	0.091015	0.096248	0.096249	0.108262
(0.60,0.40)	3	0.234118	0.182798	0.070530	0.077119	0.086908	0.091246	0.088958	0.094069	0.094068	0.105635
(0.61,0.39)	3	0.238019	0.183884	0.068792	0.075206	0.084901	0.089149	0.086899	0.091888	0.091885	0.103008
(0.62,0.38)	3	0.241921	0.184970	0.067055	0.073291	0.082895	0.087042	0.084848	0.089701	0.089704	0.100390
(0.63,0.37)	3	0.245823	0.185929	0.065319	0.071380	0.080889	0.084938	0.082792	0.087521	0.087520	0.097762
(0.64,0.36)	3	0.249725	0.186883	0.063583	0.069465	0.078884	0.082835	0.080739	0.085339	0.085333	0.095126
(0.65,0.35)	3	0.253627	0.187839	0.061858	0.067545	0.076877	0.080726	0.078684	0.083150	0.083154	0.092499
(0.66,0.34)	3	0.257529	0.188794	0.060071	0.065618	0.074873	0.078614	0.076621	0.080966	0.080968	0.089876
(0.67,0.33)	3	0.261431	0.189748	0.058304	0.063688	0.072792	0.076476	0.074537	0.078757	0.078753	0.087234
(0.68,0.32)	3	0.263905	0.190703	0.056537	0.061758	0.070603	0.074159	0.072280	0.076367	0.076368	0.084599
(0.69,0.31)	3	0.266241	0.191659	0.054770	0.059832	0.068397	0.071842	0.070019	0.073980	0.073980	0.081955
(0.70,0.30)	3	0.268578	0.192613	0.053004	0.057900	0.066191	0.069524	0.067761	0.071594	0.071594	0.079312
(0.71,0.29)	3	0.270914	0.193568	0.051237	0.055969	0.063984	0.067208	0.065502	0.069215	0.069207	0.076667
(0.72,0.28)	3	0.273250	0.194459	0.049470	0.054238	0.061780	0.064889	0.063243	0.066821	0.066823	0.074017
(0.73,0.27)	3	0.275586	0.195337	0.047703	0.052108	0.059575	0.062574	0.060984	0.064434	0.064435	0.071374
(0.74,0.26)	3	0.277922	0.196215	0.045936	0.050180	0.057369	0.060256	0.058726	0.062048	0.062049	0.068730
(0.75,0.25)	3	0.280259	0.197093	0.044170	0.048250	0.055160	0.057938	0.056467	0.059663	0.059661	0.066087
(0.76,0.24)	3	0.282595	0.197970	0.042403	0.046321	0.052953	0.055619	0.054208	0.057275	0.057275	0.063443
(0.77,0.23)	3	0.284931	0.198848	0.040636	0.044389	0.050746	0.053302	0.051950	0.054888	0.054892	0.060803
(0.78,0.22)	3	0.287267	0.199714	0.038869	0.042460	0.048540	0.050986	0.049691	0.052508	0.052507	0.058156
(0.79,0.21)	3	0.289604	0.200521	0.037102	0.040529	0.046333	0.048667	0.047436	0.050117	0.050124	0.055513
(0.80,0.20)	3	0.291940	0.201283	0.035336	0.038599	0.044128	0.046353	0.045174	0.047732	0.047729	0.052870
(0.81,0.19)	3	0.294276	0.202044	0.033569	0.036669	0.041921	0.044032	0.042921	0.045343	0.045349	0.050226
(0.82,0.18)	3	0.296612	0.202806	0.031802	0.034741	0.039716	0.041714	0.040656	0.042956	0.042956	0.047585
(0.83,0.17)	3	0.298948	0.203568	0.030036	0.032809	0.037508	0.039397	0.038400	0.040570	0.040570	0.044939
(0.84,0.16)	3	0.301285	0.204330	0.028268	0.030879	0.035302	0.037080	0.036141	0.038183	0.038183	0.042296
(0.85,0.15)	3	0.303621	0.205091	0.026503	0.028949	0.033095	0.034762	0.033880	0.035800	0.035798	0.039657
(0.86,0.14)	3	0.305957	0.205853	0.024735	0.027019	0.030893	0.032445	0.031625	0.033411	0.033416	0.037017
(0.87,0.13)	3	0.308293	0.206616	0.022968	0.025093	0.028683	0.030127	0.029364	0.031025	0.031024	0.034365
(0.88,0.12)	3	0.310629	0.207376	0.021201	0.023159	0.026478	0.027810	0.027110	0.028637	0.028637	0.031729
(0.89,0.11)	3	0.312966	0.208138	0.019435	0.021229	0.024270	0.025492	0.024852	0.026251	0.026256	0.029087
(0.90,0.10)	3	0.315302	0.208899	0.017668	0.019299	0.022064	0.023177	0.022587	0.023865	0.023865	0.026437
(0.91,0.09)	3	0.317638	0.209661	0.015902	0.017370	0.019857	0.020858	0.020329	0.021478	0.021478	0.023791
(0.92,0.08)	3	0.319974	0.210422	0.014134	0.015440	0.017655	0.018540	0.018073	0.019095	0.019098	0.021148
(0.93,0.07)	3	0.322310	0.211184	0.012367	0.013510	0.015446	0.016225	0.015811	0.016705	0.016705	0.018507
(0.94,0.06)	3	0.324647	0.211945	0.010603	0.011580	0.013238	0.013905	0.013552	0.014319	0.014319	0.015869
(0.95,0.05)	3	0.326983	0.212706	0.008834	0.009650	0.011036	0.011592	0.011297	0.011939	0.011932	0.013217
(0.96,0.04)	3	0.329319	0.213468	0.007067	0.007720	0.008825	0.009270	0.009035	0.009546	0.009551	0.010574

Table 3: Continued

(0.97,0.03)	3	0.331655	0.214229	0.005300	0.005790	0.006622	0.006957	0.006782	0.007160	0.007163	0.007931
(0.98,0.02)	3	0.333992	0.214988	0.003534	0.003861	0.004417	0.004636	0.004518	0.004773	0.004774	0.005287
(0.99,0.01)	3	0.336328	0.215748	0.001767	0.001930	0.002207	0.002322	0.002259	0.002387	0.002391	0.002645
MCVaR(T)	3	0.003902	0.003126	0.001767	0.001849	0.002207	0.002200	0.002139	0.002284	0.002283	0.002622
WCVaR(T)	3	0.336328	0.2157480	0.083551	0.091601	0.103723	0.108751	0.106224	0.112610	0.112610	0.129362

In Tab. 4, the numerical results of the minimum CVaR and VaR, proportion of lending within 3 periods are given for $(\lambda_1, \lambda_2) = (0.99, 0.01)$, $(\alpha_1, \alpha_2) = (0.99, 0.99)$ at $t = 1, 2, 3$. $(y^*_{t1}(\boldsymbol{x}^*_t), y^*_{t2}(\boldsymbol{x}^*_t))$ is $\boldsymbol{\alpha}$-VaR loss vector with confidence level $\boldsymbol{\alpha} = (0.99, 0.99)$ to $\boldsymbol{x}^*_t$ at period t. All values of $y^*_{t1}(\boldsymbol{x}^*_t)$ are zero at $t = 1, 2, 3$. But, all risk value of the first VaR are more than zero at $t = 1, 2, 3$. If we take into account the risk weight of the first objective (or surplus loss) for $\lambda_1 = 0.99$, bank will control loan for borrowers. This means that insufficient loss possibly take place.

In Tab. 5, the numerical results of the worst CVaR and VaR, proportion of lending within 3 periods are given for $(\lambda_1, \lambda_2) = (0.51, 0.49)$, $(\alpha_1, \alpha_2) = (0.99, 0.99)$ at $t = 1, 2, \cdots, 9$. All risk value of the VaR are more than zero at $t = 1, 2, 3$. The bank may control loan for borrowers.

Table 4. Ompare var with cvar at $(\lambda_1, \lambda_2) = (0.99, 0.01)$

t	$T^*_M = 9$	$y^*_{t1}(\boldsymbol{x}^*_t)$	$y^*_{t2}(\boldsymbol{x}^*_t)$	$\boldsymbol{x}^*_t = (x^*_{t1}, x^*_{t2}, x^*_{t3}, x^*_{t4})$
1	0.001767	0.00	0.031482	(0.006839, 0.948613, 0.005493, 0.000246)
2	0.001767	0.00	0.031290	(0.008767, 0.006576, 0.010851, 0.000105)
3	0.001767	0.00	0.113906	(0.011047, 0.000133, 0.000979, 0.000351)

Table 5. Compare var and cvar at $(\lambda_1, \lambda_2) = (0.51, 0.49)$

t	$T^*_W = 10$	$y^*_{t1}(\boldsymbol{x}^*_t)$	$y^*_{t2}(\boldsymbol{x}^*_t)$	$\boldsymbol{x}^*_t = (x^*_{t1}, x^*_{t2}, x^*_{t3}, x^*_{t4})$
1	0.083551	0.000749	0.030703	(0.008165, 0.814247, 0.009072, 0.000928)
2	0.083551	0.000472	0.030427	(0.009595, 0.006803, 0.010812, 0.001255)
3	0.083551	0.072625	0.032522	(0.131326, 0.000581, 0.004938, 0.002278)

4 Conclusion

We introduce the concept of α-VaR, α-CVaR for multiobjective loss based (or weight of objective) in time interval. Then, we define a multiobjective CVaR model (P2-1), another multiobjective programming model (P2-2) and single objective programming (P2-3) in time interval and prove that we can obtain the solution of (P2-1) and (P2-2) by solving the solution of (P2-3). For all weight of objective, we consider the minimum CVaR by solving (P2-4) and the worst CVaR by solving (P2-5). Approximate solution of (P2-1) to (P2-5) can be obtained by approximate problem (P2-1)$'$ to (P2-5)$'$ respectively. The problem (P2-3)$'$ is single objective programming. Then, we build multiobjective CVaR model for lending plan of bank in time interval. The multiobjective CVaR model can be transformed into linear programming. As an example, we give numerical results to find out the best period and proportion for lending plan. Numerical results show that the best period may be found out to be stabile for given confidence level and weight. The best period and proportion may direct and reference for loan plan of bank.

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